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Lévy Processes: Absolute Continuity of Hitting Times for Points

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1. Introduction

Let $X = (\Omega, \mathfrak{F}, \mathfrak{F}_t, X_t, \theta_t, P^x)$ be a real valued process with stationary independent increments and right continuous paths with left limits. For each real number x, let

$$T_x = \inf\{t > 0 : X_t = x\}$$

denote the first hitting time of $\{x\}$. We will assume that $P^0\{X_t = 0\} = 0$ for $t > 0$, and that 0 is regular for $\{0\}$, i.e. that $P^0\{T_0 = 0\} = 1$. Under these assumptions $P^0\{T_x < \infty\} > 0$ for all x ([3]).

In Section 5 of [1] Blumenthal and Getoor investigate the absolute continuity of the Lévy measure ν of the pathwise inverse to the local time of a general Markov process X at zero.

Their main result is that ν is absolutely continuous under the assumption that for each $t > 0$ and each $x \in R$ the transition function $P_t(x, dy) = P^x\{X_t \in dy\}$ is absolutely continuous with respect to Lebesgue measure. In order to prove this they have to show that the hitting time distributions $P^y\{T_0 \in dt\}$ are absolutely continuous for $y \neq 0$. However, the proof they give for this fact is incorrect. $\left(\text{The functions } t -> \int_0^t h_n(t-u)\, p(u)\, du \text{ are not non-decreasing.}\right)$

In this paper it is shown that for a Lévy process the absolute continuity of the transition functions $P_t(x, dy)$ is indeed both necessary and sufficient for the absolute continuity of ν.

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2. Preliminaries

Let $X = \{X(t),\ t > 0\}$ be a real valued process with stationary independent increments and right continuous paths. Then $E^0\ \{\exp(iuX(t))\} = \exp\{-t\Psi(u)\}$, where $\Psi(u) = iau + (\sigma^2/2)\,u^2 + \int [1 - e^{iux} + iux/(1+x^2)]\,\mu(dx)$. The measure μ is called the Lévy measure, and the function Ψ the exponent of the process X.

If $\sigma^2>0$, X is said to have a Gaussian component.

For $t>0$ let $P_t(x,dy)=P^x\{X(t)\in dy\}$. If $\sigma^2>0$, then $P_t(x,dy)$ is absolutely continuous and has a continuous positive density on R.

The following facts can be proved using the ideas of the proof of Proposition 2.1. of [9]. (See also the proof of Theorem 2. of [7].)

(2.1) **Proposition.** If $\sigma^2=0$ and $\mu(R)=\infty$, then $P_t(x,dy)$ is non atomic and the support of $P_t(0,dy)$ is an interval of the form $(-\infty,\infty)$, $(-\infty,ct]$ or $[ct,\infty)$. If $\int |x|/(1+x^2)\,\mu(dx)=\infty$, then the support of $P_t(x,dy)$ is $(-\infty,\infty)$.

The next proposition generalizes Theorem 1., 2. and 3. of [7]. (We need not assume that μ is absolutely continuous, μ may even be purely atomic. See the examples at the end of this paper.)

(2.2) **Proposition.** If the measures $P_t(x,dy)$ are absolutely continuous for each $t>0$, then $P_t(x,dy)$ has a density which is lower semi continuous and positive over its support.

Remark. For fixed $t>0$, the density of $P_t(x,dy)$ may be unbounded on every interval.

Proof. If $\sigma^2=0$, we must have $\mu(R)=\infty$. The proposition is therefore a consequence of Proposition 2.1. and the following

Lemma 1. *Let* $F(t)=\int_0^t f(x)\,dx$ *be a strictly increasing probability distribution function on* $(0,\infty)$. *Then*

$$(f*f)(x)=\int_0^x f(x-y)\,f(y)\,dy$$

is positive for a.a. $x>0$. *If in addition* f *is lower semi continuous, then* $(f*f)(x)>0$ *for all* $x>0$.

Proof. Put $N=\{t>0:(f*f)(t)=0\}$ and let $g(-t)=e^{-t}I_N(t)$. $0=\int (f*f)(t)\,g(-t)\,dt=(f*f*g)(0)=\int f(t)(f*g)(-t)\,dt$. Assume that N has positive Lebesgue measure. Then there exists a subset N_0 of N such that $N\setminus N_0$ has Lebesgue measure zero, and every $x\in N_0$ has metric density 1 with respect to N, i.e.

$$\lim_{\varepsilon\to 0}(2\varepsilon)^{-1}\int_{x-\varepsilon}^{x+\varepsilon} I_N(t)\,dt=1.$$

Let M_0 be a subset of $\{f>0\}$ such that $\{f>0\}\setminus M_0$ has Lebesgue measure zero, and every $x\in M_0$ has metric density 1 w.r.t. M_0. Then N_0-M_0 is an open set meeting $(0,\infty)$. For $t\in N_0-M_0$ we have $(f*g)(-t)>0$. Since the support of F is $[0,\infty)$, this implies $\int (f*g)(-t)\,F(dt)>0$. Contradiction! This proves the first assertion. If f is lower semi continuous, then the set $C=\{t\geqq 0: f(t)\leqq 0\}$ is closed and nowhere dense. If $(f*f)(x)=0$ for some $x>0$, then we must have $C\cup(x-C)\supset[0,x]$. This contradicts the Baire category theorem.

From now on we will assume that $P^0\{T_x<\infty\}>0$ for all x, and that $P^0\{T_0=0\}=1$. Under these assumptions the support of $P_t(x,dy)$ is R.

Let $a<b$. Consider the following stopping time,

$$T=\inf\{t>0: X(t-)<a<b<X(t)\}.$$

T is the first time the process jumps across the interval $[a,b]$ from below.

Lemma 2. Assume that $\mu\{(b-a,\infty)\}>0$. Then $P^0\{T<\infty\}>0$, and for Borel sets $A\subset(-\infty,a)$ and $B\subset(b,\infty)$ we have

$$P^0\{T\leqq t,\ X(T-)\in A,\ X(T)\in B\}$$
$$=E^0\int_0^{T\wedge t} I\{X(s)\in A\}\,\mu\{B-X(s)\}\,ds.$$

Proof. Let $f(u,v)$ be a bounded Borel function from R^2 to R that vanishes for $|u-v|<\delta$. Then, according to the theory of Lévy systems (see [11], Theorem 4.3.)

$$\sum_{s\leqq t} f(X(s-),X(s))-\int_0^t ds\int f(X(s),X(s)+y)\,\mu(dy)$$

is a martingale with mean zero. Put $f(u,v)=I_A(u)\,I_B(v)$ and apply the Optional Sampling Theorem to the stopping time $T\wedge t$. This completes the proof.

(2.1) **Corollary.** *Assume that $\mu\{(b-a,\infty)\}>0$ and assume that the measures $P_t(x,dy)$ are absolutely continuous. Then the measure $P^0\{X(T-)\in dy\}$ is absolutely continuous on R and for a.a. $y<a$*

$$P^0\{T\leqq t,\ X(T)\in B\,|\,X(T-)=y\}$$
$$=P^0\{T\leqq t\,|\,X(T-)=y\}\,P^0\{X(T)\in B\,|\,X(T-)=y\}.$$

Furthermore, the conditional distribution of T given $X(T-)=y$ is absolutely continuous on R.

Proof. Let $\tilde{X}$ denote the process X killed at time T, i.e. $\tilde{X}(t)=X(t)$ if $t<T$ and $\tilde{X}(t)=\Delta$ if $t\geqq T$. Obviously, the measure $P^0\{\tilde{X}(t)\in dy\}$ is absolutely continuous on R. Let $p(t,y)$ be a jointly measurable density for $P^0\{\tilde{X}(t)\in dy\}$. Then

$$P^0\{T\leqq t,\ X(T-)\in A,\ X(T)\in B\}=\int_0^t ds\int_A p(s,y)\,\mu\{B-y\}\,dy.$$

This allows us to identify the conditional probabilities in question thus completing the proof.

(2.2) **Corollary.** *Under the assumptions of Corollary 2.1. the distribution of the stopping time*

$$T+T_z\circ\theta_T=\inf\{t>T\colon X(t)=z\}$$

is absolutely continuous on R.

Proof. Since T and $T_z\circ\theta_T$ are conditionally independent given $X(T-)=y$, it follows that $P^0\{T+T_z\circ\theta_T\in dt\,|\,X(T-)=y\}$ is absolutely continuous on R.

3. Main Results

Let L_t^x denote the local time of X at x chosen as in Section 3. of [1]. For each Borel set B and each $t>0$

$$\int_B L_t^x\,dx=\int_0^t I_B(X_u)\,du \quad \text{a.s.}$$

We define $\tau(t)=\inf\{s>0: L_s^0>t\}$. According to [1] we then have for $\lambda>0$

$$E^0\{\exp(-\lambda\tau(t)\}=\exp\{-t(C+g(\lambda))\}$$

where $0\leqq C<\infty$ and $g(\lambda)=\int_0^\infty (1-\exp(-\lambda u))\,v(du)$.

Here v is a positive Borel measure on $(0,\infty)$ such that $v\{(0,\infty)\}=\infty$ and $\int_0^\infty u/(1+u)\,v(du)<\infty$. We shall call v the Lévy measure of τ.

We are now able to state our main result. We shall indicate that a Borel measure F on R is absolutely continuous with respect to Lebesgue measure by writing $F(dx)\ll dx$.

(3.1) **Theorem.** *The following conditions are equivalent:*

(1) $\forall t>0: P_t(x,dy)\ll dy$,

(2) $\forall z\neq 0: P^0\{T_z\in dt\}\ll dt$,

(3) $v(du)\ll du$,

(4) $\forall t>0: P^0\{\tau(t)\in ds\}\ll ds$.

(5) There exists a jointly measurable non negative function $\alpha(y,x)$ such that $E^x\{L_t^0\}=\int_0^t \alpha(y,x)\,dy$ for all $x\in R$, $t>0$.

Proof. (1)⇒(2). Assume that $P_t(x,dy)$ is absolutely continuous. If X has a Gaussian component and $\mu\{R\}<\infty$, then the absolute continuity of $P^0\{T_z\in dt\}$ follows from the absolute continuity of the hitting time distributions of Brownian motion. Assume therefore that $\mu\{R\}=\infty$. Let $\varepsilon>0$ and let $B\subset(\varepsilon,\infty)$ be a Borel set of Lebesgue measure zero. Then

$$\{T_z\in B\}\subset\bigcup_T\{T+T_z\circ\theta_T\in B\}$$

where the union is taken over the countable number of stopping times, $T=\inf\{t>0: X(t-)<r_1<r_2<X(t)\}$ and $T=\inf\{t>0: X(t)<r_1<r_2<X(t-)\}$ where $r_1<r_2$ are rational numbers. In virtue of Corollary 2.2. it follows that $P^0\{T_z\in B\}=0$.
(2)⇒(3). Blumenthal and Getoor prove this in [1]. In [6] Horowitz gives the following elegant proof:

For $s>0$ let $\sigma_s=\inf\{t>s: L_t^0=L_{t-s}^0\}$. Then $P^0\{\sigma_s<\infty\}=1$, $P^0\{X(\sigma_s)=0\}=0$ and for all $t>0$

$$\frac{v\{(s+t,\infty)\}+C}{v\{(s,\infty)\}+C}=P^0\{T_0\circ\theta_{\sigma_s}>t\}=E^0\,P^{X(\sigma_s)}\{T_0>t\}.$$

For any Borel set $B\subset(0,\infty)$ we therefore get

$$v\{s+B\}=[v\{(s,\infty)\}+C]\,E^0\,P^{X(\sigma_s)}\{T_0\in B\}.$$

If we assume that $P^y\{T_0\in dt\}$ is absolutely continuous for all $y\neq 0$ it follows that v is absolutely continuous. And we see that v has an a.e. positive density over $(0,\infty)$ under the additional assumption that the hitting time distributions have a.e. positive densities over $(0,\infty)$.

(3)⇒(4). This implication is well known. (See Fisz and Varadarajan [4].)
(4)⇒(5). Let $g(t,y)$ be a jointly Borel measurable non negative density for $P^0\{\tau(t)\in dy\}$. Then

$$P^0\{L_s^0>t\}=P^0\{\tau(t)<s\}=\int_0^s g(t,y)\,dy,$$

$$E^0\{L_s^0\}=\int_0^\infty P^0\{L_s^0>t\}\,dt=\int_0^s p(y)\,dy,$$

where the Borel function $p(y)$ is defined by $p(y)=\int_0^\infty g(t,y)\,dt$. Put $\alpha(y,x)=\int_0^y p(y-s)\,P^x\{T_0\in ds\}$.

It follows that α is jointly Borel measurable. Furthermore

$$E^x\{L_t^0\}=\int_0^t E^0\{L_{t-s}^0\}\,P^x\{T_0\in ds\}=\int_0^t \alpha(y,x)\,dy.$$

(5)⇒(1). Let $B\subset R$ be a fixed Borel set. For every $t>0$

$$\int_0^t dy\int_B \alpha(y,-x)\,dx=E^0\int_B L_t^x\,dx=E^0\int_0^t I_B(X(y))\,dy$$
$$=\int_0^t P^0\{X(y)\in B\}\,dy.$$

Therefore $P^0\{X(y)\in B\}=\int_B \alpha(y,-x)\,dx$ for a.a. $y>0$.

The exceptional set depends on B. For $y\notin N$, a null set, $P^0\{X(y)\in B\}=\int_B \alpha(y,-x)\,dx$ simultaneously for all intervals B with rational endpoints. It now follows that for $y\notin N$ this identity holds for all Borel sets B. So $P^0\{X(y)\in dx\}$ has a density for a.a. y, and hence for all y. Q.E.D.

(3.2) **Theorem.** *If $P_t(x,dy)\ll dy$ for all $t>0$, then the measures $P^0\{T_z\in dt\}, z\neq 0$, and $\nu(du)$ have a.e. positive densities on $(0,\infty)$.*

Proof. Let $z\neq 0$. Consider the function

$$F(t)=P^0\{T_z\leqq t,\ T_z<T_{2z}\}.$$

According to Theorem 3.1., F is absolutely continuous. Furthermore, F is strictly increasing on $(0,\infty)$. (See the proof of Proposition 2.1. of [9] and use the fact that $x\to E^x e^{-\lambda T_z}$ is continuous for all $\lambda>0$ according to Bretagnolle [3].) Let $f(s)$ be a density for F. In virtue of Lemma 1.,

$$(f*f)(t)=\int_0^t f(t-s)\,f(s)\,ds$$

is positive for a.a. $t\in(0,\infty)$. For any Borel set $B\subset(0,\infty)$ of positive Lebesgue measure we therefore have

$$P^0\{T_{2z}\in B\}\geqq P^0\{T_z<T_{2z}\in B\}\geqq\int_B (f*f)(t)\,dt>0.$$

This shows that $P^0\{T_{2z}\in dt\}$ has an a.e. positive density on $(0, \infty)$. We have already noted in the course of the proof of Theorem 3.1. that if the hitting time distributions $P^y\{T_0\in dt\}$ have a.e. positive densities on $(0, \infty)$, so will v.

4. Examples

If the Lévy measure μ of X is absolutely continuous, then the transition functions $P_t(x, dy)$ are absolutely continuous. The first example in this section shows that not all Lévy processes with absolutely continuous transition functions have an absolutely continuous Lévy measure.

If X is symmetric, then x regular for $\{x\}$ implies that the transition functions of X are absolutely continuous. The second example shows that not all Lévy processes for which x is regular for $\{x\}$ have absolutely continuous transition functions.

Let the process X have exponent $\Psi(u)=\int_0^1 [1-e^{iux}+iux]\,\mu(dx)$ where the Lévy measure μ satisfies $\int_0^1 x\,\mu(dx)=\infty$.

Then X has only upward jumps and moves downward in a continuous manner. Furthermore, $E^0\{X_t\}=0$ and $E^0\{X_t^2\}<\infty$. (See [10]). It follows that $P^0\{T_x<\infty\}=1$ for all x, and it is obvious from "Rogozin's Theorem" that $P^0\{T_0=0\}=1$.

(Rogozin's Theorem states that if X is a Lévy process for which

$$\int |x|/(1+x^2)\,\mu(dx)=\infty,$$

and $X_0=0$, then $\inf\{t>0\colon X_t>0\}=\inf\{t>0\colon X_t<0\}=0$ a.s.)

We shall consider two examples of such measures μ.

First let $\mu=\sum_{n=1}^{\infty}\delta(n^{-1/\alpha})$, where $1\leq\alpha<2$ and $\delta(x)$ denotes the probability measure with support $\{x\}$. Then, writing Ψ_R for the real part of Ψ, we have

$$\liminf_{u\to\infty} u^{-\alpha}\,\Psi_R(u)>0.$$

This shows that $P_t(x, dy)$ is absolutely continuous with a continuous density. (See Section 4. of [1].)

Next let $\mu=\sum_{n=1}^{\infty} 2^{2^n}\,\delta(2^{-2^n})$.

In this case we have $\liminf_{u\to\infty}\Psi_R(u)<\infty$.

So $P_t(x, dy)$ can not be absolutely continuous w.r.t. Lebesgue measure.

References

1. Blumenthal, R. M., Getoor, R. K.: Local times for Markov processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **3**, 50–74 (1964)
2. Blumenthal, R. M., Getoor, R. K.: Markov processes and potential theory. New York-London: Academic Press 1968
3. Bretagnolle, J.: Resultats de Kesten sur les processus a accroissements independants. Sém. de Probab. V, Lecture Notes in Math. **191**, 21–36. Berlin-Heidelberg-New York: Springer 1971

4. Fisz, M., Varadarajan, V. S.: A condition for absolute continuity of infinitely divisible distribution functions. Z. Wahrscheinlichkeitstheorie verw. Gebiete **1**, 335–339 (1963)
5. Gikhman, I. I., Skorokhod, A. V.: Introduction to the theory of random processes. Moscow: Izv. Nauk. 1965 (in Russian) Philadelphia: Saunders 1969
6. Horowitz, J.: Semilinear Markov processes, subordinators and renewal theory. Z. Wahrscheinlichkeitstheorie verw. Gebiete **24**, 167–193 (1972)
7. Hudson, W. N., Tucker, H. G.: Equivalence of infinitely divisible distributions. Ann. Probability **3**, 70–79 (1975)
8. Kesten, H.: Hitting probabilities of single points for processes with stationary independent increments. Mem. Amer. Math. Soc. 93 (1969)
9. Millar, P. W.: Exit properties of stochastic processes with stationary independent increments. Trans., Amer. Math. Soc. **178**, 459–479 (1973)
10. Skorokhod, A. V.: Random processes with independent increments. Moscow: Izv. Nauk. 1964 (in Russian). Springfield, Va.: Clearinghouse for Federal Scientific and Technical Information, 1966
11. Watanabe, S.: On discontinuous additive functionals and Lévy measures of a Markov process. Japan. J. Math. **34**, 53–70 (1964)

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