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Geometric Convergence of Semi-Markov Transition Probabilities

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1. Introduction

Let $\{z(t);\ t \geqq 0\}$ be an irreducible semi-Markov process whose transitions occur at instants $0 = T_0, T_1, T_2, \ldots$ Suppose $z(t)$ is continuous to the right and $z_n = z(T_n)$, $n \geqq 0$. We shall consider a separable version of the process defined by the initial distribution $\alpha_i = \Pr\{z_0 = i\}$ and the transition probabilities

$$Q_{ij}(t) = \Pr\{z_n = j,\ X_n \leqq t \,|\, z_{n-1} = i\},$$

where i, j range over a denumerable state space E, and $X_n = T_n - T_{n-1}$, $n \geqq 1$. The occupation time distribution is given by

$$H_i(t) = \Pr\{X_n \leqq t \,|\, z_{n-1} = i\} = \sum_{j \in E} Q_{ij}(t).$$

It will be assumed throughout that a transition occurs at $t = 0$. Let $P_{ij}(t)$ be the transition probability from i to j in time t, $G_{ij}(t)$ $(i \neq j)$ the distribution of the first entrance time from i to j, and $G_{ii}(t)$ the distribution of the first return time to i (after leaving i at the first exit time). Furthermore, let ${}_kP_{ij}(t)$ denote the transition probability from i to j in time t under the taboo k, and ${}_kG_{ij}(t)$ the distribution of the first entrance time (or return time, if $i = j$) under the taboo k. We agree to impose the taboo only after the first exit from i. Note that ${}_jP_{ij}(t) = \delta_{ij}\{1 - H_j(t)\}$. η_j and μ_{ij} will denote the expectations of H_j and G_{ij} respectively, i.e. $\eta_j = \int_0^\infty t\, dH_j(t)$ and $\mu_{ij} = \int_0^\infty t\, dG_{ij}(t)$.

As Laplace transforms will be used extensively, we shall henceforth adopt the abbreviation $A^*(s)$ for $\int_{0-}^\infty e^{-st}\, dA(t)$.

The notation used here is similar to PYKE and SCHAUFELE's (1964), to whose paper we refer the reader for the definition of a semi-Markov process and a more detailed treatment of the quantities defined above.

SMITH (1955) proved that if $\eta_j < \infty$, then as

$$t \to \infty, \quad P_{ij}(t) \to \pi_{ij} = G_{ij}(\infty)\, \eta_j / \mu_{jj}. \qquad (\pi_{ij} = 0 \text{ if } \mu_{jj} \text{ is infinite.})$$

In section 4 of this paper, we shall establish a necessary and sufficient condition for this convergence to be exponentially fast in the special case when the state space E is finite. It turns out that when E is finite, there is geometric convergence

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if there exists $\gamma > 0$ such that $Q_{ij}^*(-\gamma)$ converges and

$$\frac{Q_{ij}^*(-\gamma + i\,\delta)}{-\gamma + i\,\delta} \in L_1(-\infty, \infty)$$

for all i, j in E.

In section 3, a 'solidarity' theorem similar to that obtained in Markov chain theory will be proved. A transition $i \to j$ is said to be geometrically ergodic if there exists a positive number λ_{ij} such that $P_{ij}(t) - \pi_{ij} = o(e^{-\lambda_{ij} t})$ as $t \to \infty$. KENDALL (1959) proved that in an irreducible discrete Markov chain, if for some k the transition $k \to k$ is geometrically ergodic, then all transitions $i \to j$ are geometrically ergodic. This result was improved upon by VERE-JONES (1962) who showed that in an irreducible, discrete, geometrically ergodic Markov chain, a common parameter of convergence, λ, can be found for all transitions $i \to j$. The continuous time analogue of VERE-JONES' result was proved by KINGMAN (April 1963, October 1963). A simple method of proof using Laplace transforms was obtained by CHEONG (1966). It is this method which will be used here to establish a similar type of solidarity property for semi-Markov processes. The weakness of this method lies in that it cannot treat cases where instantaneous states exist. We are thus forced here to assume that there are no instantaneous states.

Section 2 contains preliminary results that will be made use of later.

2. Preliminary Results

As with Markov chains, the following first entrance formulas hold:

$$P_{ij}(t) = {}_kP_{ij}(t) + \int_0^t P_{kj}(t-u)\, dG_{ik}(u) \qquad i, j, k \in E\,; \tag{2.1}$$

$$G_{ij}(t) = {}_kG_{ij}(t) + \int_0^t G_{kj}(t-u)\, d_j G_{ik}(u) \qquad i, j, k \in E,\ \ j \neq k\,. \tag{2.2}$$

G_{ij} and P_{ij} may also be represented by Q_{ij}, thus:

$$P_{ij}(t) = \delta_{ij}\{1 - H_j(t)\} + \sum_{k \in E} \int_0^t P_{kj}(t-u)\, dQ_{ik}(u)\,, \quad i, j \in E; \tag{2.3}$$

$$G_{ij}(t) = Q_{ij}(t) + \sum_{\substack{k \in E \\ k \neq j}} \int_0^t G_{kj}(t-u)\, dQ_{ik}(u)\,, \qquad i, j \in E\,. \tag{2.4}$$

Considering special cases of (2.1) and (2.2) and taking Laplace transforms, we get the following identities:

$$P_{aa}^*(s) = \frac{1 - H_a^*(s)}{1 - G_{aa}^*(s)}\,, \tag{2.5}$$

$$G_{aa}^*(s) = {}_kG_{aa}^*(s) + {}_aG_{ak}^*(s)\, G_{ka}^*(s)\,, \tag{2.6}$$

$$G_{ka}^*(s) = {}_kG_{ka}^*(s) + {}_aG_{kk}^*(s)\, G_{ka}^*(s)\,, \tag{2.7}$$

$$G_{ak}^*(s) = \frac{{}_aG_{ak}^*(s)}{1 - {}_kG_{aa}^*(s)}\,, \tag{2.8}$$

$$G_{kk}^*(s) = {}_aG_{kk}^*(s) + {}_kG_{ka}^*(s)\, G_{ak}^*(s)\,, \tag{2.9}$$

$$P_{ka}^*(s) = G_{ka}^*(s)\, P_{aa}^*(s)\,, \tag{2.10}$$

$$_aP_{kk}^*(s) = \frac{1 - H_k^*(s)}{1 - {_aG_{kk}^*(s)}}\,, \tag{2.11}$$

$$_aP_{ak}^*(s) = {_aG_{ak}^*(s)}\, {_aP_{kk}^*(s)}\,, \tag{2.12}$$

$$P_{ak}^*(s) = \frac{_aP_{ak}^*(s)}{1 - G_{aa}^*(s)}\,, \tag{2.13}$$

$$P_{kk}^*(s) = {_aP_{kk}^*(s)} + G_{ka}^*(s)\, P_{ak}^*(s)\,, \tag{2.14}$$

$$G_{ja}^*(s) = {_kG_{ja}^*(s)} + {_aG_{jk}^*(s)}\, G_{ka}^*(s)\,, \tag{2.15}$$

$$_aP_{jk}^*(s) = {_aG_{jk}^*(s)}\, {_aP_{kk}^*(s)}\,, \tag{2.16}$$

$$P_{jk}^*(s) = {_aP_{jk}^*(s)} + G_{ja}^*(s)\, P_{ak}^*(s)\,, \tag{2.17}$$

$$P_{kk}^*(s) = \frac{1 - H_k^*(s)}{1 - G_{kk}^*(s)}\,, \quad (j \neq a,\ k \neq a,\ j \neq k)\,. \tag{2.18}$$

Equations (2.5)—(2.18) hold for $Rs > 0$, but if one side of an equation has an analytic continuation in some half-plane, then so has the other side, and the equation still holds.

We quote here without proof three theorems that will be of use later. The functions $A(t)$, $B(t)$ and $C(t)$ in theorems A, B, C and C* are assumed to be of bounded variation in every finite interval.

Theorem A. (Widder, 1946: Corollary 2.1, p. 39): If $A(\infty)$ exists and if $A(t) - A(\infty) = O\,(e^{\lambda t})$ as $t \to \infty$ for some real number λ, then $A^*(s)$ converges for $Rs > \lambda$.

Theorem B. (Widder, 1946: Theorem 2.2b, p. 40): If $A^*(s)$ converges for $s = -\lambda + i\delta$ with $\lambda > 0$, then $A(\infty)$ exists and $A(t) - A(\infty) = o(e^{-\lambda t})$ as $t \to \infty$.

Theorem C. (Widder, 1946: Theorem 11.6b, p. 89): If $C(t) = \int_0^t A(t-u)\,dB(u)$, and if $A^*(s_0)$ and $B^*(s_0)$ converge, one of them absolutely, then $C^*(s_0)$ converges and $C^*(s_0) = A^*(s_0)\, B^*(s_0)$.

With the help of theorems A and B, the condition of absolute convergence in theorem C may be removed to obtain the following slightly weaker result:

Theorem C*. *Let* $A = \lim_{t\to\infty} A(t)$ *and* $B = \lim_{t\to\infty} B(t)$. *If* $C(t) = \int_0^t A(t-u)\,dB(u)$, *and if both* $A^*(-\lambda)$ *and* $B^*(-\lambda)$ *converge for some* $\lambda > 0$, *then for all* $Rs > \frac{-\lambda}{2}$, $C^*(s)$ *converges and* $C^*(s) = A^*(s)\, B^*(s)$.

Proof. It is sufficient to prove the theorem for the case when $B(0) = 0$. By theorem B, $A - A(t) = o(e^{-\lambda t})$ and $B - B(t) = o(e^{-\lambda t})$ as $t \to \infty$. Since

$$AB - C(t) = A\{B - B(t)\} + \int_0^{t/2} \{A - A(t-u)\}\,dB(u) + \int_{t/2}^{t} + \{A - A(t-u)\}\,dB(u),$$

$$AB - C(t) = o(e^{-\lambda t/2})$$

as $t \to \infty$ and so by theorem A, $C^*(s)$ converges for all $Rs > \frac{-\lambda}{2}$. We can now appeal to yet another result of WIDDER's (Theorem 11.7b, p. 91) to assert that $C^*(s) = A^*(s)\,B^*(s)$ for $Rs > \frac{-\lambda}{2}$.

We now show that the functions $P_{ij}(t)$ satisfy the condition of bounded variation required in theorems A, B, C and C*.

Lemma 2.1. *For all i, j, $P_{ij}(t)$ is a function of bounded variation in $[0, R]$ for every $R > 0$.*

Proof. From (2.1) the following identities are derived:

$$P_{jj}(t) = 1 - H_j(t) + \int_0^t \{1 - H_j(t-u)\}\,dK_1(u)\,, \tag{2.19}$$

$$P_{ij}(t) = \int_0^t \{1 - H_j(t-u)\}\,dK_2(u) \qquad (i \neq j)\,, \tag{2.20}$$

where

$$K_1(t) = G_{jj}(t) + \int_0^t G_{jj}(t-u)\,dK_1(u)$$

and

$$K_2(t) = G_{ij}(t) + \int_0^t G_{jj}(t-u)\,dK_2(u)\,.$$

Both $K_1(t)$ and $K_2(t)$ are the renewal functions of renewal processes and are therefore non-decreasing. ($K_1(t)$, for example, is the expected number of renewals in $(0, t)$ of a renewal process with life-time distribution $G_{jj}(t)$.) Since H_j is bounded, it is obvious from (2.19) and (2.20) that $P_{ij}(t)$ is of bounded variation in every finite interval.

The following theorem is due to LEADBETTER (1964):

Theorem D. Suppose $B(t)$ is a distribution function, $C^*(s) = \frac{A^*(s)}{1 - B^*(s)}$ and $L(t)$ is the residue of $\frac{A^*(s)\,e^{st}}{s(1 - B^*(s))}$ at $s = 0$. If $1 - B(t) = O\,(e^{-\lambda t})$ as $t \to \infty$, $\lambda > 0$, and if $\frac{A^*(-\lambda + i\delta)}{-\lambda + i\delta} \in L_1(-\infty, \infty)$, then there exists $\gamma > 0$ such that $C(t) = L(t) + o(e^{-\gamma t})$ as $t \to \infty$.

Corollary. If in addition $A(t)$ is an honest distribution function with a finite first moment η_1, then $C(t) = t/\mu_1 + \mu_2/2\mu_1^2 - \eta_1/\mu_1 + o(e^{-\lambda t})$ as $t \to \infty$, where μ_1 and μ_2 are the first two moments of $B(t)$ and $1/\mu_1$ is to be interpreted as zero if μ_1 is infinite.

In LEADBETTER's theorem (p. 238), $A(t) = B(t)$, but it is easily verified that his proof remains valid for a general $A(t)$ satisfying $\frac{A^*(-\lambda + i\delta)}{-\lambda + i\delta} \in L_1(-\infty, \infty)$. Furthermore, an examination of the proof reveals that γ does not depend on $A(t)$. To prove the corollary, note that

$$\lim_{s \to 0} s\,A^*(s)\,e^{st}/(1 - B^*(s)) = t/\mu_1 + \mu_2/2\,\mu_1^2 - \eta_1/\mu_1$$

when $\mu_2 < \infty$.

3. Solidarity Properties

We shall assume throughout this section that $0 < \eta_j < \infty$. A theorem of PYKE's (1961: Theorem 5.1) assures us that the states of an irreducible semi-

Markov chain are either all transient or all recurrent. The main result of this section states that if the H_i's satisfy certain regularity conditions, then either all or none of the states of the chain are geometrically ergodic. In the former case, there exists a common parameter of convergence for all transitions.

Theorem 3.1. *Suppose for each j in E, $1 - H_j(t) = O(e^{-\lambda t})$ as $t \to \infty$, $\lambda > 0$, and $H_j^*(-\lambda + i\delta)/(-\lambda + i\delta) \in L_1(-\infty, \infty)$. If there exists a state a such that $P_{aa}(t) - \pi_{aa} = O(e^{-\lambda t})$ as $t \to \infty$, then there exists $\gamma > 0$ such that for all j, k in E,*

$$P_{jk}(t) - \pi_{jk} = o(e^{-\gamma t}) \quad as \quad t \to \infty .$$

Proof. Let $j, k \in E$, $j \neq k$, $j \neq a$, $k \neq a$. We shall show that there exists $\gamma > 0$ such that in the half-plane $Rs \geqq -\gamma$, all the transforms $P_{ka}^*(s)$, $P_{ak}^*(s)$, $P_{kk}^*(s)$ and $P_{jk}^*(s)$ converge, whence the desired result follows from theorem B.

By theorem A, $P_{aa}^*(s)$ and $H_a^*(s)$ converge for $Rs > -\lambda$. Since $P_{aa}^*(s)$ is analytic in its region of convergence (WIDDER: Theorem 5a, p. 57), there exists a positive number $\gamma' < \lambda$ such that $P_{aa}^*(s)$ is analytic in the half-plane $\Omega' = Rs \geqq -\gamma'$ and non-vanishing for real values of s in the interval $[-\gamma', 0)$. By (2.5) $G_{aa}^*(s)$ has no singularity on the segment $[-\gamma', 0)$ of the real line and, as it is the transform of a monotonic function, must therefore be convergent in Ω' (see WIDDER: Theorem 5b, p. 58). Replacing s with the real variable x in (2.6), we have an identity between positive quantities from which the convergence of $G_{aa}^*(s)$ in Ω' implies the convergence of ${}_aG_{ak}^*(x)$, ${}_kG_{aa}^*(x)$ and $G_{ka}^*(x)$ in $[-\gamma', \infty)$, and hence of ${}_aG_{ak}^*(s)$, ${}_kG_{aa}^*(s)$ and $G_{ka}^*(s)$ in Ω'. By (2.7), ${}_kG_{ka}^*(s)$ and ${}_aG_{kk}^*(s)$ are also convergent in Ω'.

By (2.10) and theorem C, $P_{ka}^*(s)$ converges in Ω'. To use (2.11), we need to show first that $1 - {}_aG_{kk}^*(s)$ is non-vanishing in Ω'. Suppose $1 - {}_aG_{kk}^*(s_0) = 0$, $s_0 = \lambda_0 + i\delta_0$, $\lambda_0 \geqq -\gamma'$. Then ${}_aG_{kk}^*(\lambda_0) \geqq 1$ and so by (2.7), ${}_aG_{kk}^*(\lambda_0) = 1$ and ${}_kG_{ka}^*(\lambda_0) = 0$. But ${}_kG_{ka}^*(\lambda_0)$ cannot be zero since ${}_kG_{ka}(t)$ is non-decreasing and not identically zero. We can now rewrite

(2.11) as $\quad {}_aP_{kk}^*(s) = \{1 - H_k^*(s)\}\, {}_aM_{kk}^*(s)$, $\quad$ where $\quad {}_aM_{kk}^*(s) = \dfrac{1}{1 - {}_aG_{kk}^*(s)}$

is the Laplace transform of a non-decreasing function. (${}_aM_{kk}(t)$ is in fact the expectation of the number of visits from k to k under the taboo a.) Since ${}_aM_{kk}^*(s)$ is analytic in Ω', it converges there, and so by theorem C, ${}_aP_{kk}^*(s)$ converges in Ω'. The convergence of ${}_aP_{ak}^*(s)$ now follows from (2.12).

If $z(t)$ is transient, $G_{aa}^*(o) < 1$ and choosing γ $(o < \gamma \leqq \gamma')$ such that $1 - H_a^*(s) = 0$ has no root in the half-plane $\Omega = Rs \geqq -\gamma$ (see LEADBETTER, Lemma 3), we see from (2.5) that $1 - G_{aa}^*(s) = 0$ has no root in Ω. (2.13) then yields the convergence in Ω of $P_{ak}^*(s)$. If $z(t)$ is recurrent, we rewrite (2.13) as

$$P_{ak}^*(s) = {}_aG_{ak}^*(s)/(1 - {}_aG_{kk}^*(s)\, P_{aa}^*(s)) \cdot U_k^*(s),$$

where

$$U_k^*(s) = 1 - H_k^*(s)/(1 - H_a^*(s)).$$

Writing

$$U_k^*(s) = 1 + H_a^*(s)/(1 - H_a^*(s)) - H_k^*(s)/(1 - H_a^*(s)),$$

and appealing to the corollary to theorem D, we see that there exists $\gamma'' > 0$ such that $\frac{\eta_k}{\eta_a} - U_k(t) = o(e^{-\gamma'' t})$ as $t \to \infty$. By the remark following theorem D, γ'' is independent of k, and by theorem A, $U_k^*(s)$ is convergent for all $Rs > -\gamma''$. As ${}_aG_{kk}^*(s) \neq 1$ anywhere in Ω', ${}_aG_{ak}^*(s)/(1-{}_aG_{kk}^*(s))$ converges in Ω'. Also, $P_{aa}^*(s)$ converges in Ω'. Hence by theorem C*, there exists a positive number γ, $\gamma < \min(\gamma', \gamma'')$, such that $P_{ak}^*(s)$ converges in $\Omega = Rs \geqq -\gamma$. By (2.14) $P_{kk}^*(s)$ converges in Ω. The convergence of $P_{jk}^*(s)$ now follows easily from identities (2.15)—(2.17), thus completing the proof.

The following weaker result is obtained if the parameter of convergence of the H_k's to unity is dependent on k.

Theorem 3.2. *Suppose for each k, there exists $\lambda_k > 0$ such that $1 - H_k(t) = O(e^{-\lambda_k t})$ as $t \to \infty$, and $H_k^*(-\lambda_k + i\delta)/(-\lambda_k + i\delta) \in L_1(-\infty, \infty)$. If there exists $\lambda_a > 0$ such that*

$$P_{aa}(t) - \pi_{aa} = O(e^{-\lambda_a t}) \quad \textit{as} \quad t \to \infty,$$

then for each k, there exists $\gamma_k > 0$ such that

$$P_{jk}(t) - \pi_{jk} = o(e^{-\gamma_k t}) \quad \textit{as} \quad t \to \infty \quad \textit{for each } j.$$

The proof is exactly the same as that of theorem 3.1, except in those places where a parameter, γ' say, which is common to all states, has to be replaced by the corresponding parameter γ_k'.

The same method of Laplace transforms may be used to derive a necessary condition for geometric ergodicity. Let us assume that the chain is 'strongly' geometrically ergodic i.e.: there exists $\lambda > 0$ (λ independent of i, j) such that for each i, j, $P_{ij}(t) - \pi_{ij} = o(e^{-\lambda t})$ as $t \to \infty$. Consider firstly the transient case, and a fixed state a. Clearly $P_{aa}(t) \geqq 1 - H_a(t)$, so $1 - H_a(t) = o(e^{-\lambda t})$ as $t \to \infty$, and $H_a^*(s)$ converges for $Rs > -\lambda$. By (2.5) $G_{aa}^*(s)$ converges, and $1 - G_{aa}^*(s)$ is non-vanishing, in some half-plane $\Omega_a = Rs > -\gamma_a, \gamma_a \leqq \lambda$. Hence ${}_kG_{aa}^*(s) \neq 1$ anywhere in Ω_a, and $G_{kk}^*(s)$ converges in Ω as a consequence of identities (2.6) —(2.9). Finally, by (2.18), $H_k^*(s)$ converges in Ω_a. Suppose now that the chain is recurrent. Choose λ_a such that $P_{ij}^*(s)$ is analytic in $Rs \geqq -\lambda_a$ for all i, j, and $P_{aa}^*(s) \neq 0$ for all real s in the interval $[-\lambda_a, 0]$. Then by (2.10) $G_{ka}^*(s)$ is analytic, hence convergent, in $Rs \geqq -\lambda_a$ and so, $1 - G_{ka}(t) = o(e^{-\lambda_a t})$ as $t \to \infty$. The obvious inequality $G_{ka}(t) \leqq H_k(t)$ now yields $1 - H_k(t) = o(e^{-\lambda_a t})$ as $t \to \infty$.

In the case when the chain is geometrically ergodic, but not strongly geometrically ergodic i.e. when the parameter of convergence λ_{ij} depends on i and j, the corresponding result holds and can be proved similarly. We summarize the last two paragraphs in the following

Theorem 3.3. (i) *If $z(t)$ is strongly geometrically ergodic, then there exists $\gamma > 0$ such that for each k, $1 - H_k(t) = o(e^{-\gamma t})$ as $t \to \infty$.*

(ii) *If $z(t)$ is geometrically ergodic, then for each k, there exists $\gamma_k > 0$ such that $1 - H_k(t) = o(e^{-\gamma_k t})$ as $t \to \infty$.*

4. Geometric Ergodicity in a Finite Chain

Let $m < \infty$ be the number of states in E. It will be assumed in this section that $0 < \eta_j < \infty$ for each j. A transform $A^*(s)$ will be said to be 'analytic at the origin' if there exists $\lambda > 0$ such that $A^*(-\lambda)$ converges.

Lemma 4.1. *$G^*_{ij}(s)$ is analytic at the origin for all i, j in E if and only if $Q^*_{ij}(s)$ is analytic at the origin for all i, j in E.*

Proof. The necessity part follows directly from (2.4). Taking Laplace transforms and using the real veriable x, we obtain

$$G^*_{ij}(x) = Q^*_{ij}(x) + \sum_{\substack{k=1 \\ k \neq j}}^{m} G^*_{kj}(x)\, Q^*_{ik}(x), \qquad i, j \in E. \tag{4.1}$$

Since (4.1) is an identity between positive quantities, and its right hand side consists of a finite sum, the convergence of $G^*_{ij}(x)$ must imply the convergence of each of the transforms on the right hand side.

The proof of the sufficiency part of the lemma proceeds by induction. The lemma is trivially true for $m = 1$. Suppose the lemma is true for $m = N$. We now consider a semi-Markov process $z(t)$ of $N + 1$ states with transition probabilities $Q_{ij}(t)$ and first passage distributions $G_{ij}(t)$. Suppose $Q^*_{ij}(s)$ is analytic at the origin for all i, j.

Let k and l be two arbitrarily chosen states, $k \neq l$, $k \neq N + 1$, $l \neq N + 1$, and write $p_{ij} = Q_{ij}(\infty)$ $(i, j \in E)$. We wish to show firstly that ${}_kG^*_{kl}(s)$ is analytic at the origin. It can be assumed that at least one of $p_{k,N+1}$ and $p_{N+1,N+1}$ is less than unity for otherwise $G_{kl}(\infty) = 0$ and the result is trivially true. We consider a new semi-Markov process $\bar{z}(t)$ with N states and transition probabilities defined as follows. For $i, j = 1, 2, \ldots, N$,

$$\begin{aligned}
\bar{Q}_{ik}(t) &= \left(1 + \frac{p_{ik}}{p_{i,N+1}}\right) Q_{i,N+1}(t) \quad \text{if} \quad p_{i,N+1} \neq 0, \\
&= Q_{ik}(t) \qquad\qquad\qquad\qquad\quad\; \text{if} \quad p_{i,N+1} = 0. \\
\bar{Q}_{ki}(t) &= (1 - p_{N+1,N+1})^{-1} Q_{N+1,i}(t) \quad \text{if} \quad p_{N+1,N+1} \neq 1, \\
&= (1 - p_{k,N+1})^{-1} Q_{ki}(t) \qquad\quad\; \text{if} \quad p_{N+1,N+1} = 1. \\
\bar{Q}_{ij}(t) &= Q_{ij}(t) \quad \text{otherwise.}
\end{aligned}$$

The notation here is obvious. All quantities related to the new process will be marked by a bar, thus $\bar{G}_{ij}$ is the first passage time distribution of $\bar{z}(t)$. It is easily checked that $\sum_{j=1}^{N} \bar{p}_{ij} = 1$ for all $i = 1, \ldots, N$. Now, ${}_kG^*_{kl}(s)$ is determined by the Q_{ij}'s according to the equation

$$\begin{aligned}
{}_kG^*_{kl}(s) = Q^*_{kl}(s) &+ \sum_{\substack{j=1 \\ j \neq k,l}}^{N+1} Q^*_{kj}(s)\, Q^*_{jl}(s) + \\
&+ \sum_{\substack{j=1 \\ j \neq k,l}}^{N+1} Q^*_{kj}(s) \sum_{n=3}^{\infty} \sum_{\underset{\sim}{\alpha}_n \in S_n} Q^*_{j\alpha_1}(s)\, Q^*_{\alpha_1\alpha_2}(s) \cdots Q^*_{\alpha_{n-2}l}(s)
\end{aligned} \tag{4.2}$$

where

$$S_n = \{\underset{\sim}{\alpha}_n = (\alpha_1, \ldots, \alpha_{n-2});\quad \alpha_i = 1, \ldots, N + 1,\ \alpha_i \neq k, l\}$$

is the set of all paths of $(n-2)$ states not containing k and l. By the definition of $\bar{Q}_{ij}(t)$, (4.2) may be rewritten as

$$
\begin{aligned}
{}_kG^*_{kl}(s) &= Q^*_{kl}(s) + \sum_{\substack{j=1\\ j\neq k,l}}^{N+1} Q^*_{kj}(s)\, Q^*_{jl}(s) \\
&\quad + \sum_{\substack{j=1\\ j\neq k,l}}^{N} Q^*_{kj}(s) \sum_{n=3}^{\infty} \sum_{\underline{\alpha}_n\in\bar{S}_n} a_{\underline{\alpha}_n}\, \bar{Q}^*_{j\alpha_1}(s)\, \bar{Q}^*_{\alpha_1\alpha_2}(s)\cdots \bar{Q}^*_{\alpha_{n-2}l}(s) \qquad (4.3)\\
&\quad + Q^*_{k,N+1}(s) \sum_{n=3}^{\infty} \sum_{\underline{\alpha}_n\in\bar{S}_n} a_{\underline{\alpha}_n}\, \bar{Q}^*_{k\alpha_1}(s)\cdots \bar{Q}^*_{\sigma_{n-2}l}(s),
\end{aligned}
$$

where $a_{\underline{\alpha}_n} \leqq 1$ is a constant, and

$$\bar{S}_n = \{\underline{\alpha}_n = (\alpha_1, \ldots, \alpha_{n-2});\quad \alpha_i = 1, \ldots, N,\quad \alpha_i \neq l\}.$$

For each $j = 1, \ldots, N$,

$$\bar{G}^*_{jl}(s) = \bar{Q}^*_{jl}(s) + \sum_{n=3}^{\infty} \sum_{\underline{\alpha}_n\in\bar{S}_n} \bar{Q}^*_{j\alpha_1}(s)\, \bar{Q}^*_{\alpha_1\alpha_2}(s)\cdots \bar{Q}^*_{\alpha_{n-2}l}(s), \tag{4.4}$$

$$\bar{G}^*_{kl}(s) = \bar{Q}^*_{kl}(s) + \sum_{n=3}^{\infty} \sum_{\underline{\alpha}_n\in\bar{S}_n} \bar{Q}^*_{k\alpha_1}(s)\, \bar{Q}^*_{\alpha_1\alpha_2}(s)\cdots \bar{Q}^*_{\alpha_{n-2}l}(s). \tag{4.5}$$

By the induction hypothesis, $\bar{G}^*_{jl}(s)$ and $\bar{G}^*_{kl}(s)$ are analytic at the origin. A comparison of (4.3) with (4.4) and (4.5) now shows that ${}_kG^*_{kl}(s)$ is also analytic at the origin.

It can be similarly shown, by reversing the roles of k and l in the definition of $\bar{z}(t)$, that ${}_lG^*_{kk}(s)$ is also analytic at the origin.

From (2.2),

$$G^*_{kl}(s) = \frac{{}_kG^*_{kl}(s)}{1 - {}_lG^*_{kk}(s)}, \tag{4.6}$$

$$G^*_{kk}(s) = {}_lG^*_{kk}(s) + {}_kG^*_{kl}(s)\, G^*_{lk}(s). \tag{4.7}$$

By (4.6) and WIDDER's Th. 56 (WIDDER, 1946: p. 58), $G^*_{kl}(s)$ is analytic at the origin. Similarly $G^*_{lk}(s)$ is analytic at the origin and by (4.7), so is $G^*_{kk}(s)$.

The proof for the remaining cases when either or both of k and l is $N+1$ will be omitted. This proof can be easily obtained by modifying the definition of $\bar{z}(t)$ slightly and proceeding in exactly the same way as before.

We recall that $z(t)$ is said to be geometrically ergodic if for each i, j in E, there exists λ_j such that $P_{ij}(t) - \pi_{ij} = o(e^{-\lambda_j t})$ as $t \to \infty$, and strongly ergodic if all the λ_j's may be replaced by a common parameter λ. Obviously for a finite chain, geometric ergodicity implies strong geometric ergodicity. The main result of this section may be stated as follows:

Theorem 4.1. (i) *If $z(t)$ is geometrically ergodic, then for all i, j, $Q^*_{ij}(s)$ is analytic at the origin.*

(ii) *If for all i, j, there exists $\lambda > 0$ such that $Q^*_{ij}(-\lambda)$ converges and*

$$Q^*_{ij}(-\lambda + i\delta)/(-\lambda + i\delta) \in L_1(-\infty, \infty),$$

then $z(t)$ is geometrically ergodic.

Proof. (i) follows easily from lemma 4.1 and the identity

$$P_{ij}^*(s) = G_{ij}^*(s)\, P_{jj}^*(s)\,.$$

(ii) Firstly, $H_i^*(-\lambda) = \sum_{k=1}^{m} Q_{ik}^*(-\lambda)$ converges and $H_i^*(-\lambda + i\delta)/(-\lambda+i\delta) \in L_1(-\infty, \infty)$. Next, by lemma 4.1, for all i, j, there exists $\sigma > 0$ such that $G_{ij}^*(-\sigma)$ converges and

$$\frac{|G_{ij}^*(-\sigma + i\delta)|}{(\sigma^2+\delta^2)^{1/2}} \leq \frac{|Q_{ij}^*(-\sigma + i\delta)|}{(\sigma^2+\delta^2)^{1/2}} + \frac{\max\limits_{1\leq k\leq m} G_{kj}^*(-\sigma)}{(\sigma^2+\delta^2)^{1/2}} \sum_{\substack{k=1\\ k\neq j}}^{m} |Q_{ik}^*(-\sigma+i\delta)|\,,$$

whence

$$\frac{G_{ij}^*(-\sigma+i\delta)}{-\sigma+i\delta} \in L_1(-\infty,\infty)\,.$$

By theorem D and the identity

$$P_{ii}^*(s) = 1 + \frac{G_{ii}^*(s)}{1-G_{ii}^*(s)} - \frac{H_i^*(s)}{1-G_{ii}^*(s)}\,,$$

$P_{ii}^*(s)$ is analytic at the origin. An appeal to theorem 3.1 now completes the proof.

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