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Variations of Processes with Stationary, Independent Increments

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1. Introduction

Let $X(t)$, $t \geqq 0$ be a stochastic process in R^N with stationary independent increments. We wish to consider limiting behavior of sequences of variational sums:

$$\sum_k f\big(X(t_{n,k}) - X(t_{n,k-1})\big), \tag{1.1}$$

where $f: R^N \longrightarrow R$ is a certain function and $\{P_n\}$ is a sequence of partitions, as the mesh of $P_n = \{t_{n,k}\}$ tends to zero.

Theorem 3.3 gives sufficient conditions that (1.1) converge in probability and identifies the limit. Theorem 4.3, almost a converse of Theorem 3.3, gives conditions under which truncated expectations of (1.1) diverge. Then (1.1) diverges at the same rate. Theorems 3.1 and 3.2 give sufficient conditions that (1.1), appropriately centered, converges in probability or, in one case, in distribution. Theorems 4.1 and 4.2, on the other hand, give sufficient conditions that truncated variances of (1.1) diverge. The distribution of (1.1) appropriately centered and normalized then converges to a normal law. In Section 2 we introduce notation and state some well-known facts. In Section 5 we present the results of calculations for special cases.

In [9] Loève considered a triangular array $\{X_{n,k}: k=1, \dots, k_n;\ n=1, \dots\}$ of random variables which are independent for each fixed n. Assuming $\sum_k X_{n,k}$ to converge in law, he asked about the limiting behavior of $\sum_k f(X_{n,k})$, appropriately centered, and obtained convergence in law for most f such that $f'(0)$ exists. With $X_{n,k} = X(t_{n,k}) - X(t_{n,k-1})$, (1.1) takes the form $\sum_k f(X_{n,k})$. Remarks 3.11 and 4.2 show that our results about convergence in law cannot always be proved for triangular arrays; however we have so stated and proved Theorems 4.1 and 4.3.

In order to facilitate the statement of our results we have divided the processes with stationary, independent increments into four types which are defined in Section 2. Most previous studies have treated processes of what we call type A or type C. Conditions for the convergence of (1.1) for type A processes are discussed by Fristedt in [4]. In case $f(x) = |x|^p$, $0 < p \leqq 2$, Millar in Theorem 4.1 of [12] gives necessary and sufficient conditions that the variational sums (1.1) converge in probability if $X(t)$ is of type A or type C. Sharpe, [13] and [14], places a condition on the Lévy measure ν sufficient to prove theorems similar to our Theo-

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rems 3.1 and 3.3. For Gaussian processes the limiting behavior of (1.1) has been studied extensively, beginning with Lévy [8]. Theorem 4.3 is an extension of Theorem 4 of [7]. There, Greenwood considers $f(x)=|x|^\beta$ for a stable process of index β.

Millar, [11], gave an almost sure convergence theorem for variational sums of certain symmetric processes where the sequence of partitions is nested. Cogburn and Tucker, [3], obtained almost sure convergence in the non-symmetric case under rather restrictive conditions on f. We know of no theorem similar to our Theorem 3.3 for non-symmetric type C processes and giving almost sure convergence. Several additional problems exist connected with the suprema of families of variational sums, as opposed to the limits of sequences considered here. Some results in this direction appear in [1, 2, 5, 7, 11].

2. Preliminaries

The general process X in R^N with stationary, independent increments has the following well-known characterization [6, p. 273]:

$$\log E e^{i\langle u, X(t)\rangle} = t\left[i\langle u,\gamma\rangle - u' S u/2 + ⨍ [e^{i\langle u,x\rangle} - 1 - i\langle u, \chi(x)\rangle]\, \nu(dx)\right], \tag{2.1}$$

where $\langle , \rangle$ denotes the inner product, $⨍$ denotes integration over the domain $R^N - \{0\}$,

$$[\chi(x)]_i = \begin{cases} -1 & \text{if } x_i \leqq -1 \\ x_i & \text{if } |x_i| < 1 \\ 1 & \text{if } x_i \geqq 1, \end{cases}$$

x_i is the i-th coordinate of x, γ is a real vector, S is a non-negative definite, symmetric matrix, and ν, called the Lévy measure, is a Borel measure on $R^N - \{0\}$ such that

$$⨍ |\chi(x)|^2 \, \nu(dx) < \infty .$$

Given such a γ, S, and ν, there exists a corresponding process with stationary, independent increments. Here it is understood that $X\colon [0,\infty)\times\Omega \to R^N$ where $(\Omega, \mathscr{F}, P)$ is a complete probability space. We usually write $X(t)$ for $X(t,\omega)$. We define $\nu(0)=0$ for notational convenience.

We have $X(0,\omega)=0$. It is well-known that we may assume X to be a strong Markov process and $X(\cdot,\omega)$ to be right continuous and to have left limits everywhere.

In consequence of (2.1), if $\int\limits_{|x|>1} |x|\,\nu(dx) < \infty$, then

$$EX(t) = t\left[\gamma + \int (x - \chi(x))\, \nu(dx)\right]. \tag{2.2}$$

Also, if $\int |x|^2\, \nu(dx) < \infty$, then

$$E\,|X(t) - EX(t)|^2 = t\left[\operatorname{trace}(S) + \int |x|^2\, \nu(dx)\right]. \tag{2.3}$$

Let $J(t,\omega) = X(t,\omega) - X(t-,\omega)$. Ito's representation [6, p. 271] states that

$$X(t,\omega) = \gamma t + W(t,\omega) + \lim_{\varepsilon\to 0}\left[\sum \{J(s,\omega) : |J(s,\omega)| > \varepsilon, s \leqq t\} - t \int\limits_{|x|>\varepsilon} \chi(x)\,\nu(dx)\right] \tag{2.4}$$

where W is a Gaussian process with stationary, independent increments (i.e. Brownian motion possibly flattened in some directions and stretched in others) such that W and $X-W$ are independent.

In case $S=0$ and $\int |\chi(x)|\, \nu(dx)<\infty$, we say that X is of type A or type B, type A if $\alpha=0$ and type B if $\alpha \neq 0$, where

$$\alpha=\gamma-\int \chi(x)\, \nu(dx). \tag{2.5}$$

We say that X is a type C process if $S=0$ and it is not a type A or type B process. If $S\neq 0$, we say X is of type D. In case X is of type A or B, (2.5) simplifies:

$$X(t, \omega)=\alpha t+\sum \{J(s, \omega)\colon s \leqq t\} \tag{2.6}$$

where the sum converges absolutely. If, in addition, $\int |x|\, \nu(dx)<\infty$, (2.2) becomes

$$EX(t)=t\left[\alpha+\int x\, \nu(dx)\right]. \tag{2.7}$$

Let $f\colon R^N \to R$ be measurable with $f(0)=0$. Suppose that $\int |\chi(f(x))|\, \nu(dx)<\infty$. ($\chi$ is one-dimensional here.) Then we let $X(f, t, \omega)=X(f, t)=X(f)$ be the type A process with Lévy measure ν_f defined by

$$\nu_f(C)=\nu(f^{-1}(C)) \tag{2.8}$$

for Borel subsets C of R. We can use (2.4) to define $X(f)$ on the same probability space $(\Omega, \mathscr{F}, P)$ on which X was defined; specifically

$$X(f, t, \omega)=\sum \{f(J(s, \omega))\colon s \leqq t\}. \tag{2.9}$$

Formulas (2.7) and (2.3) become:

$$EX(f, t)=t \int f(x)\, \nu(dx), \tag{2.10}$$

$$\operatorname{Var} X(f, t)=t \int |f(x)|^2\, \nu(dx), \tag{2.11}$$

in case $\int |f(x)|\, \nu(dx)<\infty$ or $\int |f(x)|^2\, \nu(dx)<\infty$, respectively.

Let f be as above but assume only that $\int |\chi(f(x))|^2\, \nu(dx)<\infty$. Define ν_f by (2.8) and let $Y(f, t, \omega)=Y(f, t)=Y(f)$ be the process characterized by the basic formula (2.1) with X, γ, S, and ν replaced, respectively, by $Y(f), 0, 0$, and ν_f. We can use (2.4) to define $Y(f)$ on $(\Omega, \mathscr{F}, P)$:

$$Y(f, t, \omega)=\lim_{\varepsilon \to 0}\left[\sum \{f(J(s, \omega))\colon |J(s, \omega)|>\varepsilon, s \leqq t\}-t \int_{|x|>\varepsilon} \chi(f(x))\, \nu(dx)\right]. \tag{2.12}$$

Formulas (2.2) and (2.3) become:

$$EY(f, t)=t \int [f(x)-\chi(f(x))]\, \nu(dx), \tag{2.13}$$

$$\operatorname{Var} Y(f, t)=t \int f(x)^2\, \nu(dx), \tag{2.14}$$

in case $\int_{|x|>1} |f(x)|\, \nu(dx)<\infty$ or $\int f(x)^2\, \nu(dx)<\infty$, respectively.

Let $f\colon R^N \to R$ and $h\colon [0, t] \to R^N$. Let

$$V(f, h, t, P_n)=\sum f(h(t_{n,k})-h(t_{n,k-1}))$$

where $P_n = \{t_{n,k}: k=0, \ldots, k_n\}$ is an ordered partition of $[0,t]$. We are interested in the limiting behavior of the sequence $V(f, X(\cdot, \omega), t, P_n)$. Throughout, $\{P_n\}$ is some fixed sequence of partitions of $[0,t]$ with $\operatorname{mesh}(P_n) \to 0$ as $n \to \infty$. When no confusion results we write $V_n(f)$ for $V(f, X, t, P_n)$.

If $f: R^N \to R^M$, $X(f)$, $Y(f)$, and $V_n(f)$ are defined as above and Theorems 3.1 and 3.3 can be proved by considering the components of f one at a time. With a slight modification (see Remark 3.4) Theorem 3.2 remains true.

We shall need some further notation. Let $F_{n,k}$ be the distribution measure of $X(t_{n,k}) - X(t_{n,k-1}) = X_{n,k}$. For $\theta \in R^N$, $|\theta| = 1$, and $w \in [0, \infty)$ define

$$f_r(\theta, w) = f(\theta w). \tag{2.15}$$

The symbol $\int\limits_0$ denotes an integral whose domain is a bounded neighborhood of 0 in R^N. If $a, b \in R$, $a \vee b$ and $a \wedge b$ will denote $\max\{a, b\}$ and $\min\{a, b\}$, respectively.

The Central Convergence Criterion ([16, p. 184 for $N=1$] for example) will be useful. Note that an infinitely divisible random variable Y in R^N has a canonical characterization similar to (2.1):

$$\log E e^{i\langle u, Y\rangle} = i\langle u, \gamma\rangle - u' S u/2 + \int [e^{i\langle u, x\rangle} - 1 - i\langle u, \chi(x)\rangle]\, \nu(dx). \tag{2.16}$$

Central Convergence Criterion: Let $\{Y_{n,k}: k=1, \ldots, k_n,\ n=1,2,\ldots\}$ be an infinitesimal triangular array of random vectors in R^N, independent for fixed n, with distribution measures $F_{n,k}$. Let $\{\beta_n\}$ be a sequence in R^N. In order that $\sum\limits_k Y_{n,k} - \beta_n$ converge in distribution to a random variable Y having characteristic exponent (2.16) it is necessary and sufficient that:

$$\sum_k F_{n,k}(U) \to \nu(U), \qquad \text{for all Borel sets } U \text{ with } \nu(\partial U) = 0,\ 0 \notin \bar{U}, \tag{2.17}$$

$$\lim_{\varepsilon \to 0} \varlimsup_{n \to \infty} \sum_k \Big(\int\limits_{|x|<\varepsilon} x x' F_{n,k}(dx) - \Big[\int\limits_{|x|<\varepsilon} x F_{n,k}(dx)\Big]\Big[\int\limits_{|x|<\varepsilon} x' F_{n,k}(dx)\Big]\Big) = S, \tag{2.18}$$

$$\sum_k \int \chi(x) F_{n,k}(dx) - \beta_n \to \gamma. \tag{2.19}$$

Throughout, in place of χ one could write any bounded, continuous function whose difference from x is $O(|x|^2)$ as $x \to 0$. There would be a compensatory change in the vector γ.

3. Convergence of Variational Sums

We first state as a lemma an elementary fact which will be used often in the sequel.

Lemma 3.1. *Let $f: R^N \to R$ be a continuous, bounded function. Let U be a Borel subset of R^N such that $\nu(\partial U) = 0$ and $0 \notin \bar{U}$. Then*

$$\sum_k \int_U f(x) F_{n,k}(dx) \to t \int_U f(x)\, \nu(dx) \quad \text{as} \quad n \to \infty.$$

Proof. Clearly $\sum X_{n,k} \to X(t)$ in distribution. In fact, equality holds. Condition (2.17) of the Central Convergence Criterion together with the Helly-Bray Theorem [10, p. 182] implies the lemma.

In Lemmas 3.2 and 3.3 the condition $0 \notin \bar{U}$ is removed, under additional hypotheses.

If X is of type A and $f(x)=g(|x|)$ where g is concave, it is possible to obtain a necessary and sufficient condition, namely condition (3.1) below, for $\{V_n(f)\}$ to converge almost surely. We state the next lemma in order to record the resulting convergence of expected values, used later in obtaining convergence of $\{V_n(f)\}$ in probability for more general f.

Lemma 3.2. *Let g: $[0, \infty) \to [0, \infty)$. Suppose that $g(0)=0$, and that g is concave in a neighborhood of 0, continuous, and bounded. Suppose that X is a process of type A and that*

$$\int g(|x|)\, \nu(dx) < \infty. \tag{3.1}$$

Then

$$\sum \int_U g(|x|)\, F_{n,k}(dx) \to t \int_U g(|x|)\, \nu(dx) \tag{3.2}$$

as $n \to \infty$, for any Borel set $U \subset R^N$ such that $\nu(\partial U)=0$.

Proof. Suppose we prove the lemma for $U=R^N$ and g concave everywhere. Then from Lemma 3.1, the result follows first for U with $0 \notin \bar{U}$, then for complements of such sets, and finally, by a limiting argument, for U with $0 \in \partial U$.

Let $f(x)=g(|x|)$. Then,

$$V_n(f)=\sum_k g(|X_{n,k}|) \leqq \sum_k \left(\sum \{g(|J(s)|)\colon t_{n,k-1} < s \leqq t_{n,k}\}\right) = X(f, t).$$

On the other hand,

$$\liminf V_n(f) \geqq \sum \{g(|J(s)|)\colon s \leqq t\} = X(f, t).$$

The dominated convergence theorem implies that $EV_n(f) \to EX(f, t)$, which, by (2.10), is statement (3.2).

Next we prove a lemma which is related to type C processes as Lemma 3.2 is to type A processes. The hypothesis that g is concave makes Lemma 3.2 almost trivial. If X is of type C a concave g will not satisfy (3.1). However, the analogous hypothesis that $g(\sqrt{\ })$ is concave leads eventually to the same conclusion.

Lemma 3.3. *Let g: $[0, \infty) \to [0, \infty)$. Suppose that $g(0)=g'(0)=0$, that g is continuous and bounded, and that in a neighborhood of 0, g is convex and $x \to g(x^{\frac{1}{2}})$ is concave. Suppose that X is not of type D and that*

$$\int g(|x|)\, \nu(dx) < \infty. \tag{3.3}$$

Then,

$$\sum \int_U g(|x|)\, F_{n,k}(dx) \to t \int_U g(|x|)\, \nu(dx) \tag{3.4}$$

as $n \to \infty$, for any Borel set $U \subset R^N$ such that $\nu(\partial U)=0$.

Proof. With no loss of generality we may assume that $\nu(x\colon |x|>a)=0$; a limiting argument [11] for $a \to \infty$ completes the proof. For each $w_0>0$ write $g=g_1+g_2$ where

$$g_1(w)=\begin{cases} g(w), & w \leqq w_0 \\ g(w_0)+(w-w_0)\, g'(w_0), & w > w_0. \end{cases}$$

Since $\nu(x\colon |x|>a)=0$ we may apply Lemma 3.1 to $x\to g_2(|x|)$ even though g_2 grows linearly at ∞. Hence (3.4) is true with g_2 in place of g. With g replaced by g_1 and U by R^N the left side of (3.4) is, by Theorem 4.1 of [11], bounded by the right side of (3.4). But by the monotone convergence theorem $\int\limits_{R^N} g_1(|x|)\,\nu(dx)$ can be made as small as desired by a choice of $w_0>0$ sufficiently small.

Theorem 3.1. *Let $f\colon R^N\to R$ be continuous and suppose that there exists a continuous function $g\colon [0,\infty)\to[0,\infty)$ such that $f(x)^2\leqq g(|x|)$ for x in a neighborhood of 0, $g(0)=0$,*

$$\int\limits_0 g(|x|)\,\nu(dx)<\infty,$$

and: if X is of type A, g is concave;

if X is of type B, g is concave and $|f(x+y)^2-f(y)^2|\leqq g(|x|)$ for all x,y in some neighborhood of 0;

if X is of type C, $g'(0)=0$, g is convex and $x\to g(x^{\frac{1}{2}})$ is concave;

if X is of type D, $g'(0)=g''(0)=0$.

Then,

$$V_n(f)-EV_n(\chi\circ f)\to Y(f,t)$$

in probability as $n\to\infty$. If, in addition, f is bounded, then this convergence also takes place in L_2.

Proof. Write $\beta_n(f)=EV_n(\chi\circ f)$. Since we may write f as the difference of two non-negative functions and treat these two functions separately, we assume that f itself is non-negative. Let δ be a small positive number such that $\nu\{x\colon |x|=\delta\}=\nu\{x\colon |x|=1/\delta\}=0$. Write $f=f_1+f_2+f_3$, where $f_1(x)=0$ if $|x|\geqq\delta$, $f_2(x)=0$ if $|x|<\delta$ or $|x|>1/\delta$, and $f_3(x)=0$ if $|x|\leqq 1/\delta$. Let $\varepsilon>0$. We shall complete the proof by showing, for sufficiently small δ and sufficiently large n depending on δ, that:

$$EY(f_1)^2<\varepsilon; \tag{3.5}$$

$$E[V_n(f_1)-\beta_n(f_1)]^2<\varepsilon; \tag{3.6}$$

$$E[V_n(f_2)-\beta_n(f_2)-Y(f_2)]^2<\varepsilon; \tag{3.7}$$

$$P\{|Y(f_3)|>\varepsilon\}<\varepsilon; \tag{3.8}$$

$$P\{|V_n(f_3)|>\varepsilon\}<\varepsilon. \tag{3.9}$$

These inequalities constitute a proof since $\beta_n(f_3)=0$, and if f is bounded then $f_3=0$ for sufficiently small δ.

Proof of (3.5). For δ sufficiently small $EY(f_1)=0$. By (2.14), $EY(f_1)^2=t\int f_1(x)^2\,\nu(dx)<\varepsilon$ for sufficiently small δ.

Proof of (3.6). For sufficiently small δ, $\beta_n(f_1)=EV_n(f_1)$. Since the variance of a sum of independent random variables is the sum of the variances we may rewrite the left side of (3.6) as

$$\sum\{\int f_1(x)^2\,F_{n,k}(dx)-(\int f_1(x)\,F_{n,k}(dx))^2\}\leqq\sum\int\limits_{|x|<\delta} g(|x|)\,F_{n,k}(dx). \tag{3.10}$$

In case X is of type A or type C, use Lemma 3.2 or Lemma 3.3 to deduce that the right side of (3.10) approaches

$$t \int_{|x|<\delta} g(|x|)\, \nu(dx)$$

which is less than ε for sufficiently small δ.

If X is of type D we obtain the result by combining the result for type C processes with what is known about Brownian motion. The hypotheses on f are designed so that the linear deterministic part of a type B process has no effect.

Proof of (3.7). We use Sharpe's argument [14, p. 1434]. We have

$$\beta_n(f_2) = EV_n(f_2) - \sum \int [f_2(x) - \chi(f_2(x))]\, F_{n,k}(dx).$$

By Lemma 3.1 and (2.13)

$$\sum \int [f_2(x) - \chi(f_2(x))]\, F_{n,k}(dx) \to t \int [f_2(x) - \chi(f_2(x))]\, \nu(dx) = EY(f_2).$$

If suffices, therefore, to prove

$$E[(V_n(f_2) - EV_n(f_2)) - (Y(f_2) - EY(f_2))]^2 \to 0 \tag{3.11}$$

as $n \to \infty$. The left side of (3.17) equals, by (2.14),

$$\begin{aligned} &E[V_n(f_2) - EV_n(f_2)]^2 + t \int f_2(y)^2\, \nu(dy) \\ &\qquad - 2E[(V_n(f_2) - EV_n(f_2))(Y(f_2) - EY(f_2))]. \end{aligned} \tag{3.12}$$

The first term of (3.12) is the variance of the sum of the independent random variables $f_2(X_{n,k})$. It equals

$$\sum \{\int f_2(x)^2\, F_{n,k}(dx) - (\int f_2(x)\, F_{n,k}(dx))^2\} \to t \int f_2(x)^2\, \nu(dx) \tag{3.13}$$

by Lemma 3.1.

From (2.9) and (2.12) we see that

$$Y(f_2, t) + t \int \chi(f_2(x))\, \nu(dx) = X(f_2, t), \tag{3.14}$$

an increasing type A process. The negative of the third term in (3.12) equals

$$2E[(V_n(f_2) - EV_n(f_2))\, X(f_2)]. \tag{3.15}$$

It is clear that $\liminf V_n(f_2) \geqq X(f_2)$. Apply Fatou's Lemma and Lemma 3.1 to the first and second terms of (3.15) respectively:

$$\begin{aligned} \liminf 2E[(V_n(f_2) - EV_n(f_2))\, X(f_2)] &\geqq 2E[(X(f_2) - t \int f_2(y)\, \nu(dy))\, X(f_2)] \\ &= 2E[(Y(f_2) - EY(f_2))\, X(f_2)] \qquad \text{(by (2.13) and (3.14))} \\ &= 2E[Y(f_2) - EY(f_2)]^2 = 2t \int f_2(x)^2\, \nu(dx). \end{aligned}$$

From (3.12), (3.13) and the last calculatlon, (3.11) follows.

Proof of (3.8). The process $Y(f_3)$ is of type A with finite Lévy measure. By choosing δ sufficiently small, we can make the probability that $Y(f_3, s) \equiv 0$ for $0 \leqq s \leqq t$ larger than $1 - \varepsilon$.

Proof of (3.9). For δ sufficiently small

$$P\{2\sup[|X(s)|: 0 \leqq s \leqq t] > 1/\delta\} < \varepsilon. \tag{3.16}$$

Now $V_n(f_3)=0$ unless the event on the left side of (3.16) occurs. The proof of the theorem is complete.

Remark 3.1. Certainly the L_2 convergence mentioned in Theorem 3.1 is often obtained even when f is not bounded. A "natural" conjecture might be that it is obtained whenever $\int f(x)^2\, \nu(dx) < \infty$. Here is a counter-example. Let X be the Poisson process (type A) with $\nu(\{1\})=1$ and $\nu(R-\{1\})=0$ and define f so that $f(x)=x!\, x^x$ if x is a positive integer. Then $EV_n(f)=\infty$.

Remark 3.2. The most interesting special cases of Theorem 3.1 are probably those where $f(x)=|x|^p$, $p>0$. If X is not of type D and $p>\beta/2$ where β is the largest of Blumenthal and Getoor's indices [2, p. 494], the conclusion of the theorem is valid. Among other valid cases are $p=\frac{1}{2}$ if X is of type A or B and $p=1$ if X is of type C. If X is of type D the conclusion of the theorem is valid for $p>1$. The interesting case $p=1$ when X is of type D is covered in Theorem 3.2 and Remark 3.5.

The next lemma will enable us to compare two sequences of variational sums.

Lemma 3.4. *For $n=1,2,\ldots$, let $U_n=\sum U_{n,k}$ and $Y_n=\sum Y_{n,k}$ be sums of non-negative independent random variables. Suppose that, as $n\to\infty$, $\sum (EU_{n,k})^2 \to m_1$, $\sum EU_{n,k}^2 \to m_2 < \infty$, and*

$$\limsup E \sum |U_{n,k}^2 - Y_{n,k}^2| \leqq \delta. \tag{3.17}$$

Then,

$$\operatorname{Var} U_n \to m_2 - m_1; \tag{3.18}$$

$$\limsup \left|\sum (EY_{n,k})^2 - m_1\right| \leqq [12\delta(m_2+\delta)]^{\frac{1}{2}}; \tag{3.19}$$

$$\limsup \left|\sum EY_{n,k}^2 - m_2\right| \leqq \delta; \tag{3.20}$$

$$\limsup |\operatorname{Var} Y_n - (m_2 - m_1)| \leqq \delta + 12\delta(m_2+\delta). \tag{3.21}$$

Proof. Since $\operatorname{Var} U_n = \sum \operatorname{Var} U_{n,k}$, (3.18) follows. Inequality (3.20) follows by the triangle inequality. To show (3.19) we calculate

$$\begin{aligned}\left|\sum [(EY_{n,k})^2 - (EU_{n,k})^2]\right|^2 &= \left|\sum E(Y_{n,k}-U_{n,k})\, E(Y_{n,k}+U_{n,k})\right|^2 \\ &\leqq \sum [E(Y_{n,k}-U_{n,k})]^2 \cdot \sum [E(Y_{n,k}+U_{n,k})]^2 \\ &\leqq \sum E(Y_{n,k}-U_{n,k})^2 \sum E(Y_{n,k}+U_{n,k})^2.\end{aligned}$$

Apply Schwartz's inequality to $EY_{n,k}U_{n,k}$ and (3.20) to obtain

$$\limsup \sum E(Y_{n,k}+U_{n,k})^2 \leqq 4(m_2+\delta). \tag{3.22}$$

For the other factor we have

$$\begin{aligned}\sum E(Y_{n,k}-U_{n,k})^2 &\leqq \sum E(Y_{n,k}^2 - U_{n,k}^2) + \sum 2E[U_{n,k}|U_{n,k}-Y_{n,k}|] \\ &\leqq \sum E|Y_{n,k}^2 - U_{n,k}^2| + 2\sum E|U_{n,k}^2 - Y_{n,k}^2|.\end{aligned}$$

Then (3.19) follows from (3.17) and (3.22). Since $\operatorname{Var} Y_n = \sum \operatorname{Var} Y_{n,k}$, (3.21) follows from (3.19) and (3.20).

Theorem 3.2. *Let X be a process of type D. Let $f\colon R^N \to R$ be a continuous function such that $f_r(\theta, w)/w \to f_r'(\theta, 0)$* [Def. 2.15] *uniformly in θ as $w \to 0$. Suppose that, for some $k > 0$,*

$$k\,|f(x+y)^2 - f(y)^2| \leqq |x|^2 + |f(y)|\,|x| \tag{3.23}$$

for all x, y in a neighborhood of 0. Then

$$V_n(f) - E V_n(\chi \circ f) \to Y(f, t) + B_c(t) \tag{3.24}$$

in distribution as $n \to \infty$, where $B_c(t)$ is one-dimensional Brownian motion with variance $c\,t$ and

$$\begin{aligned} c = {} & N \int f_r'(S^{\frac{1}{2}}\theta/|S^{\frac{1}{2}}\theta|, 0)^2\, |S^{\frac{1}{2}}\theta|^2\, \mu(d\theta) \\ & - 2\left[\frac{\Gamma((N+1)/2)}{\Gamma(N/2)}\right]^2 \left[\int f_r'(S^{\frac{1}{2}}\theta/|S^{\frac{1}{2}}\theta|, 0)\, |S^{\frac{1}{2}}\theta|\, \mu(d\theta)\right]^2. \end{aligned} \tag{3.25}$$

The measure μ is the uniform probability measure on the unit $N-1$-dimensional sphere.

Proof. As in the proof of Theorem 3.1 we may assume $f \geqq 0$. We wish to show that conditions (2.17), (2.18), and (2.19) of the Central Convergence Criterion are satisfied by the infinitesimal triangular array of random variables $\{f(X_{n,k})\}$ with the centering sequence $\{E V_n(\chi \circ f)\}$, and ν_f, c, and 0 in place of ν, S, and γ, respectively.

Condition (2.17) follows from Lemma 3.1. Condition (2.19) takes the form $0=0$.

For each $\varepsilon > 0$, define $f_\varepsilon(x) = f(x)$ for $|x| < \varepsilon$ and $f_\varepsilon(x) = 0$ for $|x| \geqq \varepsilon$. Let $M_\varepsilon = \sup f_\varepsilon(x)^2$. We wish to prove (2.18), which can be written as

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \operatorname{Var} V(f_\varepsilon, X, t, P_n) = c\,t. \tag{3.26}$$

Let $X(t) = W(t) + Z(t)$ where W is Gaussian and Z is not of type D. Let $f_0(x) = f_r'(x/|x|, 0)\,|x|$. A straight-forward calculation gives

$$\operatorname{Var} V(f_0, W, t, P_n) = c\,t.$$

Also $\sum (E f_0(W_{nk}))^2 = m_1$ and $\sum E\big(f_0(W_{n,k})^2\big) = m_2$ are independent of n.

By the use of the Gaussian density and (3.23) it can be shown that

$$\limsup \sum E\,|f_0(W_{n,k})^2 - f_\varepsilon(W_{n,k})^2| = 0 \tag{3.27}$$

and

$$k \limsup \sum E\,|f_\varepsilon(W_{n,k})^2 - f_\varepsilon(X_{n,k})^2| \leqq 2H_\varepsilon + (m_1 H_\varepsilon)^{\frac{1}{2}}, \tag{3.28}$$

where $h_\varepsilon(z) = 2\,|z|^2 \wedge M_\varepsilon$ and $H_\varepsilon = \int h_\varepsilon(z)\,\nu(dz)$. These two facts enable us to apply Lemma 3.4 twice: U_n, Y_n, and δ are replaced first by $V(f_0, W, t, P_n)$, $V(f_\varepsilon, W, t, P_n)$, and 0, respectively; and, second by $V(f_\varepsilon, W, t, P_n)$, $V(f_\varepsilon, X, t, P_n)$, and $[2H_\varepsilon + (m_1 H_\varepsilon)^{\frac{1}{2}}]\,k^{-1}$, respectively. In each case $m_2 - m_1 = c\,t$. On the second application of the lemma, (3.21) yields (3.26), since $H_\varepsilon \to 0$ as $\varepsilon \to 0$.

Remark 3.3. Fix t and $\{P_n\}$. Define

$$U_n(f, t_{n,k}) = \sum_{j=1}^{k} f(X_{n,j}).$$

For $s \in (t_{n,k-1}, t_{n,k})$, define $U_n(f, s) = U_n(f, t_{n,k-1})$. Then from Theorem 2, p. 477 of [6] we see that $U_n(f) \to Y(f) + B_c$ in the sense of weak convergence of processes in $D[0,1]$ with Skorohod's ρ_D metric.

Remark 3.4. If $f: R^N \to R^M$, $M > 1$, Theorem 3.2 remains true. The process $B_c(t)$ is then an M-dimensional Gaussian process with stationary, independent increments, mean 0, and covariance matrix ct. The square of each vector on the right side of (3.25) should be interpreted as the matrix product.

Remark 3.5. (See also Remark 3.2.) Let $X(t) = W(t)$ be a standard one-dimensional Brownian motion with $\operatorname{Var} X(t) = t$ and let $f(x) = |x|$. Let P_n be the partition of $[0,1]$ into 2^n equal intervals. It follows from Theorem 3.2 that $U_n = V_n(f) - (2^{1+n}/\pi)^{\frac{1}{2}}$ converges in distribution to a normal random variable with variance $1 - 2/\pi$. We shall show that U_n does not converge in probability. Suppose to the contrary that $U_n \to U$ in probability. We may assume $U_n \to U$ a.s. by considering a subsequence if necessary. Let $B_n = \{\omega: U U_n \geqq 0\}$. Then $P\{B_n\} \to 1$. Let I be the indicator function. By Fatou's Lemma,

$$\liminf_{m,n \to \infty} E\big(U_n U_m I(B_n B_m)\big) \geqq E(U^2) = 1 - 2/\pi. \tag{3.29}$$

Also,

$$E\big(|U_n U_m| \, I(\Omega - B_n B_m)\big) \leqq \big[E\big(U_n^2 I(\Omega - B_n B_m)\big) \, E\big(U_m^2 I(\Omega - B_n B_m)\big)\big]^{\frac{1}{2}}. \tag{3.30}$$

An easy calculation shows that $E(U_n^2) = 1 - 2/\pi$ and Fatou's Lemma implies that

$$\liminf_{m,n \to \infty} E\big(U_n^2 I(B_n B_m)\big) \geqq 1 - 2/\pi.$$

Therefore the right side of (3.30) goes to 0. From (3.29), (3.30), and Schwartz's inequality we conclude that $E U_n U_m \to 1 - 2/\pi$ as $m, n \to \infty$. But for fixed m one can calculate $E U_n U_m \to 0$ as $n \to \infty$, a contradiction.

In the following theorem we consider type C processes only. Almost sure convergence of $V_n(f)$ with appropriate f for type A, and therefore type B, processes is an immediate corollary of Theorem 3 of [4]. Classes of functions for which $V_n(f)$ converges in probability and for which the convergence is almost sure are also known for Brownian motion.

Theorem 3.3. *Let X be of type C. Let $f: R^N \to R$ be continuous and suppose that there exists a continuous function $g: [0, \infty) \to [0, \infty)$ such that $g(0) = g'(0) = 0$,*

$$\int_0 g(|x|) \, \nu(dx) < \infty, \tag{3.31}$$

g is convex, $x \to g(x^{\frac{1}{2}})$ is concave, and, for x in a neighborhood of 0, $|f(x)| \leqq g(|x|)$. Then $V_n(f) \to X(f, t)$ in probability as $n \to \infty$. If, in addition, f is bounded, this convergence takes place in L_2.

Proof. From Theorem 3.1,

$$V_n(f) - \sum \int \chi(f(x)) \, F_{n,k}(dx) \to Y(f, t). \tag{3.32}$$

Since $|f(x)| \leqq g(|x|)$ for small $|x|$, condition (3.31) implies that $X(f, t)$ can be defined; moreover, from (2.9) and (2.12),

$$Y(f, t) = X(f, t) - t \int \chi(f(x)) \, \nu(dx). \tag{3.33}$$

We see from (3.32) and (3.33) that it suffices to show

$$\sum \int \chi(f(x))\, F_{n,k}(dx) \to t \int \chi(f(x))\, \nu(dx) + ct. \tag{3.34}$$

Lemma 3.1 allows as to restrict the domain of integration in (3.34) to A, a small neighborhood of 0 with $\nu(\partial A)=0$. Then Lemma 3.3, together with the Helly-Bray Theorem [10, p. 182] applied to the function $(\chi \circ f)/g \leqq 1$ on A, yields (3.34).

Remark 3.6. The convergence in distribution contained in Theorem 3.3 can be proved directly by using the Central Convergence Criterion and Lemma 3.3.

Remark 3.7. In Theorem 3.3 we require $|f|$ to be bounded by a radial function $g(|\cdot|)$. The question arises whether the assumption that the bounding function is radial is needed. The following example shows that it is. Let X be a one-dimensional stable process of index $\frac{3}{2}$ and mean 0 and such that $X(t+) \leqq X(t-)$ for all t; this can be done by letting $\nu(-\infty, x) = |x|^{-\frac{3}{2}}$ if $x<0$ and $\nu(0,\infty)=0$. Let $f(x) = 0 \vee x$. Because of the above-mentioned assumption Theorem 3.3 does not apply; however $\int |f(x)|\, \nu(dx) < \infty$. We show that the conclusion of Theorem 3.3 is not true. Theorem 3.1 is applicable so we have only to show that

$$\sum \int_0^1 x F_{n,k}(dx) \to \infty.$$

This is true since $\sum \int_{-1}^{1} x F_{n,k}(dx)$ converges by condition (2.19) of the Central Convergence Criterion and

$$\limsup \sum \int_{-1}^{0} x F_{n,k}(dx) \leqq \int_{-1}^{0} x\, \nu(dx) = -\infty.$$

Remark 3.8. The conclusion of Theorem 3.3, if $f(x)=|x|^p$, $p>0$, is part of the conclusion of Theorem 4.1 of [12]. The conclusion of Theorem 3.3 holds if $p>\beta$ where β is the largest of Blumenthal and Getoor's indices [2, p. 494].

Remark 3.9. Theorem 3.3 cannot be generalized in the obvious manner to infinitesimal triangular arrays $\{X_{n,k}\}$ of random variables. In fact even the corresponding result in the type A case does not hold as the following example shows. Let $F_{n,k}$, the distribution function of $X_{n,k}$, be one-dimensional and given by

$$F_{n,k}(x) = \begin{cases} 0 & \text{if } x \leqq n^{-\frac{3}{2}} \\ 1 - \dfrac{1}{n x^{\frac{1}{2}}} & \text{if } x > n^{-\frac{3}{2}}. \end{cases}$$

By using the Central Convergence Criterion one checks that $\sum X_{n,k}$ converges in distribution to a "type A" stable, non-negative random variable of index $\frac{1}{2}$. Even though $\int_0^1 x^{\frac{3}{5}}\, \nu(dx) < \infty$, where ν is the Lévy measure for this limiting random variable, the Central Convergence Criterion shows that $\sum X_{n,k}^{\frac{3}{5}}$ does not converge in distribution to a finite random variable.

Remark 3.10. The hypothesis that f is continuous which appears in Lemma 3.1 and Theorems 3.1, 3.2, and 3.3 can be relaxed. The hypothesis

$$\nu\{x: f \text{ is discontinuous at } x\} = 0 \tag{3.35}$$

suffices.

4. Divergence Theorems

We can prove Theorems 4.1 and 4.3 in a setting more general than that of the previous section. Let $\{X_{n,k}\}$ be a infinitesimal triangular array of random variables, independent for each fixed n, such that $\sum X_{n,k}$ converges in distribution to the infinitely divisible random variable X with distribution characterized by ν, S, and γ. We may still speak of the *type* of X and retain the notation established in Section 2 except where pertinent only to processes. By $V_n(f)$ we mean $\sum f(X_{n,k})$. Note that Lemma 3.1 is still valid with $F_{n,k}$ the distribution measure of $X_{n,k}$. Although Lemma 4.1 is needed only in the proof of Theorem 4.3, it appears first because its method of proof is used in the proof of Theorem 4.1.

Lemma 4.1. *Let $f\colon R^N \to [0,\infty)$ be continuous with $f(0)=0$. Suppose either that $\int_0 f(x)\,\nu(dx)=\infty$ or that X is of type B (type D, respectively) and for each M there exists $\varepsilon>0$ such that if $|x|<\varepsilon$, $f(x)\geqq M|x|$ $\big(f(x)\geqq M|x|^2$, respectively$\big)$. Then $EV_n(\chi\circ f)\to\infty$ as $n\to\infty$.*

Proof. We may assume $f\leqq 1$. We wish to prove that

$$\sum \int f(x)\,F_{n,k}(dx)\to\infty. \tag{4.1}$$

Suppose $\int f(x)\,\nu(dx)=\infty$. There exists $\{f_m\}$ with f_m continuous, $f_m\leqq f$, $f_m(x)=0$ for x in a neighborhood of 0, and $\int f_m(x)\,\nu(dx)\to\infty$. By Lemma 3.1

$$\sum \int f(x)\,F_{n,k}(dx)\geqq \sum \int f_m(x)\,F_{n,k}(dx)\to \int f_m(x)\,\nu(dx),$$

from which (4.1) follows.

Suppose X is of type B. Modifying (2.19) by using Lemma 3.1 we obtain for a proper choice of $\varepsilon>0$,

$$\liminf |\alpha| \sum \int_{|x|<\varepsilon} M|x|\,F_{n,k}(dx)\geqq M|\alpha|^2/2,$$

and (4.1) follows.

Suppose X is of type D. From (2.18) and the hypothesis,

$$\liminf_n \sum \int_{|x|<\varepsilon} M|x|^2\,F_{n,k}(dx)\geqq M\big(\operatorname{trace}(S)\big), \tag{4.2}$$

from which (4.1) follows.

Theorem 4.1. *Let $f\colon R^N\to R$ be continuous with $f(0)=0$ and suppose that $\int_0 f(x)^2\,\nu(dx)=\infty$. Then* Var $V_n(\chi\circ f)\to\infty$ *and*

$$\frac{V_n(f)-EV_n(\chi\circ f)}{[\operatorname{Var} V_n(\chi\circ f)]^{\frac{1}{2}}}\to \mathfrak{N}(0,1) \tag{4.3}$$

in distribution as $n\to\infty$. $\mathfrak{N}$ denotes the normal distribution.

Proof. We prove

$$\operatorname{Var} V_n(\chi\circ f)=\sum \int \big(\chi(f(x))-\beta_{n,k}\big)^2\,F_{n,k}(dx)\to\infty. \tag{4.4}$$

Suppose first that $\int_0 (f(x)\vee 0)^2\,\nu(dx)=\infty$. Were it not for the terms $\beta_{n,k}=E\chi\big(f(X_{n,k})\big)$, Lemma 4.1 would give the result immediately. Note that $\max_k \beta_{n,k}\to 0$.

Let $\delta>0$, $f_\delta(x)=(\chi(f(x))-\delta)\vee 0$, and $f_{n,k}(x)=(\chi(f(x))-\beta_{n,k})\vee 0$. For sufficiently large n, $f_{n,k}(x)>f_\delta(x)$. In a neighborhood of 0, f_δ is 0, and Lemma 3.1 implies

$$\sum \int f_\delta(x)^2\, F_{n,k}(dx) \to \int f_\delta(x)^2\, \nu(dx) \to \infty$$

as $\delta \to 0$, and (4.4) follows. The proof is similar if $\int\limits_0 (f(x)\wedge 0)^2\, \nu(dx)=\infty$.

Let $\beta_n = EV_n(\chi\circ f)$ and $\rho_n^2=\operatorname{Var} V_n(\chi\circ f)$.

To prove (4.3) we apply the Central Convergence Criterion to the triangular array $\{f(X_{n,k})/\rho_n\}$ with centering constants β_n/ρ_n. We verify (2.17), (2.18), (2.19) with 0, 1, and 0 in place of ν, S, and γ, respectively. Since $\rho_n\to\infty$, (2.17) follows from Lemma 3.1. Condition (2.19) is equivalent to

$$\int \sum \left[\chi(f(x)/\rho_n)-\chi(f(x))/\rho_n\right] F_{n,k}(dx)\to 0. \tag{4.5}$$

To prove (4.5), let $b\in(0,1)$ and for each n such that $b\,\rho_n>1$ write the integral as the sum of three integrals, I_1, I_2, I_3, with domains

$$A_1=\{x: |f(x)|\leqq 1\},\qquad A_2=\{x: 1<|f(x)|\leqq b\,\rho_n\},\qquad A_3=\{x: b\,\rho_n<|f(x)|\}.$$

Clearly, $I_1=0$. We have

$$\limsup_n |I_2|\leqq b \limsup_n \int\limits_{A_2} \sum F_{n,k}(dx)\leqq b\,\nu\{x: |f(x)|\geqq 1\}. \tag{4.6}$$

For sufficiently large n, $|I_2|<2b\,\nu\{x: |f(x)|\geqq 1\}$. Since $\rho_n\to\infty$, it follows from Lemma 3.1 that $I_3\to 0$ as $n\to\infty$. Since b can be chosen small and then n large, (4.5) follows.

Let $\varepsilon>0$. Condition (2.18) will follow from

$$\Big[\sum \int\limits_{|f(x)|>\varepsilon} [\chi(f(x))^2+|\chi(f(x))|]\, F_{n,k}(dx)\Big]\Big/\rho_n\to 0$$

which is true since Lemma 3.1 implies that the numerator is bounded.

In particular Theorem 4.1 is applicable to processes, with $X_{n,k}=X(t_{n,k})-X(t_{n,k-1})$.

Remark 4.1. Let $f(x)=|x|^p$, $p>0$. If $p>\beta/2$ where β is the largest of Blumenthal and Getoor's indices [2, p. 494] then Theorem 4.1 implies that $|V_n(f)-a_n|\to\infty$ in probability for any sequence $\{a_n\}$.

Theorem 4.2. *Let X be a process of type D. Let $f: R^N\to R$ be continuous, $f(0)=0$, and such that for each M there exists $\delta>0$ such that $|f_r'(\theta,w)|>M$ for $w<\delta$ and all θ. Then* $\operatorname{Var} V_n(\chi\circ f)\to\infty$ *and*

$$\frac{V_n(f)-EV_n(\chi\circ f)}{(\operatorname{Var} V_n(\chi\circ f))^{\frac{1}{2}}}\to \mathfrak{N}(0,1)$$

in distribution as $n\to\infty$.

Proof. Fix M and choose $\delta>0$ as in the hypothesis. Let $f_\delta(x)=f(x)$ if $|x|<\delta$, $=0$ if $|x|\geqq\delta$. Then, by Lemma 3.1

$$\operatorname{Var} V_n(\chi\circ f)=\operatorname{Var} V_n(f_\delta)+O(1).$$

By writing each $\operatorname{Var} f_\delta(X_{n,k})$ in terms of the expectation conditioned on $\theta = X_{n,k}/|X_{n,k}|$, we calculate

$$\operatorname{Var} f_\delta(X_{n,k}) \geqq M^2 \operatorname{Var} |X_{n,k}|_\delta,$$

where $w_\delta = w$ if $w < \delta$, $=0$ if $w \geqq \delta$. The function $|x|_\delta$ satisfies the hypothesis of Theorem 3.2 with $g(x) = 4|x|^2$. From (3.27) with $|x|_\delta$ in place of f_ε we conclude that

$$\lim_n \operatorname{Var} V_n(f_\delta) \geqq M^2 ct,$$

where c is given by (3.26) with $f_r' \equiv 1$. Since M was arbitrary, $\operatorname{Var} V_n(\chi \circ f) \to \infty$. The remainder of the theorem follows as in the proof of Theorem 4.1.

Remark 4.2. Theorem 4.2 cannot be proved for triangular arrays. Let $X_{n,k} = \pm \frac{1}{n}$ each with probability $\frac{1}{2}$, $k = 1, 2, \ldots, n^2$. By the Central Convergence Criterion $\sum X_{n,k}$ converges to $\mathfrak{N}(0,1)$ in distribution. If $f(x) = f(-x)$, $V_n(f) - EV_n(\chi \circ f) = 0$.

Theorem 4.3. *Let $f: R^N \to [0, \infty)$ be continuous with $f(0) = 0$. Suppose either that $\int_0 f(x)\, \nu(dx) = \infty$ or that X is of type B (type D, respectively) and for each M there exists $\varepsilon > 0$ such that if $|x| < \varepsilon$, $f(x) \geqq M|x|$ ($f(x) \geqq M|x|^2$, respectively). Then $EV_n(\chi \circ f) \to \infty$ and*

$$\frac{V_n(f)}{EV_n(\chi \circ f)} \to 1 \tag{4.7}$$

in probability as $n \to \infty$.

Proof. That $EV_n(\chi \circ f) \to \infty$ is the conclusion of Lemma 4.1. To prove (4.7) apply the Central Convergence Criterion to the triangular array $\{f(X_{n,k})/EV_n(\chi \circ f)\}$ with centering constants 0. Condition (2.17) with ν replaced by 0 is a consequence of $EV_n(\chi \circ f) \to \infty$. Condition (2.18) with $S = 0$ will follow from (2.19) with $\gamma = 1$. As for (2.19), it is equivalent to (4.5) with ρ_n replaced by $EV_n(\chi \circ f)$, and the proof is then identical to the proof of (4.5).

Remark 4.3. (See Remark 3.10.) In this section the hypothesis that f is continuous can be relaxed to $\nu\{x: f \text{ is discontinuous at } x\} = 0$ and f is continuous at 0.

5. Examples

The hypotheses in Theorem 3.3 enabled us to show that $\{EV_n(\chi \circ f)\}$ has a finite limit and to evaluate that limit. A similar remark applies to Theorems 3.1 and 3.2 with the sequence $\{\operatorname{Var} V_n(\chi \circ f)\}$. The proofs of Theorems 4.1, 4.2, and 4.3 consist primarily in showing that $\operatorname{Var} V_n(\chi \circ f) \to \infty$ or that $EV_n(\chi \circ f) \to \infty$. The conclusions of these theorems, except Theorem 3.3, involve $EV_n(\chi \circ f)$ and $\operatorname{Var} V_n(\chi \circ f)$. Evaluations of $EV_n(\chi \circ f)$ and $\operatorname{Var} V_n(\chi \circ f)$ for a variety of special cases follow. In all these examples X will be a process rather than just the limit of row-sums of a triangular array and P_n will always be the partition of $[0, 1]$ into n equal intervals. The last assumption simplifies the resulting formulas.

Example 5.1. Let X be Gaussian, i.e. $\nu = \gamma = 0$. Let $p > 0$. By using the scaling property and the exponential tail of the Gaussian density we obtain

$$EV_n(\chi(|x|^p)) = E|X(1)|^p n^{(2-p)/2} + o(1),$$

$$\operatorname{Var} V_n(\chi(|x|^p)) = \operatorname{Var} |X(1)|^p n^{1-p} + o(1),$$

$$EV_n(\chi(|x|^2 \cdot |\log |x|^2|)) = E|X(1)|^2 \log n - E|X(1)|^2 \log |X(1)|^2 + o(1).$$

Example 5.2. Let X be a strictly stable process of index $\beta<2$ with

$$E\,e^{i\langle u, X(t)\rangle}=e^{-|u|^{\beta} h(u/|u|)}$$

with $\operatorname{Re} h(\theta)=\int|\langle\phi,\theta\rangle|^{\beta}\eta(d\phi)$ where η is a finite measure on the $N-1$-dimensional unit sphere. We normalize X by specifying that η be a probability measure. It is known [15, p. 157 for $N=1$; 5 for $N\geqq 1$] that $|X(1)|$ has a density ψ with

$$\psi(r)=C\,r^{-(1+\beta)}+O(r^{-(1+2\beta)}) \quad \text{as} \quad r\to\infty, \tag{5.1}$$

where $C=(2/\pi)\Gamma(1+\beta)\sin(\pi\beta/2)$. Let $p>0$. We use the scaling property and (5.1) to obtain

$$EV_n(\chi(|x|^p))=E|X(1)|^p n^{(\beta-p)/\beta}-\frac{2pC}{\beta(\beta-p)}+o(1) \quad \text{if} \quad p<\beta,$$

$$\operatorname{Var} V_n(\chi(|x|^p))\sim\operatorname{Var}|X(1)|^p n^{(\beta-2p)/\beta} \quad \text{if} \quad 2p<\beta,$$

$$EV_n(\chi(|x|^{\beta}))=\frac{C}{\beta}(\log n+1)+E\left(|X(1)|^{\beta}-\frac{C}{(|X(1)|+1)\psi(|X(1)|)}\right)+o(1). \tag{5.2}$$

Theorem 4.3 with $f(x)=|x|^{\beta}$ and formula (5.2) is Theorem 4 of [7].

Define $\lg^+=\lg_1^+=1\vee\log$ and $\lg_j^+=\lg^+\circ\lg_{j-1}^+$. Then

$$EV_n\left(\chi\left(\frac{|x|^{\beta}}{\lg^+|x|^{-1}}\right)\right)=C\left(\log\log n^{1/\beta}+\frac{1+\beta}{\beta}\right)+o(1),$$

$$EV_n\left(\chi\left(\frac{|x|^{\beta}}{\pi_1^{\infty}\lg_j^+|x|^{-1}}\right)\right)=C\left[q(n)+\frac{1}{\beta}+\log(\lg_{q(n)}^+ n)\right]+o(1)$$

where $q(n)=\max\{j\colon \lg_j^+ n>1\}$. Note that for Theorem 4.3 the quantity $\frac{1}{\beta}+\log(\lg_{q(n)}^+ n)$ is irrelevant since $\log(\lg_{q(n)}^+ n)\leqq 1$.

Example 5.3. Let $N=1$ and let X be the increasing type A process with

$$\nu(x,\infty)=\int_x^{\infty} y^{-1}e^{-y}\,dy.$$

The distribution of $X(t)$ is given by $F(t,dx)=e^{-x}x^{t-1}dx/\Gamma(t)$, $x>0$. With $\lg^+$ as in Example 5.2 we use $\int_0^1\frac{x^{1/n}-x}{x\log x^{-1}}\,dx=\log n$ to obtain

$$EV_n\left(\chi\left(\left|\lg^+\frac{1}{x}\right|^{-1}\right)\right)=\log n+C+o(1)$$

where

$$C=\int_{e^{-1}}^{\infty}\frac{e^{-x}dx}{x}-\int_{e^{-1}}^{1}\frac{(1-x)\,dx}{x\log x^{-1}}-\int_0^{e^{-1}}\frac{(1-x-e^{-x})\,dx}{x\log x^{-1}}.$$

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