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# On the Lower Limits of Maxima and Minima of Wiener Process and Partial Sums

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Let $M^+(t)$ and $-M^-(t)$ be the maximum and minimum of a Wiener process on the interval $(0, t)$. This paper gives an integral test for $P(M^+(t) < a(t)\sqrt{t}$, $M^-(t) < b(t)\sqrt{t}$ i.o.$)=0$ or 1. The case of i.i.d. random variables is also treated here. If $a(t)=b(t)$, then our result gives Chung's law of the iterated logarithm [5], while $b(t)=\infty$ corresponds to Hirsch's theorem [9]. Finally, a converse to Chung's $LIL$ is given.

## 1. Introduction

Let $X$, $X_1$, $X_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) random variables having finite second moment. Assume that $EX=0$, $EX^2=1$ and put $S_n = X_1 + \cdots + X_n$, $n=1,2,\ldots$, $M_n^+ = \max_{1 \leq i \leq n} S_i$, $M_n^- = -\min_{1 \leq i \leq n} S_i$, $M_n = \max(M_n^+, M_n^-)$.

Consider also a standard Wiener process $W(t)$ and introduce the notations $M^+(t) = \max_{0 \leq u \leq t} W(u)$, $M^-(t) = -\min_{0 \leq u \leq t} W(u)$, $M(t) = \max(M^+(t), M^-(t))$.

The usual law of the iterated logarithm gives precise upper bounds for the growth rate of all the variables $M_n^+$, $M_n^-$, $M_n$; $M^+(t)$, $M^-(t)$, $M(t)$. The corresponding precise lower bounds however are not the same for the above variables. The first result in this respect is due to Chung [5]. His result, specialised for i.i.d. variables says that under $E|X|^3 < \infty$.

$$P(M_n < a_n \sqrt{n} \text{ i.o.}) = \begin{matrix} 1 \\ 0 \end{matrix} \Leftrightarrow \sum_n \frac{1}{n a_n^2} e^{-\frac{\pi^2}{8 a_n^2}} \begin{matrix} = \infty \\ < \infty \end{matrix}. \tag{1.1}$$

It follows from (1.1) that

$$P(\liminf_{n \to \infty} M_n (\log\log n / n)^{\frac{1}{2}} = \pi 8^{-\frac{1}{2}}) = 1. \tag{1.2}$$

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On the other hand, Hirsch [9] investigated the lower limit of $M_n^+$. For the particular case of i.i.d. variables his result states that under $E|X|^3 < \infty$

$$P(M_n^+ < a_n \sqrt{n} \text{ i.o.}) = \begin{matrix} 1 \\ 0 \end{matrix} \Leftrightarrow \sum_n \frac{a_n}{n} \begin{matrix} = \infty \\ < \infty \end{matrix}. \tag{1.3}$$

Obviously the same holds true for $M_n^-$. Comparing (1.2) and (1.3) it is apparent that there is a difference between the lower limit of $M_n$ and $M_n^+$, especially $M_n^+$ can be much smaller than $M_n$ and furthermore there is no liminf sequence for $M_n^+$ (i.e. for any non-random sequence $\alpha_n$, $\liminf_{n\to\infty} \alpha_n M_n^+$ is either 0 or $\infty$).

Similar results hold also for $M(t)$ (Robbins and Siegmund [20], Jain and Taylor [12]) and $M^+(t)$ (or $M^-(t)$).

The object of this paper is to give a deeper insight into the above phenomenon by investigating the joint behavior of $(M_n^+, M_n^-)$ and $(M^+(t), M^-(t))$, resp., i.e. we consider events $A_n$ and $A(t)$ of the type

$$A_n = \{M_n^+ < a_n \sqrt{n}, M_n^- < b_n \sqrt{n}\}, \tag{1.4}$$

$$A(t) = \{M^+(t) < a(t)\sqrt{t}, M^-(t) < b(t)\sqrt{t}\} \tag{1.5}$$

and decide whether $P(A_n, n \to \infty \text{ i.o.})$ and $P(A(t), t \to \infty \text{ i.o.})$ is 0 or 1.

In Section 2 we prove our results in the case of Wiener process and by invariance it follows that the same holds for $(M_n^+, M_n^-)$, provided $E|X|^{2+\delta} < \infty$ for some $\delta > 0$.

In case when nothing more than the existence of the second moment is assumed, Jain and Pruitt [11] have shown that (1.2) (called sometimes the other or the converted law of the iterated logarithm) still holds true. In Section 3 we extend the results of Section 2 with certain modifications. In particular, we show that (1.3) is still valid in this case too. Finally, a converse of Chung's $LIL$ will be proved:

$$\liminf_{n\to\infty} M_n (\log\log n)^{\frac{1}{2}} < \infty \qquad \text{a.s.}$$

implies that $EX^2 < \infty$.

Throughout the paper $K, K_1, K_2, \ldots$ will denote (not necessarily the same) constants, whose values are unimportant in our investigations.

## 2. The Case of Wiener Process

In this section we consider events defined by (1.5) with $a(t)$, $b(t)$ satisfying certain regularity conditions. We may assume for symmetrical reason that $a(t) \leq b(t)$ and in view of (1.2) and (1.3) that for some $0 < \delta < 1$.

$$(\log t)^{-1-\delta} \leq a(t) \leq (1+\delta)\,\pi(8 \log\log t)^{-\frac{1}{2}} \tag{2.1}$$

and

$$(1-\delta)\pi(8\log\log t)^{-\frac{1}{2}} \leqq b(t). \tag{2.2}$$

We prove the following

**Theorem 2.1.** *Let $a(t)>0$, $b(t)>0$ be continuous nonincreasing functions such that $a(t)\sqrt{t}$ and $b(t)\sqrt{t}$ are increasing. Set $c(t)=a(t)+b(t)$ and let $A(t)$ be defined by* (1.5).

(i) *Put $b(t)=\infty$ in* (1.5). *Then*

$$P(A(t),\ t\to\infty \text{ i.o.}) = \begin{matrix}1\\0\end{matrix} \Leftrightarrow \int^{\infty} \frac{a(t)}{t}\,dt \begin{matrix}=\infty\\<\infty\end{matrix}. \tag{2.3}$$

(ii) *Let $b(t)<\infty$. Then*

$$P(A(t),\ t\to\infty \text{ i.o.}) = \begin{matrix}1\\0\end{matrix} \Leftrightarrow \int^{\infty} \frac{a(t)}{t\,c^3(t)}\,e^{-\frac{\pi^2}{2c^2(t)}}\,dt \begin{matrix}=\infty\\<\infty\end{matrix}. \tag{2.4}$$

Part (i) of this theorem is Hirsch's result for Wiener process, of which a simple proof will be given below. In part (ii) the particular case $a(t)=b(t)=c(t)/2$ gives Chung's theorem for Wiener process. This part allows also $b(t)=b$ (constant) $0<b<\infty$ and in this case formally we get the same integral test as in part (i).

For the proof of Theorem 2.1 we need the joint distribution of $M^+(t)$ and $M^-(t)$. Let $a>0$, $b>0$, $c=a+b$ and define

$$G(a,b)=P(M^+(t)<a\sqrt{t},\ M^-(t)<b\sqrt{t}). \tag{2.5}$$

$G(a,b)$ is given e.g. in Feller [7].

**Lemma 2.1.**

$$G(a,b)=\sum_{j=-\infty}^{\infty}(-1)^j[\phi(jc+a)-\phi(jc-b)] \tag{2.6}$$

$$=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}e^{-\frac{(2j+1)^2\pi^2}{2c^2}}\sin\frac{(2j+1)\pi a}{c}, \tag{2.7}$$

*where $\phi(x)$ denotes the standard normal distribution function.*

For $b=\infty$, i.e. for the (marginal) distribution of $M^+(t)$, (2.6) gives the well known expression

$$G(a,\infty)=2\phi(a)-1. \tag{2.8}$$

*Proof of the Convergent Part of* (i). Since both $M^+(t)$ and $a(t)\sqrt{t}$ are nondecreasing, by virtue of the Borel-Cantelli lemma it suffices to show that for suitably chosen $t_k$ with the property $t_k<t_{k+1}$, $\lim_{k\to\infty} t_k=\infty$

$$\sum_k P(M^+(t_k)<a(t_{k+1})\sqrt{t_{k+1}})<\infty. \tag{2.9}$$

Using (2.8) and the inequality $2\phi(a)-1\leqq a\sqrt{\frac{2}{\pi}}$, we obtain for $t_k=2^k$ that

$$P(M^+(t_k)<a(t_{k+1})\sqrt{t_{k+1}})$$
$$=2\phi(\sqrt{2}\,a(t_{k+1}))-1\leqq a(t_{k+1})\frac{2}{\sqrt{\pi}}. \tag{2.10}$$

But $\int^{\infty} a(t)/t\,dt<\infty$ implies the convergence of $\sum_k a(t_{k+1})$, proving that $P(A(t), t\to\infty \text{ i.o.})=0$.

*Proof of the Convergent Part of* (ii). Now assume that the integral in (2.4) converges. Define the sequence $t_k$ by $t_1=1$,

$$t_{k+1}=(1+c^2(t_k))\,t_k, \qquad k=1,2,\ldots \tag{2.11}$$

For brevity put $a_k=a(t_k)$, $b_k=b(t_k)$, $c_k=c(t_k)$ and define the event $A_k^*$ by

$$A_k^*=\{M^+(t_k)<a_{k+1}\sqrt{t_{k+1}},\, M^-(t_k)<b_{k+1}\sqrt{t_{k+1}}\}. \tag{2.12}$$

We show that $\sum_k P(A_k^*)<\infty$.

By (2.5) we have

$$P(A_k^*)=G\left(a_{k+1}\sqrt{\frac{t_{k+1}}{t_k}},\, b_{k+1}\sqrt{\frac{t_{k+1}}{t_k}}\right). \tag{2.13}$$

Applying (2.7) and the inequality $|\sin u|\leqq u\,(u>0)$, we obtain

$$G(a,b)\leqq\frac{4a}{c}\sum_{j=0}^{\infty} e^{-\frac{(2j+1)^2\pi^2}{2c^2}}. \tag{2.14}$$

Since the right hand side of (2.14) is a lacunary geometric series we get further that for $c\leqq c_0$,

$$G(a,b)\leqq\frac{4a}{c}e^{-\frac{\pi^2}{2c^2}}(1-e^{-\frac{\pi^2}{2c^2}})^{-1}\leqq K\frac{a}{c}e^{-\frac{\pi^2}{2c^2}}. \tag{2.15}$$

Hence by (2.11), (2.13) and (2.15), due to the boundedness of $c(t)$,

$$P(A_k^*)\leqq K\frac{a_{k+1}}{c_{k+1}}\exp\left(-\frac{\pi^2 t_k}{2c_{k+1}^2\,t_{k+1}}\right)$$
$$=K\frac{a_{k+1}}{c_{k+1}}\exp\left(-\frac{\pi^2}{2c_{k+1}^2(1+c_k^2)}\right)$$
$$\leqq K\frac{a_{k+1}}{c_{k+1}}\exp\left(-\frac{\pi^2}{2c_{k+1}^2}+\frac{\pi^2 c_k^2}{2c_{k+1}^2}\right)$$
$$\leqq K_1\frac{a_{k+1}}{c_{k+1}}e^{-\frac{\pi^2}{2c_{k+1}^2}}. \tag{2.16}$$

The last step follows from the fact that $tc^2(t)$ is increasing and $c^2(t)$ is bounded $(c_k^2/c_{k+1}^2 \leqq t_{k+1}/t_k = 1 + c_k^2)$.

But from our assumptions it follows that

$$\int_1^\infty \frac{a(t)}{t c^3(t)} e^{-\frac{\pi^2}{2c^2(t)}} dt = \sum_k \int_{t_k}^{t_{k+1}} \frac{a(t)}{t c^3(t)} e^{-\frac{\pi^2}{2c^2(t)}} dt$$
$$\geqq \sum_k \frac{a_{k+1}}{c_k^3} e^{-\frac{\pi^2}{2c_{k+1}^2}} \int_{t_k}^{t_{k+1}} \frac{dt}{t}$$
$$= \sum_k \frac{a_{k+1}}{c_{k+1}} e^{-\frac{\pi^2}{2c_{k+1}^2}} \frac{c_{k+1}}{c_k} \frac{\log(1+c_k^2)}{c_k^2}$$
$$\geqq K_2 \sum_k \frac{a_{k+1}}{c_{k+1}} e^{-\frac{\pi^2}{2c_{k+1}^2}}, \tag{2.17}$$

proving the convergence of the last sum, which in turn implies that $\sum_k P(A_k^*) < \infty$. This completes the proof of the convergent part of Theorem 2.1.

For the proof of the divergent part of Theorem 2.1, we need the joint distribution of $(M^+(t_1), M^+(t_2))$ and that of $(M^+(t_1), M^-(t_1), M^+(t_2), M^-(t_2))$. The required formulae are given by

**Lemma 2.2.** *Let* $0 < a_1\sqrt{t_1} < a_2\sqrt{t_2}$, $0 < b_1\sqrt{t_1} < b_2\sqrt{t_2}$, $c_1 = a_1 + b_1$, $c_2 = a_2 + b_2$. *Then*

$$P(M^+(t_1) < a_1\sqrt{t_1},\, M^+(t_2) < a_2\sqrt{t_2})$$
$$= \iint_T \frac{1}{\sqrt{t_1(t_2-t_1)}} \varphi\left(\frac{x}{\sqrt{t_1}}\right) \varphi\left(\frac{y}{\sqrt{t_2-t_1}}\right) dx\, dy, \tag{2.18}$$

*where* $\varphi(x)$ *denotes the standard normal density,* $T = T_1 \cup T_2 \cup T_3$ *and the domains* $T_i$ *are given by*

$$T_1 = \{-a_1\sqrt{t_1} < x < a_1\sqrt{t_1},\; x - a_2\sqrt{t_2} < y < a_2\sqrt{t_2} - x\},$$
$$T_2 = \{x < -a_1\sqrt{t_1},\; a_2\sqrt{t_2} - 2a_1\sqrt{t_1} < y + x < a_2\sqrt{t_2}\},$$
$$T_3 = \{x < -a_1\sqrt{t_1},\; -a_2\sqrt{t_2} < y - x < -a_2\sqrt{t_2} + 2a_1\sqrt{t_1}\}. \tag{2.19}$$

*Moreover*

$$P(M^+(t_1) < a_1\sqrt{t_1},\, M^-(t_1) < b_1\sqrt{t_1},\, M^+(t_2) < a_2\sqrt{t_2},\, M^-(t_2) < b_2\sqrt{t_2})$$
$$= \frac{4}{\pi} \sum_{j=0}^\infty \frac{1}{2j+1} e^{-\frac{(2j+1)^2\pi^2}{2c_2^2}\left(1-\frac{t_1}{t_2}\right)}$$
$$\times \frac{2}{c_1} \sum_{r=1}^\infty e^{-\frac{r^2\pi^2}{2c_1^2}} \sin\frac{r\pi a_1}{c_1}$$
$$\times \int_{-b_1\sqrt{t_1}}^{a_1\sqrt{t_1}} \sin\left(\frac{r\pi(a_1\sqrt{t_1} - y)}{c_1\sqrt{t_1}}\right) \sin\left(\frac{(2j+1)\pi(a_2\sqrt{t_2} - y)}{c_2\sqrt{t_2}}\right) dy. \tag{2.20}$$

The proof of Lemma 2.2 can be given by considering $W(t)$ $(0 \leqq t \leqq t_2)$ under the condition $W(t_1)=y$, under which $(W(t), 0 \leqq t < t_1)$ and $(W(t), t_1 \leqq t \leqq t_2)$ are independent. Applying the well known formulae for the joint distributions of $(M^+(t_1), M^-(t_1), W(t_1))$ and of $(M^+(t_2-t_1), M^-(t_2-t_1))$ (see Feller [7]) further straightforward calculations lead to (2.18) and (2.20). Details are omitted.

From Lemma 2.2 we obtain the following estimations:

$$P(M^+(t_1) < a_1\sqrt{t_1},\ M^+(t_2) < a_2\sqrt{t_2}) \leqq \frac{2a_1 a_2\sqrt{t_2}}{\pi\sqrt{t_2-t_1}} + 2a_1\sqrt{\frac{2}{\pi}}\sqrt{\frac{t_1}{t_2}} \tag{2.21}$$

and

$$\begin{aligned} &P(M^+(t_1) < a_1\sqrt{t_1},\ M^-(t_1) < b_1\sqrt{t_1},\ M^+(t_2) < a_2\sqrt{t_2},\ M^-(t_2) < b_2\sqrt{t_2}) \\ &\leqq \frac{16}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{2j+1} e^{-\frac{(2j+1)^2\pi^2}{2c_2^2}\left(1-\frac{t_1}{t_2}\right)} \\ &\quad\times\left(\sin\frac{(2j+1)\pi a_2}{c_2} + \frac{(2j+1)\pi c_1\sqrt{t_1}}{c_2\sqrt{t_2}}\right) \\ &\quad\times\sum_{r=0}^{\infty}\frac{1}{2r+1} e^{-\frac{(2r+1)^2\pi^2}{2c_1^2}}\sin\frac{(2r+1)\pi a_1}{c_1}. \end{aligned} \tag{2.22}$$

We prove first (2.21). Since the area of $T_1$ given by (2.19) is $4a_1 a_2\sqrt{t_1 t_2}$, the double integral in (2.18) over $T_1$ can be estimated by $4a_1 a_2\sqrt{t_1 t_2}\,\varphi^2(0)\, t_1^{-\frac{1}{2}}(t_2-t_1)^{-\frac{1}{2}}$, which is the first term on the right hand side of (2.21). The double integral over $T_2$ is equal to $P(U < -a_1\sqrt{t_1},\ a_2\sqrt{t_2}-2a_1\sqrt{t_1} < U+V < a_2\sqrt{t_2})$, where $U$ and $V$ are independent normal variables with mean 0 and variances $t_1$ and $t_2-t_1$, resp. Hence $U+V$ is $N(0,t_2)$ and the above probability can be estimated by

$$\begin{aligned} &P(a_2\sqrt{t_2}-2a_1\sqrt{t_2} < U+V < a_2\sqrt{t_2}) \\ &= \phi(a_2) - \phi\left(a_2 - 2a_1\sqrt{\frac{t_1}{t_2}}\right) \leqq \frac{2a_1}{\sqrt{2\pi}}\sqrt{\frac{t_1}{t_2}}. \end{aligned} \tag{2.23}$$

The same holds for the integral on $T_3$, thus (2.21) is proved.

The inequality (2.22) is a simple consequence of (2.20) and the following estimation:

$$\begin{aligned} &\sin\left(\frac{(2j+1)\pi(a_2\sqrt{t_2}-y)}{c_2\sqrt{t_2}}\right) \leqq \sin\frac{(2j+1)\pi a_2}{c_2} \\ &\quad+\frac{(2j+1)\pi|y|}{c_2\sqrt{t_2}} \leqq \sin\frac{(2j+1)\pi a_2}{c_2} + \frac{(2j+1)\pi c_1\sqrt{t_1}}{c_2\sqrt{t_2}}. \end{aligned} \tag{2.24}$$

*Proof of the Divergent Part of* (i). Now let $\int_1^\infty a(t)/t\,dt=\infty$ and without loss of generality we may assume that $\int_1^\infty a^2(t)/t\,dt<\infty$. If follows that $\sum_k a(2^k)=\infty$ and $\sum_k a^2(2^k)<\infty$. Define the event $B_k$ by

$$B_k=\{M^+(2^k)<a(2^k)\sqrt{2^k}\}. \tag{2.25}$$

Since

$$P(B_k)=2\phi(a(2^k))-1=\sqrt{\frac{2}{\pi}}a(2^k)+O(a^2(2^k)) \tag{2.26}$$

therefore $\sum_k P(B_k)=\infty$.

To show that $P(B_k \text{ i.o.})=1$, we apply the Erdös-Rényi version of Borel-Cantelli lemma (see Rényi [19]):

**Lemma 2.3.** *If* $\sum_k P(B_k)=\infty$ *and*

$$\liminf_{n\to\infty}\frac{\sum_{k=1}^{n}\sum_{l=1}^{n}P(B_kB_l)}{\left(\sum_{k=1}^{n}P(B_k)\right)^2}\leqq 1 \tag{2.27}$$

*then* $P(B_k \text{ i.o.})=1$.

From (2.21) and (2.26) we obtain for $k<l$:

$$\begin{aligned}P(B_kB_l)&\leqq\frac{2}{\pi}a(2^k)\,a(2^l)(1+2^{-(l-k)})+2a(2^k)\sqrt{\frac{2}{\pi}}2^{-\frac{1}{2}(l-k)}\\&\leqq P(B_k)\,P(B_l)+K_1P(B_k)\,2^{-\frac{1}{2}(l-k)}\\&\quad+K_2\,a^2(2^k),\end{aligned} \tag{2.28}$$

from which (2.27) is easy to verify. This proves the divergent part of (i).

*Proof of the Divergent Part of* (ii). Now assume that

$$\int_1^\infty\frac{a(t)}{t\,c^3(t)}e^{-\frac{\pi^2}{2c^2(t)}}dt=\infty. \tag{2.29}$$

Let $t_1=1$,

$$t_{k+1}=(1+c^2(t_k))\,t_k,\qquad k=1,2,\ldots. \tag{2.30}$$

Put $a_k=a(t_k)$, $b_k=b(t_k)$, $c_k=c(t_k)$ and define the event $B_k$ by

$$B_k=\{M^+(t_k)<a_k\sqrt{t_k},M^-(t_k)<b_k\sqrt{t_k}\}. \tag{2.31}$$

**Lemma 2.4.** $\sum_k P(B_k)=\infty$.

*Proof.* By (2.7)

$$P(B_k)=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}e^{-\frac{(2j+1)^2\pi^2}{2c_k^2}}\sin\frac{(2j+1)\pi a_k}{c_k}. \tag{2.32}$$

Since the summands are all nonnegative for $1\leqq j\leqq\frac{1}{2}\left(\frac{c_k}{a_k}-1\right)$, we estimate $P(B_k)$ by omitting these terms from the above sum:

$$P(B_k)\geqq\frac{4}{\pi}e^{-\frac{\pi^2}{2c_k^2}}\sin\frac{\pi a_k}{c_k}$$

$$+\frac{4}{\pi}\sum_{j>\frac{1}{2}\left(\frac{c_k}{a_k}-1\right)}\frac{1}{2j+1}e^{-\frac{(2j+1)^2\pi^2}{2c_k^2}}\sin\frac{(2j+1)\pi a_k}{c_k}. \tag{2.33}$$

Using the inequality $\sin x\geqq x/2$ for $0<x\leqq\pi/2$ and applying the same argument for the above sum as used to derive the inequalities (2.14) and (2.15), we get further

$$P(B_k)\geqq 2\frac{a_k}{c_k}e^{-\frac{\pi^2}{2c_k^2}}-K\frac{a_k}{c_k}e^{-\frac{\pi^2}{2a_k^2}}. \tag{2.34}$$

Since by (2.1), $\sum_k e^{-\frac{\pi^2}{2a_k^2}}<\infty$, Lemma 2.4 will be proved by showing that $\sum_k\frac{a_k}{c_k}e^{-\frac{\pi^2}{2c_k^2}}=\infty$. This can be seen as follows:

$$\int_1^\infty\frac{a(t)}{t\,c^3(t)}e^{-\frac{\pi^2}{2c^2(t)}}dt=\sum_k\int_{t_k}^{t_{k+1}}\frac{a(t)}{t\,c^3(t)}e^{-\frac{\pi^2}{2c^2(t)}}dt$$

$$\leqq\sum_k\frac{a_k}{c_{k+1}^3}e^{-\frac{\pi^2}{2c_k^2}}\log(1+c_k^2)$$

$$\leqq K\sum_k\frac{a_k}{c_k}e^{-\frac{\pi^2}{2c_k^2}}. \tag{2.35}$$

In the last step we used that $c_k/c_{k+1}$ is bounded (see (2.16)). The proof of Lemma 2.4 is complete.

Next we are going to apply Lemma 2.3. In order to verify (2.27), write

$$\sum_{k=1}^{n}\sum_{l=1}^{n}P(B_kB_l)=\sum_{k=1}^{n}P(B_k)+2\sum\sum_{1\leqq k<l\leqq n}P(B_kB_l) \tag{2.36}$$

and split $\sum_{k<l\leqq n}$ into three parts. For given $k\geqq 1$ define

$$L_1=\{l\colon 1\leqq l-k<c_l^{-2}\}$$
$$L_2=\{l\colon c_l^{-2}\leqq l-k<k^{\alpha/2}\} \tag{2.37}$$
$$L_3=\{l\colon k^{\alpha/2}\leqq l-k\}$$

where $\alpha$ will be defined in Lemma 2.6, below.

We shall repeatedly use the inequality

$$\frac{t_k}{t_l}\leqq(1+c_l^2)^{-(l-k)} \quad (k<l) \tag{2.38}$$

easily obtained from (2.30). Without loss of generality we assume that

$$\frac{a_l}{c_l}e^{-\frac{\pi^2}{2c_l^2}}\leqq\frac{1}{\sqrt{l}}. \tag{2.39}$$

**Lemma 2.5.** *If* $1\leqq l-k<c_l^{-2}$, *then*

$$e^{-\frac{\pi^2}{2c_l^2}\left(1-\frac{t_k}{t_l}\right)}\leqq e^{-\frac{\pi^2}{4}(l-k)}. \tag{2.40}$$

*Proof.* Using the inequality (2.38) and $1-(1+u)^{-s}\geqq us/2$ for $0\leqq u<1/s$ with $s=l-k$, $u=c_l^{-2}$, we have

$$e^{-\frac{\pi^2}{2c_k^2}\left(1-\frac{t_k}{t_l}\right)}\leqq e^{-\frac{\pi^2}{2c_l^2}(1-(1+c_l^2)^{-(l-k)})}\leqq e^{-\frac{\pi^2}{4}(l-k)}.$$

**Lemma 2.6.** *If* $c_l^{-2}\leqq l-k$, *then there exists a number* $\alpha(0<\alpha<1)$ *such that*

$$e^{-\frac{\pi^2}{2c_l^2}\left(1-\frac{t_k}{t_l}\right)}\leqq e^{-\frac{\pi^2}{2c_l^2}\alpha}. \tag{2.41}$$

*Proof.* $1-\frac{t_k}{t_l}\geqq 1-(1+c_l^2)^{-(l-k)}\geqq 1-(1+c_l^2)^{-\frac{1}{c_l^2}}\geqq\alpha,$ where

$$1-\alpha=\max_l\,(1+c_l^2)^{-\frac{1}{c_l^2}}. \tag{2.42}$$

**Lemma 2.7.** *Given any* $\varepsilon>0$, *there exists* $k_0$ *such that for* $k\geqq k_0$ *and* $l-k\geqq k^{\alpha/2}$, *the following inequality holds:*

$$e^{\frac{\pi^2}{2c_l^2}}\leqq 1+\varepsilon. \tag{2.43}$$

*Proof.* For given $k$, define $l_0$ as the largest $l$ for which $k<l<k+(\log l)^2$, and $l_1$ as the smallest $l$ for which $l\geqq k+k^{\alpha/2}$. Then for $k$ large enough, $l_1>l_0$, because otherwise $k^{\alpha/2}\leqq l_1-k<(\log l_1)^2<\log(k+k^{\alpha/2}+1)$, which is impossible for large $k$. Hence $l\geqq k+k^{\alpha/2}$ will imply that $l-k\geqq(\log l)^2$.

But from (2.2) and (2.30) it follows that $c_l^{-2}=0(\log l)$ and for $l$ large enough

$$(\log l)^2\geqq\frac{\log\frac{1}{c_l^2}+\log\frac{\pi^2}{2}-\log\log(1+\varepsilon)}{\log(1+c_l^2)}. \tag{2.44}$$

Hence there exists $k_0$ such that $k \geqq k_0$ and $l-k \geqq k^{\alpha/2}$ together imply that $l-k$ is greater than the right hand side of (2.44), which by using (2.38) yields (2.43).

Upon replacing $t_1$ by $t_k$ and $t_2$ by $t_l$ in the inequality (2.22), we have

$$P(B_k B_l) \leqq P(B_k) \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{1}{2j+1} e^{-\frac{(2j+1)^2 \pi^2}{2c_l^2}\left(1-\frac{t_k}{t_l}\right)} \cdot \left(\sin \frac{(2j+1)\pi a_l}{c_l} + \frac{(2j+1) c_k \sqrt{t_k}}{c_l \sqrt{t_l}}\right). \tag{2.45}$$

If $l \in L_1$, then from (2.45) and by Lemma 2.5,

$$P(B_k B_l) \leqq K P(B_k) e^{-\frac{\pi^2}{2c_l^2}\left(1-\frac{t_k}{t_l}\right)} \leqq K P(B_k) e^{-\frac{\pi^2}{4}(l-k)}, \tag{2.46}$$

hence

$$\sum_{l \in L_1} P(B_k B_l) \leqq K_1 P(B_k). \tag{2.47}$$

If $l \in L_2$, then from (2.45) and by Lemma 2.6,

$$P(B_k B_l) \leqq 4 P(B_k) e^{-\frac{\pi^2}{2c_l^2}\alpha} \left(\frac{a_l}{c_l} + \frac{c_k \sqrt{t_k}}{c_l \sqrt{t_l}}\right). \tag{2.48}$$

By using (2.39) and by the definition of $L_2$ (2.37)

$$\sum_{l \in L_2} e^{-\frac{\pi^2}{2c_l^2}\alpha} \frac{a_l}{c_l} \leqq \sum_{l \in L_2} \left(\frac{a_l}{c_l} e^{-\frac{\pi^2}{2c_l^2}}\right)^{\alpha} \leqq \sum_{l \in L_2} l^{-\alpha/2} \leqq \sum_{l \in L_2} k^{-\alpha/2} \leqq 1. \tag{2.49}$$

therefore

$$\sum_{l \in L_2} P(B_k B_l) \leqq 4 P(B_k) \left(1 + \sum_{l \in L_2} \frac{c_k \sqrt{t_k}}{c_l \sqrt{t_l}}\right). \tag{2.50}$$

If $l \in L_3$, then from (2.45) and by Lemmas 2.6 and 2.7, for $k \geqq k_0$,

$$P(B_k B_l) \leqq (1+\varepsilon) P(B_k) P(B_l) + 4 P(B_k) \frac{c_k \sqrt{t_k}}{c_l \sqrt{t_l}}, \tag{2.51}$$

$$\sum_{l \in L_3} P(B_k B_l) \leqq (1+\varepsilon) P(B_k) P(B_l) + 4 P(B_k) \sum_{l \in L_3} \frac{c_k \sqrt{t_k}}{c_l \sqrt{t_l}}. \tag{2.52}$$

Since the sequence $t_k$ defined by (2.30) is nearly exponential (grows at least as fast as $\exp(\beta k/\log k)$) it is easy to see that

$$\sum_{l=k+1}^{\infty} \frac{c_k \sqrt{t_k}}{c_l \sqrt{t_l}} < K_2, \tag{2.53}$$

where $K_2$ does not depend on $k$. By (2.36), (2.47), (2.50), (2.52) and (2.53), we obtain

$$\sum_{k=1}^{n} \sum_{l=1}^{n} P(B_k B_l)$$
$$\leqq K_3 \sum_{k=1}^{n} P(B_k) + (1+\varepsilon) \sum_{k=1}^{n} \sum_{l=1}^{n} P(B_k) P(B_l), \tag{2.54}$$

and since $\varepsilon$ is arbitrary, (2.27) is verified. This completes the proof of Theorem 2.1.

Theorem 2.1 yields immediately the following

**Corollary 2.1.** *Assume that for a sequence $S_n$, $n \geqq 1$ of random variables a standard Wiener process $W(t)$ can be given on the same probability space such that for some $\eta > 0$,*

$$P(\sup_{k \leqq n} |S_k - W(k)| = O(n^{\frac{1}{2}-\eta})) = 1. \tag{2.55}$$

*Let $a(t)$ and $b(t)$ satisfy the conditions of Theorem 2.1. Then*

$$\text{(i)} \quad P(M_n^+ < a(n)\sqrt{n} \text{ i.o.}) = {1 \atop 0} \Leftrightarrow \sum_n \frac{a(n)}{n} {= \infty \atop < \infty} \tag{2.56}$$

$$\text{(ii)} \quad P(M_n^+ < a(n)\sqrt{n}, M_n^- < b(n)\sqrt{n} \text{ i.o.})$$
$$= {1 \atop 0} \Leftrightarrow \sum_n \frac{a(n)}{n c^3(n)} e^{-\frac{\pi^2}{2c^2(n)}} {= \infty \atop < \infty}. \tag{2.57}$$

Example of $S_n$ satisfying (2.55) can be given by the partial sums of i.i.d. random variables with $E|X|^{2+\delta} < \infty$ for some $\delta > 0$ (Major [13]). For partial sums of certain dependent random variables satisfying (2.55) we refer to Berkes [2, 3] and Philipp and Stout [18].

## 3. The Case of i.i.d. Summands

In this section we are concerned with the maxima and minima of the sums $S_n$ of i.i.d. random variables $X_i$ with $EX_i = 0$, $EX_i^2 = 1$. In this case Strassen's invariance principle [22] says that with probability one

$$\sup_{1 \leqq k \leqq n} |S_k - W(k)| = o((n \log\log n)^{\frac{1}{2}}) \tag{3.1}$$

and Major [14] have shown that in general this result can not be improved. To handle the lower limits of $M_n^+$ and $M_n^-$ by invariance however a rate better than $o(n^{\frac{1}{4}})$ would be required (see Breiman [4]). Hence neither Chung's $LIL$ (1.2) nor Hirsch's theorem (1.3) is a consequence of the invariance. Jain and Pruitt [11] have shown that (1.2) is still valid when nothing more than the existence of the second moment is assumed. We shall prove the validity of (1.3) in this case too and give the convergent class and divergent class concerning the joint behavior of $M_n^+$ and $M_n^-$. In this case however we do not expect that the convergent and divergent classes can be separated so nicely as in Theorem 2.1. (For similar phenomenon in case of the usual law of the iterated logarithm we refer e.g. to Petrov [17].)

We consider events $A_n$ defined by (1.4).

**Theorem 3.1.** *Let $a_n > o$, $b_n > o$ be nonincreasing, $a_n\sqrt{n}$ and $b_n\sqrt{n}$ be increasing and put $c_n = a_n + b_n$.*

$$\text{(i)}\quad P(M_n^+ < a_n\sqrt{n}\,\text{i.o.}) = {1 \atop 0} \Leftrightarrow \sum_n \frac{a_n}{n} {= \infty \atop < \infty} \tag{3.2}$$

(ii) *For each $\varepsilon > 0$, small enough*

$$\sum_n \frac{a_n}{n c_n^3} e^{-\frac{\pi^2}{2c_n^2}(1+\varepsilon)} = \infty \Rightarrow P(A_n\,\text{i.o.}) = 1 \tag{3.3}$$

$$\sum_n \frac{a_n}{n c_n} e^{-\frac{\pi^2}{2c_n^2}(1-\varepsilon)} < \infty \Rightarrow P(A_n\,\text{i.o.}) = 0. \tag{3.4}$$

*Proof of the Divergent Part.* To prove the divergent part of both (i) and (ii) we follow Jain and Pruitt [11], i.e. use Skorohod embedding and then apply the result proved already for the Wiener process.

By Skorohod embedding, instead of $S_n$, we can consider $W(T_1 + \dots + T_n)$, where $T_i \geqq 0$ are i.i.d. random variables with $ET_i = 1$. By the law of large numbers, with probability 1, for given $\varepsilon > 0$, $T_1 + T_2 + \dots + T_n < (1+\varepsilon)n$ eventually. Hence

$$M_n^+ = \max_{1 \leqq k \leqq n} W(T_1 + \dots + T_k) \leqq \max_{t \leqq n(1+\varepsilon)} W(t) = M^+(n(1+\varepsilon)). \tag{3.5}$$

Similarly,

$$M_n^- \leqq M^-(n(1+\varepsilon)). \tag{3.6}$$

It is then obvious that $A_n^* \subseteq A_n$, where the event $A_n^*$ is defined by

$$A_n^* = \left\{ M_n^+(n(1+\varepsilon)) < \frac{a_n}{\sqrt{1+\varepsilon}}\sqrt{n(1+\varepsilon)},\, M_n^-(n(1+\varepsilon)) < \frac{b_n}{\sqrt{1+\varepsilon}}\sqrt{n(1+\varepsilon)} \right\} \tag{3.7}$$

and an appeal to Theorem 2.1 shows that the divergence of the series in (3.2) and (3.3), resp. implies that $P(A_n^*\ \text{i.o.}) = 1$, hence $P(A_n\ \text{i.o.}) = 1$, proving the divergent parts of both (i) and (ii).

*Proof of the Convergent Part.* Here we employ a method developed by Heyde [8] and used to give a simple proof of the Hartman-Wintner law of the iterated logarithm (see also Petrov [17]). This method relies on the Berry-Esseen bound for the speed of convergence in the central limit theorem. Therefore we need also a Berry-Esseen type bound for the deviation between the joint distribution of $(M_n^+\sqrt{n}, M_n^-\sqrt{n})$ and their limiting distribution. For the case of $M_n^+$, i.e. for one-sided maximum, Arak [1] has given a bound that could be used to prove the convergent part of (i). The two-sided case has been considered by Nagaev [15, 16], and Sahanenko [21], who by improving Nagaev's method, has given a bound satisfactory for our purpose. Since this involves also the one-sided case, we quote only Sahanenko's result as

**Lemma 3.1.** *Let $X_i$, $1\leqq i\leqq n$ be a sequence of i.i.d. random variables with $EX_i=0$, $EX_i^2=1$, $\gamma_3=E|X_i|^3<\infty$. Suppose that functions $g_1(t)$ and $g_2(t)$ are given that satisfy the following conditions:*

$$\begin{aligned}&\text{(i)}\quad g_2(t)<g_1(t)\quad \text{for } 0\leqq t\leqq 1\\ &\text{(ii)}\quad g_2(0)<0<g_1(0)\\ &\text{(iii)}\quad |g_i(t+h)-g_i(t)|<K_1 h\quad \text{for all } h>0;\ i=1,2.\end{aligned}\tag{3.8}$$

*Then*

$$\left|P\left(g_2\left(\frac{k}{n}\right)<\frac{S_k}{\sqrt{n}}<g_1\left(\frac{k}{n}\right),\ 1\leqq k\leqq n\right)\right.$$
$$\left.-P(g_2(t)<W(t)<g_1(t), 0\leqq t\leqq 1)\right|<K_2\frac{\gamma_3}{\sqrt{n}},\tag{3.9}$$

*where $S_k=X_1+\cdots+X_k$ $(k=1,\ldots,n)$; $K_1$ and $K_2$ are numerical constants.*

The next lemma is an analogue of Theorem 4 of Heyde [8].

**Lemma 3.2.** *Let $X$, $X_1,\ldots$ be a sequence of i.i.d. random variables with $EX=0$, $EX^2=1$ and denote by $F(x)$ the distribution function of $X$. Put $S_n=X_1+\cdots+X_n$,*

$$\sigma_n^2=\int_{|x|<\sqrt{n}} x^2\,dF(x)-\Big(\int_{|x|<\sqrt{n}} x\,dF(x)\Big)^2\tag{3.10}$$

*and*

$$G_n(a,b)=P(M_n^+<a\,\sigma_n\sqrt{n}, M_n^-<b\,\sigma_n\sqrt{n}).\tag{3.11}$$

*Let $G(a,b)$ be given by (2.5) and $n_0>0$, $\lambda>1$ be given constants and $n_k\sim n_0\lambda^k$. Then*

$$\sum_k \sup_{a\geqq 0, b\geqq 0}|G_{n_k}(a,b)-G(a,b)|<\infty.\tag{3.12}$$

*Proof.* Define the truncated variables $\tilde{X}_{kn}$ by

$$\tilde{X}_{kn}=\begin{cases}X_k & \text{if } |X_k|<\sqrt{n}\\ 0 & \text{if } |X_k|\geqq\sqrt{n}\end{cases}\tag{3.13}$$

and put $\tilde{S}_{in}=\tilde{X}_{1n}+\cdots+\tilde{X}_{in}$, $\tilde{M}_n^+=\max_{1\leqq i\leqq n}\tilde{S}_{in}$, $\tilde{M}_n^-=-\min_{1\leqq i\leqq n}\tilde{S}_{in}$.

Then by a standard argument for truncated variables (see e.g. Petrov [17]), for all $a>0$, $b>0$ we have

$$|P(M_n^+<a\sqrt{n}, M_n^-<b\sqrt{n}) \\ \quad -P(\tilde{M}_n^+<a\sqrt{n}, \tilde{M}_n^-<b\sqrt{n})| \leqq nP(|X|\geqq\sqrt{n}). \tag{3.14}$$

Let

$$\mu_n=E\tilde{X}_{kn}=\int_{|x|<\sqrt{n}} x\,dF(x) \tag{3.15}$$

Since

$$P(\tilde{M}_n^+<a\sqrt{n}, \tilde{M}_n^-<b\sqrt{n}) \\ =P\left(-\frac{b}{\sigma_n}-\frac{k\mu_n}{\sigma_n\sqrt{n}}<\frac{\tilde{S}_{kn}-k\mu_n}{\sigma_n\sqrt{n}}<\frac{a}{\sigma_n}-\frac{k\mu_n}{\sigma_n},\ 1\leqq k\leqq n\right), \tag{3.16}$$

we obtain, by using Lemma 3.1 and $\lim_{n\to\infty}\sigma_n=1$ that

$$|P(\tilde{M}_n^+<a\sqrt{n}, \tilde{M}_n^-<b\sqrt{n}) \\ \quad -P\left(\frac{-b-t\mu_n\sqrt{n}}{\sigma_n}<W(t)<\frac{a-t\mu_n\sqrt{n}}{\sigma_n},\ 0\leqq t\leqq 1\right)\Bigg| \\ \leqq\frac{K_2}{\sqrt{n}}E\left|\frac{\tilde{X}_{1n}-\mu_n}{\sigma_n}\right|^3\leqq\frac{4K_2}{\sigma_n^3\sqrt{n}}(E|\tilde{X}_{1n}|^3+|\mu_n|^3) \\ \leqq\frac{K_3}{\sqrt{n}}(E|\tilde{X}_{1n}|^3+|\mu_n|^3). \tag{3.17}$$

It is readily seen that

$$G\left(\frac{a}{\sigma_n}-|u_n|, \frac{b}{\sigma_n}-|u_n|\right) \\ \leqq P\left(-\frac{b}{\sigma_n}-t\,u_n<W(t)<\frac{a}{\sigma_n}-t\,u_n, 0\leqq t\leqq 1\right) \\ \leqq G\left(\frac{a}{\sigma_n}+|u_n|, \frac{b}{\sigma_n}+|u_n|\right), \tag{3.18}$$

where $u_n=\mu_n\sqrt{n}/\sigma_n$, and since $G(a,b)$ has bounded partial derivatives, we obtain further that for sufficiently large $n$,

$$\Bigg|P\left(-\frac{b}{\sigma_n}-t\,u_n<W(t)<\frac{a}{\sigma_n}-t\,u_n,\ 0\leqq t\leqq 1\right) \\ \quad -G\left(\frac{a}{\sigma_n}, \frac{b}{\sigma_n}\right)\Bigg|\leqq K_4|u_n|\leqq K_5|\mu_n|\sqrt{n}. \tag{3.19}$$

(3.14), (3.17), and (3.19) together yield the following inequality

$$\frac{1}{n}\left|P(M_n^+ < a\sqrt{n}, M_n^- < b\sqrt{n}) - G\left(\frac{a}{\sigma_n}, \frac{b}{\sigma_n}\right)\right|$$
$$\leqq P(|X| \geqq \sqrt{n}) + K_3 n^{-\frac{3}{2}}(E|\tilde{X}_{1n}|^3 + |\mu_n|^3) + K_5 |\mu_n| n^{-\frac{1}{2}}. \tag{3.20}$$

We can complete the proof of Lemma 3.2 in exactly the same way as Heyde has proved his Theorem 4 in [8]. (See also Petrov [17].)

Now we are ready to prove the convergent part of Theorem 3.1.

First assume that $\sum_n a_n/n < \infty$. Choose a subsequence $n_k = 2^k$. Then $\sum_k a_{n_{k+1}} < \infty$. We want to show that $\sum_k P(M_{n_k}^+ < a_{n_{k+1}} \sqrt{n_{k+1}}) < \infty$. By Lemma 3.2, this sum and $\sum_k G(a_{n_{k+1}} \sqrt{n_{k+1}}/\sigma_{n_k} \sqrt{n_k}, \infty)$ converge together. But

$$G\left(\frac{a_{n_{k+1}}\sqrt{n_{k+1}}}{\sigma_{n_k}\sqrt{n_k}}, \infty\right) = 2\phi\left(\sqrt{2}\frac{a_{n_{k+1}}}{\sigma_{n_k}}\right) - 1 = O(a_{n_{k+1}}), \tag{3.21}$$

proving the convergent part of (i).

To prove the convergent part of (ii), we show that for $n_k = [(1+\varepsilon)^k]$,

$$\sum_k P(M_{n_k}^+ < a_{n_{k+1}} \sqrt{n_{k+1}}, M_{n_k}^- < b_{n_{k+1}} \sqrt{n_{k+1}}) < \infty. \tag{3.22}$$

By Lemma 2.3, the sum in (3.22) and

$$G\left(\frac{a_{n_{k+1}}\sqrt{n_{k+1}}}{\sigma_{n_k}\sqrt{n_k}}, \frac{b_{n_{k+1}}\sqrt{n_{k+1}}}{\sigma_{n_k}\sqrt{n_k}}\right) \tag{3.23}$$

converge together. By the inequality (2.15),

$$G\left(\frac{a_{n_{k+1}}\sqrt{n_{k+1}}}{\sigma_{n_k}\sqrt{n_k}}, \frac{b_{n_{k+1}}\sqrt{n_{k+1}}}{\sigma_{n_k}\sqrt{n_k}}\right)$$
$$\leqq K\frac{a_{n_{k+1}}}{c_{n_{k+1}}} \exp\left(-\frac{\pi^2 \sigma_{n_k}^2 n_k}{2c_{n_{k+1}}^2 n_{k+1}}\right). \tag{3.24}$$

For $k$ large enough, one has $\sigma_{n_k}^2 \geqq 1-\varepsilon^2$ and since $n_k/n_{k+1} = 1/(1+\varepsilon)$, the right hand side of (3.24) can be estimated from above by

$$K\frac{a_{n_{k+1}}}{c_{n_{k+1}}} \exp\left(-\frac{\pi^2(1-\varepsilon)}{2c_{n_{k+1}}^2}\right). \tag{3.25}$$

$n_k$ is an exponential sequence, therefore (3.25) and the sum in (3.4) converge together. This completes the proof of Theorem 3.1.

Finally we remark that the following converse to Chung's law of the iterated logarithm holds true:

**Theorem 3.2.** *Assume that $X$, $X_1$, $X_2$,... is a sequence of i.i.d. random variables with $EX^2=\infty$. Then*

$$P(\lim_{n\to\infty} M_n(\log\log n/n)^{\frac{1}{2}}=\infty)=1. \tag{3.26}$$

A partial converse is implicit in Jain and Pruitt [10]. It is clear that (3.26) holds in cases treated in [10]. But using their method, together with an inequality of Esseen [6], it is not hard to see that (3.26) is true in general. A sketch of proof is given below.

We have to show that for all $\alpha>0$.

$$P(M_n<\alpha(n/\log\log n)^{\frac{1}{2}}\text{ i.o.})=0. \tag{3.27}$$

Jain and Pruitt [10] show that

$$P(M_n<\alpha(n/\log\log n)^{\frac{1}{2}})\leqq[P(|S_m|<2\alpha(n/\log\log n)^{\frac{1}{2}})]^N, \tag{3.28}$$

where $m=[n/\log\log n]$, $N=[n/m]$.

From Esseen [6] (see also Petrov [17]) it is easy to see that

$$P(|S_m|\leqq 2\alpha\sqrt{m})\leqq K\left(\int_{-2\alpha\sqrt{m}}^{2\alpha\sqrt{m}} x^2\,d\tilde{F}(x)\right)^{-\frac{1}{2}}, \tag{3.29}$$

where $\tilde{F}(x)$ is the distribution function of the symmetrized $X$. But $EX^2=\infty$ implies that the right hand side of (3.29) is arbitrary small for $m$ large enough, therefore

$$P(|S_m|<2\alpha\sqrt{m})<e^{-(1+\varepsilon)}, \quad n\geqq n_0 \tag{3.30}$$

consequently

$$P(M_n<\alpha(n/\log\log n)^{\frac{1}{2}})<e^{-(1+\varepsilon)N}, \quad n\geqq n_0 \tag{3.31}$$

and to complete the proof of (3.27) we may refer to Jain and Pruitt [10].

Hence $\liminf_{n\to\infty} M_n(\log\log n/n)^{\frac{1}{2}}<\infty$ a.s. implies $EX^2<\infty$ and, as easily seen, also $EX=0$.

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