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On the Convergence of the Kadanoff Transformation Towards Trivial Fixed Points*

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Summary. We consider the Kadanoff transformation T (depending on a positive parameter p) acting on probability measures μ on the space $\{+1, -1\}^{\mathbb{Z}^d}$. A measure μ is called a non-trivial fixed point of T, if it is extremal in the set of T-invariant measures but is not a product measure. We describe the set of trivial fixed points and show that non-trivial fixed points exist provided that $d \geqq 2$ and p large enough. A strong mixing condition on μ implies convergence of $T^n\mu$ towards a trivial fixed point. In particular this applies to the two-dimensional Ising model except at the critical point. What happens at the critical point still remains unknown.

1. Introduction

The ideas of the renormalization group, as described for example in the recent report [9], pose many mathematical problems. In order to clarify one of them, Griffiths and Pearce [4] consider simple cell-type transformations for Ising models like the Kadanoff transformation (which we will define precisely a little later). They find out peculiarities in the behavior of these transformations as acting on Hamiltonians. Hence it is somewhat obscure how it should be possible to make precise the approximations physicists use. In this paper, however, instead of working with Hamiltonians themselves, we consider the simpler questions of how these transformations act on probability measures.

We consider the configuration space $\Omega = \{+1, -1\}^{\mathbb{Z}^d}$ of an Ising spin system. The associated family of σ-algebras $\{\mathscr{F}_\Lambda; \Lambda \subset \mathbb{Z}^d\}$ is defined by

$$\mathscr{F}_\Lambda = \sigma\{\omega(x); x \in \Lambda\}.$$

$\mathscr{F}_\Lambda$ is the information from inside spin configurations of Λ. $\mathscr{F}$ is used for $\mathscr{F}_{\mathbb{Z}^d}$ for simplicity.

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For each $x\in\mathbb{Z}^d$, a subset ("cell") V_x of $\mathbb{Z}^d$ is defined by

$$V_x \equiv \{y\in\mathbb{Z}^d;\ 2x^i \leqq y^i < 2(x^i+1),\ i=1,2,\ldots,d\},$$

where $x=(x^1,x^2,\ldots,x^d)\in\mathbb{Z}^d$.

The (d-dimensional) Kadanoff transformation T is a mapping from the space $\mathcal{M}$ of probability measures on $(\Omega,\mathcal{F})$ into itself which satisfies the relation

$$T\mu(\omega(x)=+1,\ x\in\Lambda)=\int_\Omega \prod_{x\in\Lambda} P_x(+|\omega)\,\mu(d\omega) \tag{1}$$

for every finite subset $\Lambda\subset\mathbb{Z}^d$, where

$$\begin{aligned} P_x(+|\omega)&=[Z_x(\omega)]^{-1}\exp\{p\sum_{y\in V_x}\omega(y)\},\\ Z_x(\omega)&=\exp\{p\sum_{y\in V_x}\omega(y)\}+\exp\{-p\sum_{y\in V_x}\omega(y)\}, \end{aligned} \tag{2}$$

where $p>0$ is a given parameter.

We are going to consider the following problems:

1. What are the possible invariant probability measures (fixed points) for T, i.e. what is the structure of

$$\mathcal{I}=\{\mu\in\mathcal{M};\ T\mu=\mu\}?$$

2. For which μ does $\lim_{n\to\infty} T^n\mu$ exist, in particular what happens if we take μ to be a Gibbs state $\mu_{\beta,h}$ (with inverse temperature β and magnetic field h)? Is the critical point $(\beta_c,0)$ distinguished by a special limiting behavior of $T^n\mu_{\beta_c,0}$ different from that of $\{T^n\mu_{\beta,h}\}$, $(\beta,h)\neq(\beta_c,0)$?

We were not able to give a satisfactory answer to these questions, the main problems are left open. In order to describe our partial results we use the following notation:

Let $F=F(x_1,x_2,\ldots,x_s)$, $s=2^d$ be the $[0,1]$-valued function defined on $[0,1]^s$ by

$$F(x_1,x_2,\ldots,x_s)=\sum_{k=0}^{s}\alpha_k \sum_{1\leqq i_1<\ldots<i_k\leqq s}\ \prod_{v=1}^{k} x_{i_v} \prod_{\substack{j\neq i_1,\ldots,i_k\\ 1\leqq j\leqq s}}(1-x_j), \tag{3}$$

where

$$\begin{aligned}\alpha_k&=e^{2p(2k-s)}/[1+e^{2p(2k-s)}]\\ &=P_x(+|\omega)\quad \text{for } \omega \text{ such that } |\{y\in V_x\colon \omega(y)=+1\}|=k.\end{aligned}$$

Then it is obvious that

$$T\mu(\omega(x)=+1,\ x\in\Lambda)=\int\prod_{x\in\Lambda}F\left(\left\{\frac{1+\omega(y)}{2},\ y\in V_x\right\}\right)\mu(d\omega). \tag{4}$$

In Sect. 2 we prove

Theorem 1. (i) *Assume that* $s \geqq 3$ *(or* $d \geqq 2$*). Then there is a critical* $p_c > 0$, *depending on* s *(or* d*), such that* $\mathscr{I} = \{\nu_{1/2}\}$ *if* $0 < p \leqq p_c$, *and* $\mathscr{I} \supsetneqq \{\nu_{1/2}\}$ *if* $p > p_c$, *where* $\nu_{1/2}$ *is* $\frac{1}{2}$*-product measure.*

(ii) *Let* $p > p_c$ *and* $\mathscr{P}$ *be the set of all product measures on* $(\Omega, \mathscr{F})$. *Then we have*

$$\mathscr{P} \cap \mathscr{I} = \{\nu_h; h \in \mathscr{H}\},$$

where

$$\mathscr{H} = \{h: \mathbb{Z}^d \to [0,1] \text{ with } h(x) = F(\{h(y), y \in V_x\}) \text{ for any } x \in \mathbb{Z}^d\}$$

and

$$\nu_h(\omega(x) = +1, x \in \Lambda) = \prod_{x \in \Lambda} h(x)$$

for every finite $\Lambda \subset \mathbb{Z}^d$.

For the proof we use a recursion formula defined by means of F and study the fixed points of the function

$$f_s(x) \equiv F(\overbrace{x, x, \ldots, x}^{s}) \qquad (x \in [0,1]). \tag{5}$$

In Sect. 3 we investigate the structure of

$$\mathscr{I}_{\text{ex}} \equiv \{\mu \in \mathscr{I}; \mu \text{ is extremal in } \mathscr{I}\}$$

and prove

Theorem 2. *If* $p > p_c$, *then*

$$\mathscr{P} \cap \mathscr{I} \subsetneqq \mathscr{I}_{\text{ex}}.$$

We can only show that non-trivial fixed points $\mu \in \mathscr{I}_{\text{ex}} \setminus (\mathscr{P} \cap \mathscr{I})$ exist but have no information about the structure of this set. We have no example for a translation invariant non-trivial fixed point.

As for the second question, we were able to show only very few things. In the two-dimensional Ising model case, we can give our best result, while for $d \geqq 3$ there are strong restrictive conditions which we will discuss in detail in Sects. 4 and 5.

Theorem 3. *Let* $\mu_{\beta,h}$ *be an extremal Gibbs state in the two-dimensional Ising model. The limit* $\lim_{n\to\infty} T^n \mu_{\beta,h}$ *exists and is a product measure if* $(\beta, h) \neq (\beta_c, 0)$.

This theorem follows from a more general statement proved in Sect. 4 (Theorem 4), which is based on the strong mixing property of the initial measure μ. In the case of the two-dimensional Ising model, this property holds exactly for $(\beta, h) \neq (\beta_c, 0)$, see [5, 2]. If it is violated for μ we have no result on the behavior of $T^n\mu$. In particular the most interesting problem whether $T^n \mu_{\beta_c, 0}$ converges towards a non-trivial fixed point, is unsolved. In the last section we discuss some conjectures and give partial results on the identification of $\lim_{n\to\infty} T^n \mu_{\beta,h}$ for some values of (β, h) (Theorem 5, Theorem 6).

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2. Proof of Theorem 1

The proof of Theorem 1 is carried by the following three Lemmas.

Lemma 1. *$F(x_1, x_2, \ldots, x_s)$ is strictly increasing in each variable $x_j \in [0, 1]$, $j = 1, 2, \ldots, s$.*

Lemma 2. *Let $s \geqq 3$ and f_s be defined by (5). Then*

$$\frac{d^2}{dx^2} f_s(x) \begin{cases} <0 & \text{if } x > \frac{1}{2} \\ =0 & \text{if } x = \frac{1}{2} \\ >0 & \text{if } x < \frac{1}{2} \end{cases}$$

Lemma 3. *Let $s \geqq 3$. Then*

$$\frac{d}{dx} f_s(x)|_{x=\frac{1}{2}} \quad \text{is increasing in } p>0 \text{ and}$$

$$\lim_{p \to \infty} \left(\frac{d}{dx} f_s(x)|_{x=\frac{1}{2}} \right) > 1.$$

Proof of Lemma 1. It suffices to know that

$$\frac{\partial}{\partial x_1} F(x_1, x_2, \ldots, x_s) > 0. \tag{6}$$

The left hand side of (6) is equal to

$$\sum_{k=0}^{s-1} (\alpha_{k+1} - \alpha_k) \sum_{2 \leqq i_1 < \ldots < i_k \leqq s} \prod_{v=1}^{k} x_{i_v} \prod_{\substack{j \neq i_1, \ldots, i_k \\ 2 \leqq j \leqq s}} (1 - x_j).$$

By the definition of α_k:

$$\alpha_k = e^{2p(2k-s)} / [1 + e^{2p(2k-s)}],$$

it is easy to see that $\alpha_{k+1} - \alpha_k > 0$ for every k. This proves (6) for $(x_1, \ldots, x_s) \in [0, 1]^s$. Q.e.d.

Proof of Lemma 2. We first note that

$$\frac{d}{dx} f_s(x) = s \sum_{k=0}^{s-1} (\alpha_{k+1} - \alpha_k) \binom{s-1}{k} x^k (1-x)^{s-k-1}. \tag{7}$$

By differentiating both sides, we get

$$\begin{aligned} \frac{d^2}{dx^2} f_s(x) &= s(s-1) \sum_{k=0}^{s-2} (\alpha_{k+2} - 2\alpha_{k+1} + \alpha_k) \binom{s-2}{k} x^k (1-x)^{s-k-2} \\ &= s(s-1) \sum_{k=0}^{[s/2-1]} \binom{s-2}{k} (\alpha_{k+2} - 2\alpha_{k+1} + \alpha_k) x^k (1-x)^k \\ &\quad \cdot [(1-x)^{s-2-2k} - x^{s-2-2k}]. \end{aligned} \tag{8}$$

In the last equality we used the fact that

$$\alpha_{s-j}=1-\alpha_j \quad \text{for any } j.$$

From the definition of α_k, we get by direct calculation that

$$\alpha_{k+2}-2\alpha_{k+1}+\alpha_k \begin{cases} >0 & \text{if } k<\dfrac{s-2}{2} \\ =0 & \text{if } k=\dfrac{s-2}{2}. \end{cases}$$

Thus, if $s\geqq 3$, there is at least one $k\geqq 0$ with

$$\alpha_{k+2}-2\alpha_{k+1}+\alpha_k>0,$$

which proves

$$\frac{d^2}{dx^2}f_s(x)\begin{cases} <0 & \text{for } x>\frac{1}{2} \\ =0 & \text{for } x=\frac{1}{2}. \\ >0 & \text{for } x<\frac{1}{2} \end{cases} \quad \text{Q.e.d.}$$

Proof of Lemma 3. By direct differentiation we get

$$\frac{d}{dx}f_s(x)=s(\alpha_s x^{s-1}-\alpha_0(1-x)^{s-1})$$
$$+\sum_{k=1}^{s-1}\alpha_k\binom{s}{k}x^{k-1}(1-x)^{s-1-k}[k(1-x)-(s-k)x].$$

Hence

$$\frac{d}{dx}f_s(x)|_{x=\frac{1}{2}}=\sum_{k=0}^{s}\alpha_k(2k-s)\binom{s}{k}(\tfrac{1}{2})^{s-1}$$
$$=(\tfrac{1}{2})^{s-1}\sum_{k=0}^{[\frac{s-1}{2}]}(\alpha_k-\alpha_{s-k})(2k-s)\binom{s}{k}.$$

But $\alpha_k-\alpha_{s-k}=\operatorname{th} p(2k-s)$ and $(2k-s)\operatorname{th} p(2k-s)$ are positive for all k and increasing in $p>0$.

Finally we have

$$\lim_{p\to\infty}\frac{d}{dx}f_s(x)|_{x=\frac{1}{2}}=(\tfrac{1}{2})^{s-1}\sum_{k=0}^{[\frac{s-1}{2}]}\binom{s}{k}(s-2k).$$

If s is odd, then $s-2k\geqq 1$ for any $k\leqq\left[\frac{s-1}{2}\right]$, and we have

$$\lim_{p\to\infty}\left(\frac{d}{dx}f_s(x)|_{x=\frac{1}{2}}\right)>(\tfrac{1}{2})^{s-1}\sum_{k=0}^{[\frac{s-1}{2}]}\binom{s}{k}=1$$

if $\frac{s-1}{2} \geqq 1$, i.e. $s > 3$. If s is even, then $s-2k \geqq 2$ for any $k \leqq \left[\frac{s-1}{2}\right] = \frac{s}{2}-1$, and

$$2\sum_{k=0}^{\frac{s}{2}-1} \binom{s}{k} = 2^s - \binom{s}{s/2} > 2^{s-1} \qquad (s \geqq 4, s \text{ even})$$

by easy calculation. Q.e.d.

Now we are going to prove Theorem 1, we begin with part (ii):

Let $\mu \in \mathscr{P}$ and $\mu(x) = \mu(\omega(x) = +1)$. Using (4) and $V_{x_1} \cap V_{x_2} = \emptyset$ if $x_1 \neq x_2$, we get

$$\begin{aligned} T\mu(\omega(x) = +1, x \in \Lambda) &= \int \prod_{x \in \Lambda} F\left(\left\{\frac{1+\omega(y)}{2}, y \in V_x\right\}\right) \mu(d\omega) \\ &= \prod_{x \in \Lambda} \int F\left(\left\{\frac{1+\omega(y)}{2}, y \in V_x\right\}\right) \mu(d\omega) \\ &= \prod_{x \in \Lambda} F(\mu(y), y \in V_x). \end{aligned} \tag{9}$$

This proves (ii). For the proof of (i) we use the following properties of $f_s(x)$:

$$\begin{aligned} &f_s(\tfrac{1}{2}) = \tfrac{1}{2} \\ &f_s(0) = \alpha_0 > 0 \\ &f_s(1) = \alpha_s < 1. \end{aligned} \tag{10}$$

Furthermore, by Lemma 2, f_s is concave on $[\frac{1}{2}, 1]$ and convex on $[0, \frac{1}{2}]$. Therefore the equation $x = f_s(x)$ has unique solution $x = \frac{1}{2}$ in $[0, 1]$ iff $\frac{d}{dx} f_s(x)|_{x=\frac{1}{2}} \leqq 1$, and three solutions $x = \underline{r}$, $x = \frac{1}{2}$, $x = \bar{r}$ $(0 < \underline{r} < \frac{1}{2} < \bar{r} < 1, \underline{r} = 1 - \bar{r})$

$$\text{iff } \frac{d}{dx} f_s(x)|_{x=\frac{1}{2}} > 1.$$

Thus, by Lemma 3, there is $p_c > 0$ for $s \geqq 3$ such that

$$\frac{d}{dx} f_s(x)|_{x=\frac{1}{2}} \leqq 1 \qquad \text{iff } 0 < p \leqq p_c.$$

For each $n \geqq 1$, we define a family $\{P_x^n(+|\omega), x \in \mathbb{Z}^d\}$ by

$$\begin{aligned} P_x^1(+|\omega) &= P_x(+|\omega) \\ P_x^{n+1}(+|\omega) &= F(\{P_y^n(+|\omega), y \in V_x\}). \end{aligned} \tag{11}$$

Then it is easy to see that for any finite set $\Lambda \subset \mathbb{Z}^d$

$$T^n \mu(\omega(x) = +1, x \in \Lambda) = \int \prod_{x \in \Lambda} P_x^n(+|\omega) \mu(d\omega).$$

From Lemma 1 and (11), $P_x^n(+|\omega)$ is increasing in ω. Thus if we take $+\mathbb{1}$, $-\mathbb{1}\in\Omega$ with

$$+\mathbb{1}(x)=+1,\quad -\mathbb{1}(x)=-1 \qquad \text{for every } x\in\mathbb{Z}^d,$$

we have

$$P_x^n(+|-\mathbb{1})\leqq P_x^n(+|\omega)\leqq P_x^n(+|+\mathbb{1}).$$

$$\{P_x^n(+|+\mathbb{1}\}_{n=1}^{\infty} \quad \text{and} \quad \{P_x^n(+|-\mathbb{1})\}_{n=1}^{\infty}$$

satisfy

$$P_x^n(+|\pm\mathbb{1})=f_s(P_x^{n-1}(+|\pm\mathbb{1})) \qquad \text{for all } n\geqq 2. \tag{12}$$

If $f_s(x)$ has unique solution $x=\frac{1}{2}$, then $x>f_s(x)$ if $x>\frac{1}{2}$ and $x<f_s(x)$ if $x<\frac{1}{2}$. This implies that $\{P_x^n(+|+\mathbb{1})\}_{n=1}^{\infty}$ is decreasing and $\{P_x^n(+|-\mathbb{1})\}_{n=1}^{\infty}$ is increasing and

$$\lim_{n\to\infty} P_x^n(+|\pm\mathbb{1})=\tfrac{1}{2}. \tag{13}$$

By the monotonicity of $P_x^n(+|\omega)$ in ω we get

$$\lim_{n\to\infty} T^n\mu(\omega(x)=+1, x\in\Lambda)=(\tfrac{1}{2})^{|\Lambda|}. \tag{14}$$

If $x=f_s(x)$ has solutions $\underline{r}<\frac{1}{2}<\bar{r}$, the product measures $\nu_{\underline{r}}$ resp. $\nu_{\bar{r}}$ with $\nu_{\underline{r}}(\omega(x)=+1)=\underline{r}$ resp. $\nu_{\bar{r}}(\omega(x)=+1)=\bar{r}$ $(x\in\mathbb{Z}^d)$ are invariant under T because of (9). Thus the uniqueness of the solution for $f_s(x)$ is equivalent to the uniqueness of $\mathscr{I}$. This completes the proof of Theorem 1.

3. Proof of Theorem 2

In order to prove Theorem 2 we show

Proposition 1. *If* $\nu\in\mathscr{I}$ *is trivial on*

$$\mathscr{F}_{\infty}\equiv \bigcap_{\substack{\Lambda:\text{finite}\\ \Lambda\subset\mathbb{Z}^d}} \mathscr{F}_{\Lambda^c},$$

then $\nu\in\mathscr{I}_{\text{ex}}$.

Proof. Let $C(\Omega)$ be the set of all continuous functions on Ω with supremum norm. For any finite $\Lambda\subset\mathbb{Z}^d$, C_Λ denotes the set of functions depending on $\{\omega(x), x\in\Lambda\}$.

$$K\equiv \bigcup_{\substack{\Lambda:\text{finite}\\ \Lambda\subset\mathbb{Z}^d}} C_\Lambda$$

is dense in $C(\Omega)$. For fixed finite $\Lambda\subset\mathbb{Z}^d$ we have for $f\in C_\Lambda$

$$\int f(\omega)(T\mu)(d\omega)=\int(fT)(\omega)\,\mu(d\omega),$$

where

$$(fT)(\omega)=\sum_{\eta\in\Omega_\Lambda} f(\eta)\prod_{x\in\Lambda} P_x(\eta(x)|\omega),$$

where $P_x(-|\omega)=1-P_x(+|\omega)$. It is easily seen that T can be uniquely extended to $C(\Omega)$ as a positive contraction operator. The following ergodic theorem holds for T:

A measure $\nu\in\mathscr{I}$ is ergodic for T iff $\nu\in\mathscr{I}_{\text{ex}}$ (see e.g. [7], p. 359 or [8], Theorem 3.8).

In order to prove Proposition 1 we therefore have to show: if $\nu\in\mathscr{I}$ is trivial on $\mathscr{F}_\infty$ then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(fT^n)(\omega)=\int f\,d\nu \qquad \nu-\text{a.s.} \tag{15}$$

for $f\in K$.

We denote by

$$g(\omega)\equiv\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(fT^n)(\omega), \tag{16}$$

$$\Lambda_0\equiv\{x\in\mathbb{Z}^d;\, x\in V_x\}=\{x\in\mathbb{Z}^d;\; -1\leqq x^i\leqq 0 \text{ for } 1\leqq i\leqq d\}. \tag{17}$$

What we only need to show is that g is $\mathscr{F}_{\Lambda_0^c}$-measurable; because then for $n\geqq 2$

$$gT^n \text{ is } \mathscr{F}_{\Lambda_0^c}\text{-measurable,}$$

where

$$\Lambda_n=\{x\in\mathbb{Z}^d;\, \|x-(-\tfrac{1}{2},-\tfrac{1}{2},\dots,-\tfrac{1}{2})\|\leqq 2^{n-1}+\tfrac{1}{2}\}$$

and $\|x-y\|=\max\limits_{1\leqq i\leqq d}|x^i-y^i|$.

Since g is T-invariant we get that g is $\mathscr{F}_\infty$-measurable. Hence (15) follows because ν is trivial on $\mathscr{F}_\infty$.

Let x be such that $x\in V_x$. We are going to show that g is $\mathscr{F}_{\{x\}^c}$-measurable ($\mathscr{F}_{\Lambda_0^c}$-measurability follows similarly).

The proof is based on the following

Lemma 4. *Assume that η and $\omega\in\Omega$ satisfy*

$$\eta(y)=\omega(y) \qquad \textit{for all } y\in V_a^{n+1}\setminus V_b^n \tag{18}$$

for some $n\geqq 1$, $a\in\mathbb{Z}^d$ and $b\in V_a$, where

$$V_z^n\equiv\left\{\begin{matrix} y;\text{ there exist } z=u_0,u_1,u_2,\dots,u_{n-1},u_n=y \\ \text{such that } u_{i+1}\in V_{u_i},\, i=0,1,\dots,n-1\end{matrix}\right\}.$$

Then we have

$$|P_a^{n+1}(+|\omega)-P_a^{n+1}(+|\eta)|\leqq(\tfrac{1}{2}\operatorname{th}2p)|P_b^n(+|\omega)-P_b^n(+|\eta)|. \tag{19}$$

Proof. From the definition, the left hand side of (19) is smaller than

$$\begin{aligned}&\left|\frac{\partial F}{\partial x_1}(x_1,\{P_\alpha^n(+|\omega),\alpha\in V_a\setminus\{b\}\})\right|\cdot|P_b^n(+|\omega)-P_b^n(+|\eta)|\\ &\leqq\max_{0\leqq k\leqq s-1}(\alpha_{k+1}-\alpha_k)|P_b^n(+|\omega)-P_b^n(+|\eta)|.\end{aligned}$$

As is easily seen,

$$\max_{0\leq k\leq s-1}(\alpha_{k+1}-\alpha_k)=\alpha_{\frac{s}{2}+1}-\alpha_{\frac{s}{2}}=\tfrac{1}{2}\,\mathrm{th}\,2p.\quad \text{Q.e.d.}$$

In order to apply Lemma 4 we denote by

$$U_x^{\pm}\,\omega(y)=\begin{cases}\pm 1 & \text{if } y=x\\ \omega(y) & \text{if } y\neq x.\end{cases}$$

Then $\omega(y)=(U_x^+\omega)(y)$ for all $y\in V_x^{n+1}\setminus V_x^n$, hence assumption (18) is fulfilled trivially. Therefore we get from Lemma 4

$$\max_{\omega\in\Omega}|P_x^n(+|\omega)-P_x^n(+|U_x^+\omega)|\to 0 \quad \text{as } n\to\infty. \tag{20}$$

We write

$$f(\omega)=f(U_x^+\omega)\,\chi_{\{\omega(x)=+1\}}(\omega)+f(U_x^-\omega)\,\chi_{\{\omega(x)=-1\}}(\omega)$$

and get from this

$$fT^n(\omega)=[(f\circ U_x^+)\,T^n](\omega)\cdot P_x^n(+|\omega)+[(f\circ U_x^-)\,T^n](\omega)\cdot P_x^n(-|\omega).$$

By (20) the last expression is approximately equal to

$$[(f\circ U_x^+)\,T^n](\omega)\cdot P_x^n(+|U_x^+\omega)+[(f\circ U_x^-)\,T^n](\omega)\cdot P_x^n(-|U_x^-\omega)$$

which is $\mathscr{F}_{\{x\}^c}$-measurable. This proves that g is $\mathscr{F}_{\{x\}^c}$-measurable. Q.e.d.

Kolmogorov's $0-1$ law and Proposition 1 imply

Corollary 1. $\mathscr{I}\cap\mathscr{P}\subset\mathscr{I}_{\mathrm{ex}}$.

In order to show the strict inclusion, we construct an example in the following way: Let Λ_n be

$$\Lambda_n\equiv\{y;\, y\in V_x^n \text{ for some } x\in\Lambda_0\},\qquad n\geq 1.$$

Define a measure μ on $(\Omega,\mathscr{F})$ by

$$\mu\equiv\bigotimes_{n=0}^{\infty}\left[\tfrac{1}{2}(\delta_{+\mathbb{1}}+\delta_{-\mathbb{1}})|_{\Lambda_n\setminus\Lambda_{n-1}}\right].$$

Then it is easy to see that $\lim_{m\to\infty}T^m\mu$ exists and satisfies

$$\lim_{m\to\infty}T^m\mu(\omega(x)=+1,\, x\in A)=\prod_{i=1}^{k}\{\tfrac{1}{2}(\bar{r}^{|A_i|}+\underline{r}^{|A_i|})\}, \tag{21}$$

where $A=A_1\cup A_2\cup\ldots\cup A_k$, such that $A_i\subset\Lambda_{n_i}\setminus\Lambda_{n_i-1}$ for some $n_1<n_2<\ldots<n_k$; $|A_i|$ is the cardinality of A_i. $\nu\equiv\lim_{m\to\infty}T^m\mu$ is invariant under T and

$$\nu(\omega(x)=\omega(y)=+1)=\tfrac{1}{2}(\bar{r}^2+\underline{r}^2)\neq\mu(\omega(x)=+1)\,\mu(\omega(y)=+1)$$

if x and y are in the same $\Lambda_n \setminus \Lambda_{n-1}$ for some $n \geqq 1$. From (21), it is easy to see that ν is trivial on $\mathscr{F}_\infty$. Thus, $\nu \in \mathscr{I}_{\text{ex}}$. Q.e.d.

Remark 1. Proposition 1 can also be proved via the more general Theorem (4.6) of Föllmer [3]. It implies that two measures $\nu_1, \nu_2 \in \mathscr{I}$ are identical whenever they coincide on $\mathscr{F}_\infty$. However in order to check Föllmer's condition (4.7) one needs an estimate similar to (20). Therefore we prefer to give the above self-contained proof.

4. Theorem 4

Let us turn to our final topic. Before we state the result, we need some definitions. Let $\mathscr{F}_a^b$, $-\infty \leqq a < b \leqq \infty$ be

$$\mathscr{F}_a^b \equiv \sigma\{\omega(x);\, a \leqq x^1 \leqq b\}.$$

Definition. Let $\mu \in \mathscr{M}$ be translation invariant. μ is said to be *strongly mixing* in x^1-direction iff

$$\psi(l) \equiv \sup_{A \in \mathscr{F}_{-\infty}^0} \sup_{B \in \mathscr{F}_l^\infty} |\mu(A \cap B) - \mu(A)\,\mu(B)|$$

converges to zero as $l \to \infty$.

Theorem 4. *Assume that $p > p_c$ and*

$$\delta = \delta(s, p) = \frac{s}{4} \operatorname{th} 2p < 1, \qquad s = 2^d. \tag{22}$$

Then, for any translation invariant μ, which is strongly mixing in each direction, the limit $\lim_{m \to \infty} T^m \mu$ exists and is equal to either $\nu_{1/2}$, $\nu_{\underline{r}}$ or $\nu_{\bar{r}}$, where $\nu_{1/2}$, $\nu_{\underline{r}}$, $\nu_{\bar{r}}$ are product measures with constant density $\nu_{1/2}(\omega(x) = +1) = \frac{1}{2}$, $\nu_{\underline{r}}(\omega(x) = +1) = \underline{r}$, $\nu_{\bar{r}}(\omega(x) = +1) = \bar{r}$, and $\underline{r} < \frac{1}{2} < \bar{r}$ are the solutions for the equation $x = f_s(x)$.

Remark 2. This theorem states a somewhat stronger fact than what we stated in Theorem 3. The dimension d comes in only through (22) which is trivial when $d = 2$ ($s = 4$). Since every Gibbs state is translation invariant in two dimensions ([1, 6]), Theorem 3 is completely included in Theorem 4.

Remark 3. The condition (22) is not satisfied when p is too large for more than two dimensions. Unfortunately we were not able to give a better condition.

Let us define $n_0 = n_0(l)$ as

$$n_0(l) \equiv \min\{n;\, 2^n \geqq l\} \qquad \text{for every } l \geqq 1.$$

For the proof of Theorem 4 we need the following two lemmas.

Lemma 5. *Let $\Lambda \subset \mathbb{Z}^d$ be a finite subset. Under the conditions of Theorem 4, we have*

$$\gamma_n(A) \equiv |T^n \mu(\omega(x) = +1, x \in A) - \prod_{x \in A} T^n \mu(\omega(x) = +1)|$$
$$\leqq 2d|A|(\delta^{n-n_0(l)} + \psi(l))$$

for any $l \geqq 1$ *and any* $n \geqq n_0(l)$.

Proof. We prove this by induction on $|A|$. $|A| = 1$ is trivial. Let

$$M_i(A) \equiv \max\{x^i; x = (x^1, x^2, \ldots, x^d) \in A\},$$
$$m_i(A) \equiv \min\{x^i; x = (x^1, x^2, \ldots, x^d) \in A\}$$

for $1 \leqq i \leqq d$.

Without loss of generality we can assume that

$$M_d(A) + m_d(A) \geqq 0, \quad m_i(A) \geqq 0 \quad \text{for } 1 \leqq i \leqq d-1.$$

Let $A_1 = \{x \in A; x^d = M_d(A)\}$, $A_2 = A \setminus A_1$.

Both A_1 and A_2 are non-empty. For fixed $l \geqq 1$ and $n \geqq n_0(l)$, we define a configuration $\omega^+_{n,l}$ by

$$\omega^+_{n,l}(y) = \begin{cases} +1 & \text{if } 2^n M_d(A) \leqq y^d \leqq 2^n M_d(A) + l \\ \omega(y) & \text{otherwise.} \end{cases}$$

Then it is clear from Lemma 4 that

$$|P^n_x(+|\omega) - P^n_x(+|\omega^+_{n,l})| \leqq [\tfrac{1}{2} \operatorname{th} 2p] \sum_{y \in \underline{V}_x} |P^{n-1}_y(+|\omega) - P^{n-1}_y(+|\omega^+_{n,l})|$$

if $n-1 \geqq n_0(l)$, where $\underline{V}_x \equiv \{y \in V_x; y^d = 2x^d\}$.

Thus, we have for every k with $n_0(l) \leqq k \leqq n$, that

$$|P^n_x(+|\omega) - P^n_x(+|\omega^+_{n,l})|$$
$$\leqq [\tfrac{1}{2} \operatorname{th} 2p]^{n-k} \sum_{y_1 \in \underline{V}_x} \sum_{y_2 \in \underline{V}_{y_1}} \cdots \sum_{y_{n-k} \in \underline{V}_{y_{n-k-1}}} |P^k_{y_{n-k}}(+|\omega) - P^k_{y_{n-k}}(+|\omega^+_{n,l})|$$
$$\leqq \left[\frac{S}{4} \operatorname{th} 2p\right]^{n-k},$$

since $|\underline{V}_x| = \frac{S}{2}$ for any $x \in \mathbb{Z}^d$.

From this we have

$$|T^n \mu(\omega(x) = +1, x \in A) - \prod_{x \in A} T^n \mu(\omega(x) = +1)|$$
$$\leqq |\int \prod_{x \in A_1} P^n_x(+|\omega^+_{n,l}) \prod_{x \in A_2} P^n_x(+|\omega)\, \mu(d\omega) - \prod_{x \in A} T^n \mu(\omega(x) = +1)| + |A_1|\, \delta^{n-n_0}.$$

The first term of the right hand side is smaller than

$$|T^n \mu(\omega(x) = +1, x \in A_1)\, T^n \mu(\omega(x) = +1, x \in A_2) - \prod_{x \in A} T^n \mu(\omega(x) = +1)| + \psi(l)$$
$$+ |A_1|\, \delta^{n-n_0}.$$

Using the induction hypothesis and noticing that A_1 is at most $(d-1)$-dimensional, we get

$$\gamma_n(A) \leqq 2(d-1)|A_1|(\delta^{n-n_0}+\psi(l))+2d|A_2|(\delta^{n-n_0}+\psi(l))+\psi(l)+2|A_1|\,\delta^{n-n_0}$$
$$\leqq 2d|A|(\delta^{n-n_0}+\psi(l)). \quad \text{Q.e.d.}$$

Lemma 6. *Under the conditions of Theorem 4, for any* $\varepsilon>0$, *there exists* $n_1(\varepsilon)>0$ *such that if* $n \geqq n_1(\varepsilon)$,

$$|\int P_x^{n+1}(+|\omega)\,\mu(d\omega)-f_s(\int P_x^n(+|\omega)\,\mu(d\omega))|<\varepsilon$$

for every $x\in\mathbb{Z}^d$.

Proof. Denoting by $T^n\mu(x)\equiv T^n\mu(\omega(x)=+1)$, we have

$$T^{n+1}\mu(x)=\int F(\{P_y^n(+|\omega), y\in V_x\})\,\mu(d\omega).$$

Combining this with the definition of F and Lemma 5 we get

$$|T^{n+1}\mu(x)-F(T^n\mu(y), y\in V_x)| \leqq (2d)\,s^2 2^s(\delta^{n-n_0(l)}+\psi(l)) \tag{23}$$

for $l \geqq 1$, $n \geqq n_0(l)$. Taking first $l \geqq 1$ large enough so that $\psi(l)<\varepsilon/(2d)\,s^2 2^{s+1}$, and then $n_1>n_0(l)$ such that $\delta^{n-n_0(l)}<\varepsilon/(2d)\,s^2 2^{s+1}$ for any $n \geqq n_1$, we obtain the desired inequality, since μ is translation invariant. Q.e.d.

Proof of Theorem 4. Assume $\overline{\lim}_{m\to\infty} T^m\mu(x)>\frac{1}{2}$ (the case $\lim_{m\to\infty} T^m\mu(x)=\frac{1}{2}$ is trivial). By Lemma 5 every limit point of $\{T^m\mu\}$ is a product measure. Since

$$\overline{\lim_{n\to\infty}}\, T^m\mu(x) \leqq \lim_{m\to\infty} P_x^m(+|+\mathbb{1})=\bar{r},$$

it suffices to show

$$\underline{\lim}_{m\to\infty}\, T^m\mu(x)>\xi \qquad \text{for any } \xi\in(\tfrac{1}{2},\bar{r}). \tag{24}$$

First we prove

$$T^m\mu(x)>\xi \qquad \text{for infinitely many } m. \tag{25}$$

Let v^* be a limit point of $\{T^m\mu\}$ satisfying $v^*(\omega(x)=+1)>\frac{1}{2}$. Since v^* is a translation invariant product measure it is easy to see that $\lim_{m\to\infty} T^m v^* = v_{\bar{r}}$. Therefore $v_{\bar{r}}(x)$ is a limit point of $\{T^m\mu(x)\}$ which implies (25).

Now (24) is easily proved by means of Lemma 6 and (25): Let $\varepsilon>0$ be sufficiently small so that

$$\xi<f(\xi)-\varepsilon.$$

For this ε we take $n_1(\varepsilon)$ as in Lemma 6. By (25) there is $m_0>n_1(\varepsilon)$ such that

$$T^{m_0}\mu(x)>\xi.$$

Using induction on l we then get

$$T^{m_0+l}\mu(x)>\xi \qquad \text{for any positive } l \geqq 1,$$

because Lemma 6 implies

$$T^{m_0+l+1}\mu(x) \geqq f(T^{m_0+l}\mu(x))-\varepsilon>f(\xi)-\varepsilon>\xi. \quad \text{Q.e.d.}$$

5. Comments on the Case of the Two-dimensional Ising Model

In this section we consider the two-dimensional Ising model. It is known ([5]) that if $(\beta, h) \neq (\beta_c, 0)$, i.e. besides the critical point, the strongly mixing property holds for every extremal Gibbs state. Thus by Theorem 4,

$$\lim_{m\to\infty} T^m \mu_{\beta,h} \quad \text{converges to some product measure.}$$

For this limiting measure it would be natural to expect that

(i) $\lim_{m\to\infty} T^m \mu_{\beta,h} = \nu_{\bar{r}}$ if $h>0$

(ii) $\lim_{m\to\infty} T^m \mu^+_{\beta,0} = \nu_{\bar{r}}$ if $h=0,\ \beta>\beta_c$

(iii) $\lim_{m\to\infty} T^m \mu^-_{\beta,0} = \nu_{\underline{r}}$ if $h=0,\ \beta>\beta_c$

(iv) $\lim_{m\to\infty} T^m \mu_{\beta,0} = \nu_{\frac{1}{2}}$ if $h=0,\ \beta<\beta_c$

(v) $\lim_{m\to\infty} T^m \mu_{\beta,h} = \nu_{\underline{r}}$ if $h<0$,

and we further hope that

$$\lim_{m\to\infty} T^m \mu_{\beta_c,0} \quad \text{exists and is in } \mathscr{I}_{\text{ex}} \backslash \mathscr{P}.$$

Unfortunately, we were not able to prove even (i)~(v), except (iv). Here, we give a result what we were able to prove.

Theorem 5. *Assume that $p>p_c$. For sufficiently large β we have*

$$\begin{aligned} &\lim_{m\to\infty} T^m \mu^+_{\beta,0} = \lim_{m\to\infty} T^m \mu_{\beta,h} = \nu_{\bar{r}} \quad (h>0) \\ &\lim_{m\to\infty} T^m \mu^-_{\beta,0} = \lim_{m\to\infty} T^m \mu_{\beta,h} = \nu_{\underline{r}} \quad (h<0). \end{aligned} \tag{26}$$

If $\beta<\beta_c$ then we have

$$\lim_{m\to\infty} T^m \mu_{\beta,0} = \nu_{\frac{1}{2}}.$$

Proof. The latter half is trivial since $\lim_{m\to\infty} T^m \mu_{\beta,0}$ is a product measure and $T^m \mu_{\beta,0}$ is invariant under the spin reversal $\omega \to -\omega$. By the *FKG* inequality we have

$$\lim_{m\to\infty} T^m \mu^+_{\beta,0}(x) \leqq \lim_{m\to\infty} T^m \mu_{\beta,h}(x) \quad \text{for } h>0.$$

Thus we only have to prove that

$$\lim_{m\to\infty} T^m \mu^+_{\beta,0}(x) = \bar{r} \quad \text{for all } x\in\mathbb{Z}^d.$$

Take $\xi_0 \in (\frac{1}{2}, \bar{r})$ as $f'_s(\xi_0)=1$ and $\varepsilon < f_s(\xi_0) - \xi_0$. For this $\varepsilon>0$, we take $n_1(\varepsilon)$ according to Lemma 6 as applied to $\mu^+_{\beta,0}$, where β is large ($n_1(\varepsilon)$ can be taken uniformly for large β's because the mixing coefficient ψ is estimated uniformly in large β's; see [5], proof of Theorem 1). Then

$$
\begin{aligned}
&T^{n_1}\mu^+_{\beta,0}(\omega(x)=+1)-T^{n_1}\mu^+_{\beta,0}(\omega(x)=-1)\\
&=\int P^{n_1}_x(+|\omega)\,\mu^+_{\beta,0}(d\omega)-\int P^{n_1}_x(-|\omega)\,\mu^+_{\beta,0}(d\omega)\\
&=\sum_{\omega:P^{n_1}_x(+|\omega)-P^{n_1}_x(-|\omega)>0}\{P^{n_1}_x(+|\omega)-P^{n_1}_x(-|\omega)\}\{\mu^+_{\beta,0}(\omega)-\mu^-_{\beta,0}(\omega)\}\\
&\geqq(\bar{r}-\underline{r})\,\mu^+_{\beta,0}(+\mathbb{1}|_{V^{n_1}_x})\\
&\quad-\mu^+_{\beta,0}(\omega\in\{+1,-1\}^{V^{n_1}_x};\ \text{there exists } y\in V^{n_1}_x \text{ such that } \omega(y)=-1)\\
&\geqq(\bar{r}-\underline{r})-2|V^{n_1}_x|\sum_{k=4}^{\infty}k^2(3e^{-\beta})^k\equiv D(n_1;\beta)
\end{aligned}
$$

by Peierls' argument. As is easily seen, $D(n_1;\beta)\to(\bar{r}-\underline{r})$ as $\beta\to\infty$.

Therefore if β is sufficiently large, $D(n_1;\beta)\geqq 2\xi_0-1$. This implies that

$$T^{n_1}\mu^+_{\beta,0}(x)\geqq\xi_0.$$

By Lemma 6 we have

$$T^{n_1+l}\mu^+_{\beta,0}(x)\geqq f_s(T^{n_1+l-1}\mu^+_{\beta,0}(x))-\varepsilon\geqq\xi_0$$

by induction. Thus, we have

$$\lim_{m\to\infty}T^m\mu^+_{\beta,0}(x)\geqq\xi_0>\tfrac{1}{2},$$

which implies that $\lim_{m\to\infty}T^m\mu^+_{\beta,0}(x)=\bar{r}$. Q.e.d.

Theorem 6. *If* $h\geqq 2$ *(*$h\leqq -2$ *resp.), then*

$$\lim_{m\to\infty}T^m\mu_{\beta,h}=\nu_{\bar{r}}\qquad(\nu_{\underline{r}}\ \textit{resp.}).$$

Proof. Note that by the *FKG* inequality we have

$$
\begin{aligned}
T^m\mu_{\beta,h}(x)&\geqq\int\prod_{z\in V^{m-1}_x}P^m_x(+|\omega)\,\mu(d\omega|-\mathbb{1}_{|\partial V_z})\\
&=\underbrace{f_s\circ f_s\circ\ldots\circ f_s}_{m-1}\left(\int P_0(+|\omega)\,\mu(d\omega|-\mathbb{1}_{|\partial V_0})\right),
\end{aligned}
\tag{27}
$$

where $\mu(\cdot|-\mathbb{1}_{|\partial V_0})$ is given by

$$\mu(\eta|-\mathbb{1}_{|\partial V_0})=Z^{-1}\exp\{\beta[\sum_{\substack{\langle x,y\rangle\\ x,y\in V_0}}\eta(x)\eta(y)+h\sum_{x\in V_0}\eta(x)-2\Sigma'\eta(x)]\},$$

where the summation Σ' is taken over all x's in V_0 such that there is a $y\in\partial V_0$ nearest neighbour to x.

$$
\begin{aligned}
&\int P_0(+|\omega)\,\mu(d\omega|-\mathbb{1}_{|\partial V_0})-\int P_0(-|\omega)\,\mu(d\omega|-\mathbb{1}_{|\partial V_0})\\
&=\sum_{\omega:P_0(+|\omega)>\frac{1}{2}}\{P_0(+|\omega)-P_0(-|\omega)\}\{\mu(\omega|-\mathbb{1}_{|\partial V_0})-\mu(-\omega|-\mathbb{1}_{|\partial V_0})\}.
\end{aligned}
$$

By direct calculation, we can show that this is positive, which implies that

$$\int P_0(+|\omega)\,\mu(d\omega| - \mathbb{1}_{|\partial V_0}) > \tfrac{1}{2}.$$

This and (27) prove our statement. Q.e.d.

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