Z. Wahrscheinlichkeitstheorie verw. Geb. 27, 105–107 (1973)

A Generalization of the Mean Ergodic Theorem in Banach Spaces

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The purpose of this note is to combine the weakly almost periodic compactification of de Leeuw and Glicksberg [1] and a mean ergodic theorem proven in [3] in an elementary fashion to obtain the following theorem.

Theorem. *Let T be a linear map of the Banach space X into itself. Suppose for each $x \in X$ the "orbit" $\{T^n x\}_0^\infty$ is weakly conditionally compact. In such a situation we call the semigroup $\{T^n\}_0^\infty$ weakly almost periodic. Then given $x \in X$, $\varepsilon > 0$, and $\{n_i\}$ an increasing sequence of non-negative integers of positive lower density; there exists $x_1, x_2, \ldots, x_k \in X$ and complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ of modulus 1 such that:*

$$\limsup_N \left\| \frac{1}{N} \sum_1^N T^{n_i} x - \sum_{j=1}^k x_j \frac{1}{N} \sum_1^N \lambda_j^{n_i} \right\| < \varepsilon.$$

We note that, since $\frac{1}{N} \sum_1^N e^{in\theta} \to 0$ for $\theta \neq 0(2\pi)$, the mean ergodic theorem follows as a corollary.

Now suppose T satisfies the conditions of the theorem. By the uniform boundedness principle $\|T^n\| \leqq M$ for some finite M and all n. We define B_p to be the closed invariant subspace of X generated by all eigenvectors of T with eigenvalue on the unit circle. B_p is called the set of almost periodic vectors. We define B_w as follows: $x \in B_w$ iff there exists $A \subset Z^+$ with density $d(A)$ of $A = 1$ such that $\underset{n \in A}{T^n x} \to 0$ weakly. B_w is called the set of weakly mixing vectors. It follows trivially that $B_p \cap B_w = \{0\}$.

Lemma 1. *Let $\{A_i\}$ be a countable family of sets of positive integers each of density 1. Then there exists $\bar{A}$ of density 1 such that $\bar{A} \setminus A_i$ is finite for each A_i.*

Proof. Simply take $\bar{A} = \bigcap_1^\infty A_i \cup \{n \leqq q_i\}$ for a suitable sequence q_i; for example such that

$$\frac{\operatorname{card}\{n : n \leqq \ell \text{ and } n \in A_i\}}{\ell} \geqq 1 - \frac{1}{2^i} \quad \text{for } \ell \geqq q_i.$$

Corollary 1. *B_w is a closed invariant subspace of X.*

Proof. $d(A) = 1 \Leftrightarrow d(A \oplus 1) = 1$ implies B_w is invariant; $d(A) = 1$ and $d(B) = 1 \Rightarrow d(A \cap B) = 1$ implies that B_w is linear. For the closure suppose $x_i \in B_w$ and $x_i \to \bar{x}$. Choose A_i such that $\underset{n \in A_i}{T^n x_i} \to 0$ weakly. Then $\underset{n \in \bar{A}}{T^n \bar{x}} \to 0$ weakly where $\bar{A}$ is constructed as in the above Lemma.

Definition 1. Let $\{x_i\}$ be a sequence of vectors in a Banach space. We define $\mathfrak{F}(\{x_i\})$ to be a countable family of linear functionals constructed with the aid of the Hahn-Banach theorem satisfying the following property: For each finite sequence of non-negative rationals $\{\alpha_i\}_1^k$, whose sum is 1, there exists $f \in \mathfrak{F}$ of norm one such that

$$f\left(\sum_1^k \alpha_i x_i\right) = \left\|\sum_1^k \alpha_i x_i\right\|.$$

Lemma 2. *Let $\{x_i\}$ be a conditionally compact sequence in the weak topology of the Banach space X. Then $x_i \to 0$ weakly iff $\lim_i f(x_i) = 0$ for all $f \in \mathfrak{F}(\{x_i\})$.*

Proof. If $x_i \not\to 0$ weakly, $\exists\, g \in X^*$ and a subsequence $\{x_\ell\}$ s.t. $g(x_\ell) > \varepsilon > 0$ for all ℓ where g has norm 1. Hence by the conditional weak compactness of the subsequence $\{x_\ell\}$ there exists a nonzero weak accumulation point of this sequence $\bar{x}$.

By a well known theorem of Mazur $\bar{x}$ may be approximated abitrarily in norm by a rational convex combination of elements in $\{x_\ell\}$. It follows that there exists $\bar{f} \in \mathfrak{F}(\{x_i\})$ with $\bar{f}(\bar{x}) \neq 0$. This contradicts $\bar{f}(x_\ell) \to 0$ weakly.

Corollary 2. $x \in B_w$ *iff* $\frac{1}{N} \sum_1^N |f(T^n x)| \to 0$ *for all* $f \in X^*$.

Proof. Apply Lemma 1 to the family

$$\mathfrak{A} = \left\{\left\{n \colon |f(T^n x)| < \frac{1}{k}\right\};\, k \in Z^+, f \in \mathfrak{F}(\{T^n x\})\right\}$$

obtaining $\bar{A}$. Then $f(\underset{n \in \bar{A}}{T^n} x) \to 0$ weakly for all $f \in \mathfrak{F}(\{T^n x\})$. Since $\mathfrak{F}(\{T^n x\}_{n \in \bar{A}}) \subset \mathfrak{F}(\{T^n x\})$, $\underset{n \in \bar{A}}{T^n} x \to 0$ weakly by Lemma 2.

Since the main result in [3] asserts that $\frac{1}{N} \sum_1^N T^{n_i} x \to 0$ for all $\{n_i\}$ of positive lower density and all $x \in B_w$, our theorem will be proven if we can show that each $x \in X$ may be written as $x = x_1 + x_2$ where $x_1 \in B_p$, $x_2 \in B_w$. To this end we consider the semigroup $\{T^n\}_0^\infty$ of operators on X in the weak operator topology, i.e. the weakest topology on $\{T^n\}_0^\infty$ in which all of the functions $f(T^n x)$ are continuous ($f \in X^*$, $n \in Z^+$, $x \in X$). The closure of $\{T^n\}_0^\infty$ in this topology is a compact commutative semigroup of operators $\bar{S}$ on X in which multiplication is separately continuous [1]. We also have $\|s\| \leqq M$ for all $s \in \bar{S}$. Since $\bar{S}$ is commutative it contains a minimal ideal K which is algebraically a group and topologically closed. By a theorem of Ellis [2], K is a compact topological group in which multiplication is jointly continuous. The identity E of K is a projection on X, i.e. $E^2 = E$. Since K is an ideal the restriction of $\bar{S}$ to the subspace "range E" is a compact topological group in the weak operator topology of uniformly bounded linear operators. By the use of weak integration and some rather delicate machinery in the theory of compact topological groups, it is proven in [1] that "range E" is contained in the closed invariant subspace generated by the finite dimensional invariant subspaces of the semigroup $\{T^n\}$ restricted to "range E". Since the action of T^n on each of these finite dimensional spaces has a uniformly bounded inverse, T may be diagonalized on such a subspace. Clearly each eigenvalue has modulus 1 and "range E" $\subset B_p$.

We now write $X = EX + (I-E)X$ and it remains to show $Ex=0 \Rightarrow x \in B_w$. Now consider $\mathfrak{F}(\{T^n x\})$. Since E is in the weak operator closure of $\{T^n\}$ there exists for each finite subfamily $\bar{\mathfrak{F}} \subset \mathfrak{F}(\{T^n x\})$ a subsequence ℓ_i such that $f(T^{\ell_i} x) \to 0$ for all $f \in \bar{\mathfrak{F}}$. By the usual diagonalization argument we may construct a sequence q_i such that $f(T^{q_i} x) \to 0$ for all $f \in \mathfrak{F}(\{T^n x\}) \supset \mathfrak{F}(\{T^{q_i} x\})$. Hence $T^{q_i} x \to 0$ weakly. Now consider the bounded class of functions $g_i(s) = |f(s\,T^{q_i} x)|$ which are in $C(\bar{S})$. Since $f(s\,y) \in X^*$ for each $s \in \bar{S}$, $g_i(s) \to 0$ pointwise in $C(\bar{S})$. Hence $g_i \to 0$ in the weak topology of $C(\bar{S})$ by the bounded convergence theorem. By Mazur's theorem for each $\varepsilon > 0$ we may find $\alpha_1, \alpha_2, \ldots, \alpha_p \geqq 0 \, (\sum \alpha_i = 1)$ such that

$$\sum_{i=1}^{p} \alpha_i \, |f(s\,T^{q_i} x)| < \varepsilon \qquad \text{for all } s \in \bar{S}.$$

Since $T^n \in \bar{S}$, we have

$$\varepsilon > \frac{1}{N} \sum_{n=1}^{N} \sum_{i=1}^{p} \alpha_i \, |f(T^{n+q_i} x)|$$

$$= \sum_{i=1}^{p} \alpha_i \left(\frac{1}{N} \sum_{1}^{N} |f(T^n x)| - \frac{1}{N} \sum_{1}^{q_i} |f(T^n x)| + \frac{1}{N} \sum_{N+1}^{N+q_i} |f(T^n x)| \right).$$

Hence $\limsup_N \frac{1}{N} \sum_{1}^{N} |f(T^n x)| \leqq \varepsilon$. Since ε, f were arbitrary $x \in B_w$ and the proof is complete.

References

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(Received November 1, 1972)