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A Functional Central Limit Theorem Connected with Extended Renewal Theory

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1. Introduction and Statement of Result

Let $(\Omega, \mathscr{A}, P)$ be a probability space, let $\xi_1, \xi_2, \ldots,$ be a sequence of i.i.d. random variables with expectation $\theta > 0$ and variance $\sigma^2 < \infty$. Set $S_0 = 0$ and $S_n = \sum_{\nu=1}^{n} \xi_\nu$, $n = 1, 2, \ldots$.

Let $X_n(t, \omega) = \frac{1}{\sigma\sqrt{n}}(S_{[nt]}(\omega) - [n t]\,\theta) = \frac{1}{\sigma\sqrt{n}} \cdot \sum_{\nu=1}^{[nt]} (\xi_\nu - \theta)$, $0 \leqq t \leqq 1$, and let W denote the Wiener measure on $(C, \mathscr{C})$, (cp. [2], Section 9).

By Donsker's theorem, (see [2], 137), $X_n \xrightarrow{\mathscr{D}} W$, as $n \to \infty$.

Let $N(c) = \min\{k; S_k > c \cdot k^p\}$, $c \geqq 0$, $0 \leqq p < 1$ and define

$$Z_n(t, \omega) = \frac{N(n\,t, \omega) - \lambda(n\,t)}{\sigma \cdot \theta^{-1} \cdot (1-p)^{-1} \cdot \sqrt{\lambda(n)}}, \quad 0 \leqq t \leqq 1, \; 0 \leqq p < 1,$$

where $\lambda(x) = (x/\theta)^{1/q}$, $x \geqq 0$, $q = 1 - p$.

Theorem. *$Z_n \xrightarrow{\mathscr{D}} W'$, as $n \to \infty$, where $W'(t) = W(t^{1/q})$, $0 \leqq t \leqq 1$.*

If $p = 0$ and if $\xi_1, \xi_2, \ldots,$ are assumed to be positive random variables, the theorem is contained in Billingsley [2], Theorem 17.3., p. 148.

If $p = 0$, the theorem has been proved by Basu [1] and Vervaat [11].

As is pointed out in [2], n may tend to infinity in a continuous manner.

Since the projections from C to R^k are continuous mappings, (see [2], 20), it follows that the finite-dimensional distributions converge to multidimensional normal distributions (cp. [2], 30). In particular, the following corollary holds.

Corollary. $Z_n(1) = \frac{N(n) - \lambda(n)}{\sigma\,\theta^{-1}(1-p)^{-1}\sqrt{\lambda(n)}} \xrightarrow{\mathscr{D}} N(0, 1)$, *as* $n \to \infty$, $0 \leqq p < 1$.

This result has already been proved by Siegmund [10] and later, differently, in [5], Theorem 3.5. If $p = 0$ the corollary reduces to Heyde [7], Theorem 4.

The method in [11] is first to prove that the maximum cumulative process converges weakly to the Wiener measure and then that its inverse, which is Z_n, converges. The present proof follows the lines of [2], (see also [1]), and is presented in Section 2. Finally, in Section 3 we show that, if $p = 0$, it is possible to derive the above theorem by an application of [2], Theorem 17.3 and its proof.

2.

Assume in the proof that $\theta > 1$. This is no restriction.

I. Define the random change of time

$$\Phi_n(t,\omega)=\begin{cases}\dfrac{N\left((n\,t)^q,\omega\right)}{n} & \text{if } \dfrac{N(n^q,\omega)}{n}\leqq 1\\ t/\theta^{1/q} & \text{otherwise}.\end{cases}$$

The first step is to prove that Φ_n converges in probability in the sense of the Skorohod topology, (see [2], 111 ff.), to φ, where $\varphi(t)=t/\theta^{1/q}$, $0\leqq t\leqq 1$.

Since

$$\sup_{0\leqq t\leqq 1}|\Phi_n(t,\omega)-\varphi(t)|\leqq \sup_{0\leqq t\leqq 1}|T_n^{(p)}(t,\omega)|,$$

where

$$T_n^{(p)}(t,\omega)=N\left((n\,t)^q,\omega\right)/n-t/\theta^{1/q},$$

this follows from the first of the following lemmata.

Lemma 1. $$\sup_{0\leqq t\leqq 1}|T_n^{(p)}(t)|\xrightarrow{\text{as}}0,\quad \textit{as } n\to\infty.$$

Lemma 2. *Suppose that* $f_n(t)$, $n=1,2,\ldots$, *is a sequence of real valued functions, such that* $f_n(t)$ *is a non-decreasing function of* t, $0\leqq t\leqq 1$, $n=1,2,\ldots$, *and suppose that* $f_n(t)\to t^\alpha$, *as* $n\to\infty$, $\alpha\geqq 0$, *for any fixed* $t\in Q$, *where* Q *is the set of rationals in* $[0,1]$. *Then, the convergence is uniform, i.e.*

$$\sup_{0\leqq t\leqq 1}|f_n(t)-t^\alpha|\to 0,\quad \textit{as } n\to\infty.$$

The proof of Lemma 2 is omitted.

Proof of Lemma 1. By Siegmund [9], Lemma 4, ($0<p<1$), and Heyde [6], Theorem 7, ($p=0$),

$$\frac{N(c)}{\lambda(c)}\xrightarrow{\text{as}}1,\quad \text{as } c\to\infty. \tag{1}$$

Therefore, $T_n^{(p)}(t)\xrightarrow{\text{as}}0$, as $n\to\infty$, for any fixed t, $0<t\leqq 1$. Since this is trivially true when $t=0$, it follows that $T_n^{(p)}(t)\xrightarrow{\text{as}}0$, as $n\to\infty$, $0\leqq t\leqq 1$, or, equivalently,

$$\frac{N\left((n\,t)^q\right)}{n}\xrightarrow{\text{as}}\frac{t}{\theta^{1/q}},\quad \text{as } n\to\infty,\ 0\leqq t\leqq 1. \tag{2}$$

Let $A_t=\{\omega;\ T_n^{(p)}(t,\omega)\not\to 0, \text{ as } n\to\infty\}$ and set $A=\bigcup_{t\in Q}A_t$. Then $P(A)=0$. Furthermore, $N\left((n\,t)^q,\omega\right)$ is non-decreasing and piecewise constant as a function of t. It follows that

$$T_n^{(p)}(t,\omega)\to 0,\quad \text{as } n\to\infty,\ t\in Q,\ \omega\notin A, \tag{3}$$

and hence, by Lemma 2, that

$$\sup_{0\leqq t\leqq 1}|T_n^{(p)}(t,\omega)|\to 0,\quad \text{as } n\to\infty,\ \text{for any } \omega\notin A,$$

i.e. Lemma 1 is proved.

Similarly,

$$\sup_{0 \leqq t \leqq 1} \left| \frac{N(n\,t)}{n^{1/q}} - \left(\frac{t}{\theta}\right)^{1/q} \right| \xrightarrow{\text{as}} 0, \quad \text{as } n \to \infty, \tag{4}$$

and

$$\sup_{0 \leqq t \leqq 1} \left| \frac{N(\sqrt{n} \cdot t)}{n^{\frac{1}{2q}}} - \left(\frac{t}{\theta}\right)^{1/q} \right| \xrightarrow{\text{as}} 0, \quad \text{as } n \to \infty, \tag{5}$$

are obtained.

II. $X_n \circ \Phi_n \xrightarrow{\mathscr{D}} W \circ \varphi$, as $n \to \infty$, where $(x \circ \varphi)(t) = x(\varphi(t))$. (Cp. [2], 144.)

III. Define $Y_n(t, \omega) = \dfrac{1}{\sigma \sqrt{n}} \cdot \sum_{\nu=1}^{N((nt)^q, \omega)} (\xi_\nu - \theta)$, $0 \leqq t \leqq 1$.

$$Y_n \xrightarrow{\mathscr{D}} W \circ \varphi, \quad \text{as } n \to \infty.$$

IV. Define $Z_n^*(t, \omega) = \dfrac{(n\,t)^q \cdot N^p((n\,t)^q, \omega) - \theta \cdot N((n\,t)^q, \omega)}{\sigma \sqrt{n}}$, $0 \leqq t \leqq 1$.

$$Z_n^* \xrightarrow{\mathscr{D}} W \circ \varphi, \quad \text{as } n \to \infty.$$

V.

$$-\theta^{\frac{1}{2q}} \cdot Z_n^* \xrightarrow{\mathscr{D}} W, \quad \text{as } n \to \infty.$$

Steps II, III, IV and V follow as in [2], (and [1]). In particular, if $p=0$, then

$$-\theta^{\frac{1}{2q}} \cdot Z_n^*(t, \omega) = Z_n(t, \omega), \quad 0 \leqq t \leqq 1, \tag{6}$$

and the theorem is proved.

VI. Assume during the rest of the proof that $0 < p < 1$ and define

$$Z_n^{**}(t, \omega) = \theta^{\frac{1}{2q}} \cdot \frac{\theta \cdot N(n\,t, \omega) - n\,t \cdot N^p(n\,t, \omega)}{\sigma \cdot n^{\frac{1}{2q}}} = \frac{N(n\,t, \omega) - \lambda^q(n\,t) \cdot N^p(n\,t, \omega)}{\sigma \cdot \theta^{-1} \cdot \sqrt{\lambda(n)}},$$

$0 \leqq t \leqq 1$.

From V it follows that

$$Z_n^{**} \xrightarrow{\mathscr{D}} W', \quad \text{as } n \to \infty, \quad \text{where} \quad W'(t) = W(t^{1/q}), \quad 0 \leqq t \leqq 1. \tag{7}$$

VII. The final step is to prove that

$$\sup_{0 \leqq t \leqq 1} |Z_n^{**}(t) - Z_n(t)| \xrightarrow{P} 0, \quad \text{as } n \to \infty. \tag{8}$$

The theorem then follows from (7), (8) and [2], Theorem 4.1. However,

$$\begin{aligned} \sup_{0 \leqq t \leqq 1} |Z_n^{**}(t) - Z_n(t)| &\leqq \sup_{0 \leqq t \leqq 1/\sqrt{n}} |Z_n^{**}(t) - Z_n(t)| \\ &+ \sup_{1/\sqrt{n} \leqq t \leqq 1} |Z_n^{**}(t) - Z_n(t)| \leqq \sup_{0 \leqq t \leqq 1/\sqrt{n}} |Z_n^{**}(t)| + \sup_{0 \leqq t \leqq 1/\sqrt{n}} |Z_n(t)| \\ &+ \sup_{1/\sqrt{n} \leqq t \leqq 1} |Z_n^{**}(t) - Z_n(t)|. \end{aligned}$$

Therefore, it suffices to show that

$$\sup_{0\leq t\leq 1/\sqrt{n}} |Z_n^{**}(t)| \xrightarrow{P} 0, \quad \text{as } n\to\infty, \tag{9}$$

$$\sup_{0\leq t\leq 1/\sqrt{n}} |Z_n(t)| \xrightarrow{P} 0, \quad \text{as } n\to\infty, \text{ and} \tag{10}$$

$$\sup_{1/\sqrt{n}\leq t\leq 1} |Z_n^{**}(t)-Z_n(t)| \xrightarrow{P} 0, \quad \text{as } n\to\infty. \tag{11}$$

(9) follows from (7) and Corollary 1 to Theorem 5.1 of [2] (cp. also [2], 70).

$$\begin{aligned}\sup_{0\leq t\leq 1/\sqrt{n}} |Z_n(t)| &= \theta\cdot(1-p)\cdot\sigma^{-1}\cdot \sup_{0\leq t\leq 1/\sqrt{n}}\left|\frac{N(n\,t)-\lambda(n\,t)}{\sqrt{\lambda(n)}}\right| \\ &= \theta^{1+\frac{1}{2q}}\cdot(1-p)\cdot\sigma^{-1}\cdot \sup_{0\leq t\sqrt{n}\leq 1}\left|\frac{N(n\,t)-(n\,t/\theta)^{1/q}}{n^{\frac{1}{2q}}}\right| \\ &= \theta^{1+\frac{1}{2q}}\cdot(1-p)\cdot\sigma^{-1}\cdot \sup_{0\leq s\leq 1}\left|\frac{N(\sqrt{n}\cdot s)}{n^{\frac{1}{2q}}}-\left(\frac{s}{\theta}\right)^{1/q}\right| \xrightarrow{\text{as}} 0, \quad \text{as } n\to\infty,\end{aligned}$$

by (5). Thus (10) holds.

Now, suppose that $c>0$. Taylor expansion gives (cp. [5])

$$\begin{aligned}&N(c,\omega)-\lambda^q(c)\cdot N^p(c,\omega)\\ &\quad = N(c,\omega)-\lambda^{1-p}(c)\cdot\{\lambda^p(c)+p\cdot(\lambda(c)+\rho\cdot(N(c,\omega)-\lambda(c)))^{p-1}\cdot(N(c,\omega)-\lambda(c))\}\\ &\quad = (N(c,\omega)-\lambda(c))\cdot(1-p+p\{1-(1+\rho\cdot(\lambda^{-1}(c)\cdot N(c,\omega)-1))^{p-1}\}),\end{aligned}$$

where $0\leq\rho=\rho(c,\omega)\leq 1$.

Therefore, if $t>0$,

$$Z_n^{**}(t,\omega)=\frac{N(n\,t,\omega)-\lambda(n\,t)}{\sigma\cdot\theta^{-1}\cdot\sqrt{\lambda(n)}}\cdot(1-p+p\cdot R_n(t,\omega))=Z_n(t,\omega)\cdot(1+p/q\cdot R_n(t,\omega)),$$

where $R_n(t,\omega)=1-(1+\rho\cdot(\lambda^{-1}(n\,t)\cdot N(n\,t,\omega)-1))^{p-1}$.

Hence, $Z_n^{**}(t,\omega)-Z_n(t,\omega)=Z_n^{**}(t,\omega)\cdot\dfrac{p\cdot R_n(t,\omega)}{q+p\cdot R_n(t,\omega)}$, and

$$\sup_{1/\sqrt{n}\leq t\leq 1}|Z_n^{**}(t)-Z_n(t)|\leq \sup_{0\leq t\leq 1}|Z_n^{**}(t)|\cdot \sup_{1/\sqrt{n}\leq t\leq 1}\left|\frac{p\cdot R_n(t)}{q+p\cdot R_n(t)}\right|. \tag{12}$$

By (7) and Corollary 1 to Theorem 5.1 of [2] (cp. also [2], p. 70),

$$\sup_{0\leq t\leq 1}|Z_n^{**}(t)| \xrightarrow{\mathscr{D}} \sup_{0\leq t\leq 1}|W'(t)| = \sup_{0\leq t\leq 1}|W(t)|, \quad \text{as } n\to\infty. \tag{13}$$

Furthermore,

$$\begin{aligned}&\sup_{1/\sqrt{n}\leq t\leq 1}|\lambda^{-1}(n\,t)\cdot N(n\,t,\omega)-1|\\ &\quad = \sup_{\sqrt{n}\leq n\,t\leq n}|\lambda^{-1}(n\,t)\cdot N(n\,t,\omega)-1|\leq \sup_{c\geq\sqrt{n}}|\lambda^{-1}(c)\cdot N(c,\omega)-1|\to 0, \quad \text{as } n\to\infty,\end{aligned}$$

for almost all ω, because of (1). Since $0 \leqq \rho \leqq 1$, it follows that $\sup_{1/\sqrt{n} \leqq t \leqq 1} |R_n(t)| \xrightarrow{\text{as}} 0$, as $n \to \infty$, which together with (12) and (13) implies (11).

The proof is complete.

Remark. The theorem remains valid if $\xi_1, \xi_2, \ldots,$ are independent random variables with common expectation $\theta > 0$ and variance $\sigma^2 < \infty$, provided they satisfy Lindeberg's condition (see also [1]).

3.

In classical renewal theory one studies sums of positive random variables and functions thereof, whereas in extended renewal theory one allows the summands to assume negative values. In this section we show that by applying [2], Theorem 17.3, and the steps of the proof which led to it, to specifically chosen sequences of positive i.i.d. random variables and a specific first passage time, the general result follows. Only the case $p=0$ is considered.

Let everything be given as before and suppose that $p=0$. An investigation of the proof of [2], Theorem 17.3 (cp. also [1]), shows that only the first step, i.e. the proof of

$$\sup_{0 \leqq t \leqq 1} \left| \frac{N(nt)}{n} - \frac{t}{\theta} \right| \xrightarrow{P} 0, \quad \text{as } n \to \infty, \tag{14}$$

depends on the fact that the summands are positive random variables. To prove the theorem it thus suffices to show that (14) also holds in the general case.

The idea of the following argument is the use of ladder variables. This method was introduced in Blackwell [3] to prove an extension of a renewal theorem.

Let $N_1, N_1+N_2, N_1+N_2+N_3, \ldots$ be the strong ascending ladder indices, i.e. $N_1 = \min\{k;\, S_k > 0\}$, $N_1+N_2 = \min\{k > N_1;\, S_k > S_{N_1}\}, \ldots$ and let $\eta_1 = S_{N_1}$, $\eta_1 + \eta_2 = S_{N_1+N_2}, \ldots$ be the corresponding ladder heights. Then $N_1, N_2, \ldots$ and $\eta_1, \eta_2, \ldots$ are two sequences of i.i.d. positive random variables. (See [3].)

Define the first passage time

$$M(c) = \min\left\{m;\, \sum_{\nu=1}^{m} \eta_\nu > c\right\}, \quad c \geqq 0.$$

By e.g. [5], Theorem 2.1, the variances of N_1, η_1, N and M are finite.

Furthermore, $\mu = E\eta_1 = \theta \cdot EN_1 \geqq \theta > 1$, by Wald's lemma and the assumptions.

Since a first passage must occur at a ladder point, it follows that

$$N(nt) = N_1 + \cdots + N_{M(nt)}. \tag{15}$$

Thus

$$\begin{aligned}
\left| \frac{N(nt)}{n} - \frac{t}{\theta} \right| &= \left| \frac{1}{n} \cdot \sum_{\nu=1}^{M(nt)} N_\nu - \frac{t}{\theta} \right| \\
&\leqq \left| \frac{1}{n} \cdot \sum_{\nu=1}^{M(nt)} N_\nu - \frac{M(nt) \cdot EN_1}{n} \right| + \left| \frac{M(nt) \cdot EN_1}{n} - \frac{t}{\theta} \right| \\
&= \left| \frac{1}{n} \cdot \sum_{\nu=1}^{M(nt)} (N_\nu - EN_\nu) \right| + \left| \frac{M(nt)}{n} - \frac{t}{\theta \cdot EN_1} \right| \cdot EN_1,
\end{aligned}$$

and so

$$\sup_{0\leq t\leq 1}\left|\frac{N(nt)}{n}-\frac{t}{\theta}\right|\leq \sup_{0\leq t\leq 1}\left|\frac{1}{n}\cdot\sum_{\nu=1}^{M(nt)}(N_\nu-EN_\nu)\right|+EN_1\cdot\sup_{0\leq t\leq 1}\left|\frac{M(nt)}{n}-\frac{t}{\mu}\right| \quad (16)$$

Set $\tau^2=\operatorname{Var} N_1$ and define

$$X_n'(t,\omega)=\frac{1}{\tau\sqrt{n}}\cdot\sum_{\nu=1}^{[nt]}(N_\nu-EN_\nu),\qquad 0\leq t\leq 1.$$

$$\Phi_n'(t,\omega)=\begin{cases}\dfrac{M(nt,\omega)}{n} & \text{if } \dfrac{M(n,\omega)}{n}\leq 1\\ t/\mu & \text{otherwise}\end{cases}$$

$$Y_n'(t,\omega)=\frac{1}{\tau\sqrt{n}}\cdot\sum_{\nu=1}^{M(nt,\omega)}(N_\nu-EN_\nu),\qquad 0\leq t\leq 1.$$

Since $\eta_1,\eta_2,\ldots$ are positive random variables with positive, finite expectation μ, and finite variance, Theorem 17.3 of [2] yields

$$V_n\xrightarrow{\mathscr{D}}W,\qquad \text{as } n\to\infty, \quad (17)$$

where

$$V_n(t,\omega)=\frac{M(nt,\omega)-nt/\mu}{(\operatorname{Var}\eta_1)^{\frac{1}{2}}\cdot\mu^{-\frac{3}{2}}\cdot\sqrt{n}},\qquad 0\leq t\leq 1.$$

Hence,

$$\sup_{0\leq t\leq 1}\left|\frac{M(nt)}{n}-\frac{t}{\mu}\right|\xrightarrow{P}0,\qquad \text{as } n\to\infty. \quad (18)$$

(Cp. [2], formula (17.22), 149.)

Since $N_1,N_2,\ldots$ are positive i.i.d. random variables, Donsker's theorem applies, i.e.

$$X_n'\xrightarrow{\mathscr{D}}W,\qquad \text{as } n\to\infty. \quad (19)$$

As in [2] it follows that

$$X_n'\circ\Phi_n'\xrightarrow{\mathscr{D}}W\circ\varphi',\quad \text{as } n\to\infty,\qquad \text{where } \varphi'(t)=t/\mu,\ 0\leq t\leq 1, \quad (20)$$

$$Y_n'\xrightarrow{\mathscr{D}}W\circ\varphi',\quad \text{as } n\to\infty, \quad (21)$$

$$\sqrt{\mu}\cdot Y_n'\xrightarrow{\mathscr{D}}W,\qquad \text{as } n\to\infty, \quad (22)$$

and

$$\sup_{0\leq t\leq 1}\left|\frac{1}{n}\cdot\sum_{\nu=1}^{M(nt)}(N_\nu-EN_\nu)\right|\xrightarrow{P}0,\qquad \text{as } n\to\infty. \quad (23)$$

Now (16), (18) and (23) yield (14), which thus holds in the general case.

In Section 2 of this paper Lemma 1 is used instead of (14). The lemma is a consequence of (1) and Lemma 2. In the case of positive summands, (again $p=0$), (1) was proved in Doob [4], Theorem 1. However, with the above method it is possible to deduce the general result from Doob's result as follows.

According to the strong law of large numbers,

$$\frac{1}{n}\cdot\sum_{\nu=1}^{n} N_\nu \xrightarrow{\text{as}} EN_1, \quad \text{as } n\to\infty.$$

From Richter [8], Theorem 1, it follows that

$$\frac{1}{M(c)}\cdot\sum_{\nu=1}^{M(c)} N_\nu \xrightarrow{\text{as}} EN_1, \quad \text{as } c\to\infty.$$

and thus, by (15),

$$\frac{N(c)}{M(c)} \xrightarrow{\text{as}} EN_1, \quad \text{as } c\to\infty.$$

Finally, by Doob's theorem and Wald's lemma,

$$\frac{N(c)}{c}=\frac{N(c)}{M(c)}\cdot\frac{M(c)}{c} \xrightarrow{\text{as}} EN_1\cdot\frac{1}{\mu}=EN_1\cdot\frac{1}{\theta\cdot EN_1}=\frac{1}{\theta}, \quad \text{as } c\to\infty.$$

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