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Endomorphisms of Substitution Minimal Sets

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§ 1. Introduction

Let $\mathscr{X}=(X,h)$ be a (discrete) *flow*, that is, X is a non-empty compact Hausdorff space and h is a homeomorphism of X onto itself. A flow $\mathscr{X}'=(X',h')$ is a *subflow* of $\mathscr{X}$ if X' is a closed, h-invariant subset of X and $h'=h|X'$. A flow is *minimal* if it has no proper subflows; a minimal flow is also called a *minimal set.* Let $\mathscr{Y}=(Y,g)$ be another flow. A *homomorphism* φ of $\mathscr{X}$ to $\mathscr{Y}$ is a continuous map of X onto Y such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow[g]{} & Y \end{array}$$

In this case $\mathscr{Y}$ is called a *factor* of $\mathscr{X}$. A homomorphism of $\mathscr{X}$ to itself is called an *endomorphism* and a one-to-one endomorphism is called an *automorphism.*

A standard problem in topological dynamics is to determine the set of endomorphisms of a given minimal set or class of minimal sets. In this paper we solve this problem for the class of minimal sets known as substitution minimal sets. The solution is given by the following theorem.

Theorem. *Let θ be a substitution of equal length on two symbols such that the substitution minimal set $\mathscr{M}(\theta)=(M,\sigma)$ coming from θ is non-periodic. If θ is non-dual, then every endomorphism of $\mathscr{M}(\theta)$ is a power of the shift σ. If θ is dual, then every endomorphism of $\mathscr{M}(\theta)$ is of the form σ^k or $\sigma^k\delta$ where k is an integer and δ is the dualizing map.*

§ 2. Substitution Minimal Sets

Let Ω denote the set of *bisequences* of zeros and ones, i.e.,

$$\Omega=\{\ldots\omega_{-1}\,\omega_0\,\omega_1\ldots|\,\omega_i=0 \text{ or } 1\}.$$

Set-theoretically, $\Omega=\prod_{-\infty}^{\infty}\{0,1\}$. Given the product topology, Ω is a compact, metrizable space homeomorphic to the Cantor discontinuum. The *shift* homeomorphism $\sigma\colon\Omega\to\Omega$ is defined by $\sigma(\omega)_i=\omega_{i+1}$. The resulting flow $\mathscr{S}=(\Omega,\sigma)$ is called the *symbolic flow* or *shift dynamical system.* It has been the object of extensive study for the past thirty years.

The *dual* of a point $\omega\in\Omega$ is defined to be the bisequence $\tilde{\omega}$ obtained from ω by interchanging zeros and ones. The *dualizing map* δ, defined by $\delta(\omega)=\tilde{\omega}$, is an automorphism of $\mathscr{S}$. The dual of a block (i.e., a finite sequence) is analogously defined.

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Let θ be a substitution of equal length n on two symbols. Thus θ is given by two n-blocks, $\theta(0)=a_0 a_1 \dots a_{n-1}$ and $\theta(1)=b_0 b_1 \dots b_{n-1}$, where $a_i, b_i = 0$ or 1. In a manner described in [2] or [3], θ gives rise to a minimal subflow $\mathscr{M}(\theta)$ of $\mathscr{S}$ called a *substitution minimal set*. In order to insure that $\mathscr{M}(\theta)$ will be non-periodic, we will assume that θ satisfies none of the following triviality conditions.

(i) $\theta(0)=00\dots0$ or $\theta(1)=11\dots1$.

(ii) $\theta(0)=11\dots1$ and $\theta(1)=00\dots0$.

(iii) $\theta(0)=\theta(1)$.

(iv) $\theta(0)=0101\dots010$ or $1010\dots101$ and $\theta(1)=\widetilde{\theta(0)}$.

The remaining substitutions will be classified according to the size of the *disagreement set* $J(\theta)=\{i \mid a_i \neq b_i\}$. We say that θ is *dual* (in the language of [2], continuous) if $J=\{0,1,\dots,n-1\}$ and *non-dual* (discrete) if $J \neq \{0,1,\dots,n-1\}$. This terminology agrees with that of Keynes [5].

Let θ be a non-dual substitution of length n and let $\mathscr{M}(\theta)=(M,\sigma)$ be the substitution minimal set coming from θ. It is shown in [3] that the maximal equicontinuous factor of $\mathscr{M}(\theta)$ is the *n-adic flow* $\mathscr{Z}(n)=(Z(n),\tau)$ where $Z(n)$ is the group of n-adic integers $\left\{\sum_{i=0}^{\infty} z_i n^i \mid z_i=0,1,\dots,n-1\right\}$ and $\tau(z)=z+1$. Suppose $z=\sum z_i n^i$ is an element of $Z(n)$. We say that z is an *integer in* $Z(n)$ if $z_i=0$ for large i or if $z_i=n-1$ for large i, that is, if z is in the τ-orbit of 0.

Let $\mathscr{X}$ be a flow and $\mathscr{Y}$ its maximal equicontinuous factor. A homomorphism $p: \mathscr{X} \to \mathscr{Y}$ is called a *structure homomorphism* of $\mathscr{X}$. In [2], a structure homomorphism $\pi: \mathscr{M}(\theta) \to \mathscr{Z}(n)$ is analyzed in some detail. It is shown that $Z(n)$ may be written as the disjoint union of two non-empty subsets Z^* and E, where $Z^* = \{z \mid \pi^{-1}(z)$ is a singleton$\}$ and $E=\bigcup_{-\infty}^{\infty} \tau^k J_\infty$, where $J_\infty=\{\sum z_i n^i \mid z_i \in J$ for all $i\}$.

In the next section, we reduce the problem of finding the endomorphisms of $\mathscr{M}(\theta)$ to that of finding which $t \in Z(n)$ satisfy $E+t \subseteq E$, and we solve this combinatoric problem in § 4.

§ 3. Reduction of the Problem

Suppose that $\mathscr{X}$ is a minimal set, $\mathscr{Y}$ a factor of $\mathscr{X}$ and $p: \mathscr{X} \to \mathscr{Y}$ a homomorphism. Suppose further that p is *somewhere one-to-one*, that is, there is a point $y \in Y$ for which $p^{-1}(y)$ is a singleton. It is easy to see that every p-fibre is pairwise proximal. Thus, if a structure homomorphism of $\mathscr{X}$ is somewhere one-to-one, then $\mathscr{X}$ is proximally equicontinuous [1] and point-distal (i.e., there is a point in X proximal only to itself).

Suppose now that $\mathscr{X}$ is proximally equicontinuous and minimal, $\mathscr{Y}$ its maximal equicontinuous factor and $p: \mathscr{X} \to \mathscr{Y}$ a structure homomorphism. Then for each endomorphism φ of $\mathscr{X}$, there is an endomorphism ψ of $\mathscr{Y}$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[\psi]{} & Y \end{array}$$

If, in addition, $\mathscr{X}$ is point-distal, then this correspondence is one-to-one. For suppose φ and φ' are endomorphisms of $\mathscr{X}$ such that $p\varphi = p\varphi' = \psi p$. Then $\varphi(x) = \varphi'(x)$ for any $x \in \varphi^{-1}(x_0)$, where x_0 is any distal point in X. By the minimality of $\mathscr{X}$, $\varphi = \varphi'$.

Since $\mathscr{Y}$ is equicontinuous and minimal, every endomorphism of $\mathscr{Y}$ is one-to-one. Thus, if the diagram above commutes, then ψ must map the set $\{y \in Y \mid p^{-1}(y)$ is a singleton$\}$ to itself.

Let $\mathscr{M}(\theta)$ be a substitution minimal set coming from a non-dual substitution θ of length n. Without loss of generality, we may assume that θ is in *normal form* [2, § 7], in particular that $0 \in J(\theta)$. The results of [2] quoted in § 2 together with the results above show that for each endomorphism φ of $\mathscr{M}(\theta)$, there is an automorphism ψ of $\mathscr{Z}(n)$ such that $\psi(Z^*) \subseteq Z^*$. Since for each integer k, the following diagram commutes,

$$\begin{array}{ccc} M & \xrightarrow{\sigma^k} & M \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Z(n) & \xrightarrow[\tau^k]{} & Z(n) \end{array}$$

to prove the first part of the theorem, we need only show that if $\psi(Z^*) \subseteq Z^*$, then $\psi = \tau^k$ for some integer k.

Since each automorphism of $\mathscr{Z}(n)$ is of the form $\psi(z) = z + t$ for some $t \in Z(n)$, we must show that if $Z^* + t \subseteq Z^*$, then t is an integer in $Z(n)$. However $Z^* + t \subseteq Z^*$ implies $E + (-t) \subseteq E$, furthermore t is an integer in $Z(n)$ if and only if $(-t)$ is an integer in $Z(n)$. Therefore it suffices to show that if $E + t \subseteq E$, then t is an integer in $Z(n)$. We state this result as a lemma to be proved in the next section.

Lemma. *Let $n \geqq 2$ and let J be a proper subset of $\{0, 1, \ldots, n-1\}$ which contains 0. Let E be as in § 2. If $t \in Z(n)$ and $E + t \subseteq E$, then t is an integer in $Z(n)$.*

§ 4. Proof of the Lemma

We will use the following notation and terminology.

(i) For any integer q, $[q]$ will denote the residue mod n of q.

(ii) The phrase "q appears in z" will mean $z_k = q$ for some k where $z = \sum z_i n^i$.

The following remarks are easily verified and will be used in the proof of the lemma.

(i) If q appears infinitely often in some member of E which is not an integer in $Z(n)$, then $q \in J$.

(ii) If $q_1 \neq q_2$ both appear infinitely often in some member of E, then both belong to J.

Assume the hypotheses of the lemma hold but that t is not an integer in $Z(n)$. Let $t = \sum t_i n^i$; then $\{k \mid t_k \neq n-1\}$ is infinite. Notice that the lemma is trivial for $n = 2$, so we assume that $n \geqq 3$.

Proposition. *Let $j \in J$ and let q appear infinitely often in t. Then for each $m \geqq 0$, $[m q + j] \in J$.*

Proof. We prove the proposition for $m=1$, the proof is then completed by induction. Since $0\in E$, $t\in E$. Therefore the result is true for $j=0$. We assume that $j\neq 0$.

Let $K=\{k_1<k_2<\cdots\}$ be an infinite subset of $\{k|t_k=q\}$ satisfying: for each i, there is an integer m, $k_i<m<k_{i+1}$, such that $t_m\neq n-1$. Let $K'=\{k_{3i}|i\geqq 1\}$, $K''=\{k_{3i-1}\}$, and $K'''=\{k_{3i-2}\}$. Define $z\in Z(n)$ by $z_k=j$ if $k\in K'$ and $z_k=0$ otherwise. Then $z\in E$ and so $z+t\in E$. Now

$$\begin{aligned}(z+t)_k&=[q+j] && \text{if } k\in K'\\ &=q && \text{if } k\in K''\\ &=q \text{ or } [q+1] && \text{if } k\in K'''.\end{aligned}$$

Since $[q+j]\neq q$, $[q+j]\in J$.

The proposition has the immediate corollary that $n-1$ does not appear infinitely often in t. If not, then taking $j=0$ in the proposition, we have $[m(n-1)]\in J$ for all $m\geqq 0$. Then $J=\{0,1,\ldots,n-1\}$ which is impossible. We may therefore assume that $n-1$ does not appear in t at all.

There are now two possibilities to consider.

(A) For some $q\neq 0$, the set $K(q)=\{k|t_k=q,\ t_{k-1}\neq 0\}$ is infinite.

(B) For each $q\neq 0$, $K(q)$ is finite.

We show that each leads to the contradiction that $J=\{0,1,\ldots,n-1\}$.

Suppose (A) holds. Let K^* be an infinite subset of $K(q)$ such that

(i) Any two members of K^* differ by at least 5.

(ii) K^* does not contain two consecutive members of $K(q)$.

Let j_0 denote the largest member of J and let j be any member of J. Define $z\in Z(n)$ by $z_k=j$, $z_{k-1}=j_0$ for $k\in K^*$ and $z_k=0$ otherwise. Then $z\in E$.

Let $k\in K^*$. Since $z_{k-1}+t_{k-1}>j_0$, there is a "carry" to place k. Thus $(z+t)_k=[q+j+1]$. If $k\in K-K^*$, then $(z+t)_{k-2}=t_{k-2}$ or $t_{k-2}+1$, so there is no "carry" to place $k-1$. Therefore $(z+t)_{k-1}=t_{k-1}\neq 0$ or $n-1$. Hence $z+t$ is not an integer in $Z(n)$ and so $[q+j+1]\in J$.

Using this and the proposition, it is easy to see that $[m_1q+m_2]\in J$ for all $m_1,m_2\geqq 1$. But then $J=\{0,1,\ldots,n-1\}$.

Suppose that (B) holds. Then there is an integer k_0 such that for all $k\geqq k_0$, $t_k\neq 0$ implies $t_{k-1}=t_{k+1}=0$. Let $K_0=\{k\geqq k_0|t_k\neq 0\}$ and let $j\in J$, $j\neq n-1$. Define $z\in Z(n)$ by $z_k=j_0$, $z_{k+1}=j$ for $k\in K_0$ and $z_k=0$ otherwise. Then $z\in E$. Let $k\in K_0$. Since $z_k+t_k>j_0$, there is a "carry" to place $k+1$. Therefore $(z+t)_{k+1}=j+1$. But $(z+t)_{k-1}=0$ or 1, neither of which is $n-1$ because $n\geqq 3$. Hence $j+1\in J$. Therefore $J=\{0,1,\ldots,n-1\}$. This proves the lemma.

§ 5. The Dual Case

Let θ be a dual substitution of length n, given by $\theta(0)=a_0a_1\ldots a_{n-1}$ and $\theta(1)=\widetilde{\theta(0)}$. Let $\mathcal{M}(\theta)=(M(\theta),\sigma)$ be the substitution minimal set coming from θ and let $\pi\colon\mathcal{M}(\theta)\to\mathcal{Z}(n)$ be the structure homomorphism described in [2]. As in [2], the four members of $\pi^{-1}(0)$ are ω^{00}, ω^{01}, ω^{10}, and ω^{11}. It is easy to see that only asymptotic pairs of orbits in $\mathscr{M}(\theta)$ are those of ω^{00} and ω^{01}, ω^{00} and ω^{10}, ω^{11} and ω^{01}, and ω^{11} and ω^{10}.

Any automorphism φ of $\mathscr{M}(\theta)$ must map ω^{00} into the orbit of some point of $\pi^{-1}(0)$. If $\varphi(\omega^{00})\in\mathcal{O}(\omega^{00})$, then $\varphi=\sigma^k$ for some integer k. If $\varphi(\omega^{00})\in\mathcal{O}(\omega^{11})$, then $\varphi=\sigma^k\delta$. If $\varphi(\omega^{00})\in\mathcal{O}(\omega^{01})$, then for some integer k, $\varphi'=\sigma^k\varphi$ is an automorphism such that ω^{00} and $\varphi'(\omega^{00})$ are asymptotic. Then φ' is the identity which is impossible. Similarly $\varphi(\omega^{00})\notin\mathcal{O}(\omega^{10})$. Therefore to complete the proof of the theorem, it suffices to show that every endomorphism of $\mathscr{M}(\theta)$ is one-to-one.

Suppose φ is an endomorphism of $\mathscr{M}(\theta)$ which is not one-to-one. It is easy to see that φ must map the four points of $\pi^{-1}(0)$ to the same point, in particular, that $\varphi(\omega^{00})=\varphi(\omega^{11})$. Using the theorem of Curtis-Hedlund-Lyndon [4] on endomorphisms of $\mathscr{S}$, we may assume that $\varphi=f_\infty|M(\theta)$ where f_∞ is an endomorphism of $\mathscr{S}$ coming from some block map f. Since $\omega^{11}=\widetilde{\omega^{00}}$, it follows that for each block B appearing in $\mathscr{M}(\theta)$, $f(B)=f(\tilde{B})$. Therefore $\varphi(\omega)=\varphi(\tilde{\omega})$ for all $\omega\in M(\theta)$.

As in [2, § 8], let $\bar{\theta}$ be the non-dual substitution associated with θ; $\bar{\theta}(0)=c_0c_1\ldots c_{n-1}$, $\bar{\theta}(1)=d_0d_1\ldots d_{n-1}$ where $c_i=d_i=a_i+a_{i+1} \pmod 2$ for $i=0,1,\ldots,n-2$ and $c_{n-1}=\tilde{d}_{n-1}=a_{n-1}+a_1+1 \pmod 2$. Let g be the block map given by $g(x_1,x_2)=x_1+x_2$. Then g_∞ is a homomorphism of $\mathscr{M}(\theta)$ to $\mathscr{M}(\bar{\theta})$ such that every g_∞-fibre consists of a pair $\{\omega,\tilde{\omega}\}$. Hence there is an endomorphism ψ of $\mathscr{M}(\bar{\theta})$ such that the following diagram commutes.

$$\begin{array}{ccc} M(\theta) & \xrightarrow{\ \varphi\ } & M(\theta) \\ {\scriptstyle g_\infty}\downarrow & & \downarrow{\scriptstyle g_\infty} \\ M(\bar{\theta}) & \xrightarrow[\ \psi\ ]{} & M(\bar{\theta}) \end{array}$$

Let $\omega\in M(\theta)$ and let $\zeta=\varphi(\omega)=\varphi(\tilde{\omega})$. Then $g_\infty^{-1}(g_\infty(\zeta))=\{\zeta,\tilde{\zeta}\}$. If $\omega'\in\varphi^{-1}(\tilde{\zeta})$, then $\psi g_\infty(\omega)=\psi g_\infty(\omega')$ but $g_\infty(\omega)\neq g_\infty(\omega')$. However this is impossible because every endomorphism of $\mathscr{M}(\bar{\theta})$ is one-to-one.

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