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Some Extensions of the Hewitt-Savage Zero-One Law

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Summary. Let $\mathscr{M}$ denote the class of infinite product probability measures $\mu = \mu_1 \times \mu_2 \times \cdots$ defined on an infinite product of replications of a given measurable space $(\mathscr{X}, \mathscr{A})$, and let $\mathscr{H}$ denote the subset of $\mathscr{M}$ for which $\mu(A) = 0$ or 1 for each permutation invariant event A. Previous works by Hewitt and Savage, Horn and Schach, Blum and Pathak, and Sendler (referenced in the paper) discuss very restrictive sufficient conditions under which a given member μ of $\mathscr{M}$ belongs to $\mathscr{H}$. In the present paper, the class $\mathscr{H}$ is shown to possess several closure properties. E.g., if $\mu \in \mathscr{H}$ and $\mu_0 \ll \mu_n$ for some $n \geqq 1$, then $\mu_0 \times \mu_1 \times \mu_2 \times \ldots \in \mathscr{H}$. While the current results do not permit a complete characterization of $\mathscr{H}$, they demonstrate conclusively that $\mathscr{H}$ is a much larger subset of $\mathscr{M}$ than previous results indicated. The interesting special case $\mathscr{X} = \{0, 1\}$ is discussed in detail.

1. Introduction

The zero-one law described by Hewitt and Savage (1955) asserts that an infinite product probability measure assigns the value zero or one to each (permutation) invariant event when the component probability measures of the infinite product are identical. This contrasts with Kolmogorov's zero-one law which makes the same claim for each tail event (a special kind of invariant event) with no restrictions upon the components. The gap between these two laws was slightly bridged when Horn and Schach (1970) showed that the assertion for invariant events held when each component occurred infinitely often in the infinite product. Blum and Pathak (1972) refined their argument and showed that it was sufficient that each component be a limit point of the sequence of components under the total variation norm. The intent of the present author's investigation was to discover whether the Blum-Pathak result leaves much room for improvement. In this regard, Sendler (1975) has pointed out that the zero-one law holds

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for invariant events when each component of the infinite product probability measure is degenerate, i.e., when each component assigns probability one to a single point. Thus, there is at least some room for improvement of the Blum-Pathak result. The main conclusion to be drawn from the present paper is that substantial improvements are possible. It should be mentioned that a zero-one law for invariant events does not hold without some restrictions upon the components. The reader can convince himself of this with very simple counter-examples of his own making, or refer to any of several counter-examples given below.

Consider the infinite product measurable space $(\mathscr{X}^\infty, \mathscr{A}^\infty)$ which is generated by a fixed measurable space $(\mathscr{X}, \mathscr{A})$ replicated a countably infinite number of times. Associated with each point $x=(x_1, x_2 \ldots)\in\mathscr{X}^\infty$ is the *orbit* $o(x)$ consisting of all points $y=(y_1, y_2, \ldots)\in\mathscr{X}^\infty$ which can be obtained by permuting a finite number of the components of x. An event $A\in\mathscr{A}^\infty$ is said to be (*permutation*) *invariant* if $o(A)=A$.

Let $\mathscr{M}$ denote the collection of all infinite product probability measures $\mu=\mu_1 \times\mu_2\times\cdots$ whose components are probability measures on $(\mathscr{X}, \mathscr{A})$. Further, let $\mathscr{H}$ denote the collection of all probability measures $\mu\in\mathscr{M}$ which possess the "Hewitt-Savage zero-one property": $\mu(A)=0$ or 1 for every invariant event $A\in\mathscr{A}^\infty$. We shall demonstrate that $\mathscr{H}$ possesses certain closure properties, some of which lead to the general conclusion that there are many members of $\mathscr{H}$ whose membership can not be established by applying Blum and Pathak's results.

The simplest closure result is the following:

Closure Property 1. *If $\mu\in\mathscr{H}$ and $\nu\in\mathscr{M}$ is absolutely continuous with respect to μ, then $\nu\in\mathscr{H}$.*

Proof. This is immediate.

A set of necessary and sufficient conditions for $\nu\ll\mu$ has been given by Kakutani: Let $\mu=\mu_1\times\mu_2\times\cdots$ and $\nu=\nu_1\times\nu_2\times\cdots$. In order that $\nu\ll\mu$ it is necessary and sufficient that (i) $\nu_n\ll\mu_n$ for $n\geqq 1$, and (ii) $\prod_{n=1}^{\infty}\int_{\mathscr{X}}\sqrt{d\nu_n/d\mu_n}\,d\mu_n>0$. (See, for instance, Neveu (1975), p. 44.)

Example 1. Suppose $\mathscr{X}$ is the real line and each μ_n is the "standard" normal distribution, whose mean is zero and whose variance is one. Further, suppose ν_n is the normal distribution whose mean is α_n and whose variance is one. Then, by applying (ii), it easily follows that $\nu\ll\mu$ (in fact $\nu\equiv\mu$) iff $\sum_{n=1}^{\infty}(\alpha_n)^2<\infty$. For such ν, it follows from the classical Hewitt-Savage zero-one law and closure property 1 that $\nu\in\mathscr{H}$. Suppose, instead, that $\alpha_n=n$ and $A=\mathbb{R}_+^\infty$, the set of points $x=(x_1, x_2, \ldots)$ all of whose components are positive. Then, by the Borel-Cantelli lemma, $\nu(\{x\in\mathscr{X}^\infty\colon x_n<0 \text{ infinitely often}\})=0$ and, hence, $\nu(A)\in(0,1)$. Consequently, $\nu\notin\mathscr{H}$.

All of the closure properties discussed herein are "structure free" in the sense that each is described without an explicit reference to the structure of the measurable space $(\mathscr{X}, \mathscr{A})$. This suggests that the Hewitt-Savage zero-one property is similarly structure free. A recent result by Aldous and Pitman (1977) points in the opposite direction. They find a condition for μ to belong to $\mathscr{H}$ which is necessary and sufficient *when $\mathscr{X}$ is finite*, but not otherwise. We will describe this condition in Section 3.

2. Other Closure Properties

Hewitt and Savage (1955) describe a proof of their zero-one law, due to Halmos, which is appropriate for doubly infinite product measures $\mu = \mathop{\times}\limits_{n=-\infty}^{\infty} \mu_n$. This suggests the following question: Is the Hewitt-Savage zero-one property order independent? For instance, if $\mu = \mu_1 \times \mu_2 \times \cdots \in \mathscr{H}$, does it follow that $\mu' = \mu_2 \times \mu_1 \times \mu_3 \times \mu_4 \times \cdots$ and $\mu'' = \mu_2 \times \mu_1 \times \mu_4 \times \mu_3 \times \mu_6 \times \mu_5 \times \cdots$ are members of $\mathscr{H}$? The answer is in the affirmative. Quite obviously $\mu' \in \mathscr{H}$ because it can be obtained from μ through a finite permutation of the indices of μ. The following result asserts that $\mu'' \in \mathscr{H}$ as well:

Closure Property 2. *If $\mu \in \mathscr{H}$ and $v \in \mathscr{M}$ is obtainable from μ by a finite or infinite permuting of the components in μ, then $v \in \mathscr{H}$.*

Proof. If A is an invariant set, then $v(A) = \mu(TA)$, where T is some permutation operator on $\mathscr{X}^\infty$. Since $\mu \in \mathscr{H}$, it suffices to show that $o(TA) = TA$. This reduces to checking that $T^{-1}ST$ is a finite permutation whenever S is a finite permutation.

Closure Property 3. *If $\mu \in \mathscr{H}$ and $v \in \mathscr{H}$, then $\mu \times v \in \mathscr{H}$.*

Remark. The expression "$\mu \times v$" will be taken to mean *any* arrangement $\tau = \tau_1 \times \tau_2 \times \cdots$ of the combined components of μ and v. Closure property 2 provides a strong justification for this convenient abuse of notation.

Proof. For the sake of definiteness, the arrangement $\tau = \mu_1 \times v_1 \times \mu_2 \times v_2 \times \cdots$ will be used. For each point $x = (x_1, x_2, \ldots) \in \mathscr{X}^\infty$, define $y = (y_1, y_2, \ldots)$ and $z = (z_1, z_2, \ldots)$ by means of the identity $x = (y_1, z_1, y_2, z_2, \ldots)$. Let A be a fixed but arbitrary invariant event, and let A_z denote the section of A defined by $\{y \in \mathscr{X}^\infty : x = (y_1, z_1, y_2, \ldots) \in A\}$, $z \in \mathscr{X}^\infty$. Applying Fubini's theorem, one has

$$\tau(A) = \int_{\mathscr{X}^\infty} \mu(A_z)\, v(dz). \tag{1}$$

Since A is an invariant event, each set A_z is an invariant event as well. Consequently, $\mu(A_z) = 0$ or 1 for each $z \in \mathscr{X}^\infty$. Moreover, A_z is unaltered by finite permutations of the components in z. Thus $E = \{z : \mu(A_z) = 1\}$ is an invariant event. Hence, $v(E) = 0$ or 1. In the first case, it follows from (1) that $\tau(A) = 0$, and, in the second case, that $\tau(A) = 1$. Therefore $\tau = \mu \times v \in \mathscr{H}$.

The proof of the next closure property is similar to the last, but is more complicated.

Closure Property 4. *If $\mu \in \mathscr{H}$ and v is a probability measure on $(\mathscr{X}, \mathscr{A})$ that is absolutely continuous with respect to one of the components of μ, then $\tau = v \times \mu \in \mathscr{H}$.*

Proof. Because of closure property 2, it will be assumed, without loss of generality, that $\nu \ll \mu_1$. Furthermore, it will be assumed that $\nu = \mu_1$. For, then, the general case $\nu \ll \mu_1$ will follow from this special case by means of closure property 1. For any event $A \in \mathscr{A}^\infty$, Fubini's theorem yields

$$\tau(A) = \int_{\mathscr{X}} \mu(A_z)\, \mu_1(dz), \tag{2}$$

where A_z is the section $\{y = (y_1, y_2, \ldots) \in \mathscr{X}^\infty : (z, y_1, y_2, \ldots) \in A\}, z \in \mathscr{X}$. Applying Fubini's theorem again, one obtains

$$\mu(A_z) = \int_{\mathscr{X}} \mu^{(1)}(A_{z, y_1})\, \mu_1(dy_1), \qquad z \in \mathscr{X}, \tag{3}$$

where $\mu^{(1)} = \mu_2 \times \mu_3 \times \cdots$ and A_{z, y_1} is the section $\{(y_2, y_3, \ldots) \in \mathscr{X}^\infty : (y_1, y_2, \ldots) \in A_z\}$. Suppose A is any invariant event. Then each A_z is an invariant event, and $\mu(A_z) = 0$ or 1. It follows from (2) that $\tau(A) = \mu_1(E)$, where $E = \{z \in \mathscr{X} : \mu(A_z) = 1\}$.

It will now be shown that $\mu_1(E) = 0$ or 1, from which it will follow that $\tau \in \mathscr{H}$. Let E^c denote the complement of E (relative to $\mathscr{X}$). From (3), it follows that

$$\int_{E^c \times \mathscr{X}} \mu^{(1)}(A_{z, y_1})\, \mu_1 \times \mu_1(d(z, y_1)) = \int_{E^c} \mu(A_z)\, \mu_1(dz) = 0.$$

Thus

$$\mu^{(1)}(A_{z, y_1}) = 0 \quad \text{on } E^c \times E \quad \text{a.e.} \quad (\mu_1 \times \mu_1). \tag{4}$$

Similarly, one can show that

$$\mu^{(1)}(A_{z, y_1}) = 1 \quad \text{on } E \times E^c \quad \text{a.e.} \quad (\mu_1 \times \mu_1). \tag{5}$$

But, since A is an invariant event, $A_{z, y_1} = A_{y_1, z}$ for each z and y_1, in $\mathscr{X}$. This leads to an incompatibility of (4) and (5) unless $\mu_1(E)\, \mu_1(E^c) = 0$, i.e., unless $\mu_1(E) = 0$ or 1.

Closure property 4 provides a condition under which membership is preserved when a component probability measure is added to a member of $\mathscr{H}$. Some such condition is necessary as the next example demonstrates.

Example 2. Suppose $\mathscr{X} = \{0, 1\}$ and $\mathscr{A}$ is the power set of $\mathscr{X}$. Let $\mu_n(\{0\}) = 1$ for $n \geqq 1$. Then $\mu = \mu_1 \times \mu_2 \times \cdots \in \mathscr{H}$. Suppose $\nu(\{0\}) = \nu(\{1\}) = 1/2$ and $\tau = \nu \times \mu$. Let A denote the invariant event $o((1, 0, 0, \ldots))$ (the orbit of $(1, 0, 0, \ldots)$). Obviously, $\tau(A) = 1/2$ and, consequently, $\tau \notin \mathscr{H}$.

Closure property 4 can be used repeatedly to add any finite number of components to a probability measure μ in $\mathscr{H}$ without removing it from $\mathscr{H}$. This fact is stated precisely with the appropriate assumptions as follows:

Closure Property 4′. *If $\mu \in \mathscr{H}$ and $\nu = \nu_1 \times \cdots \times \nu_n$ is a product probability measure on $(\mathscr{X}^n, \mathscr{A}^n)$ (for some finite $n \geqq 1$) each component of which is absolutely continuous with respect to some component of μ, then $\nu \times \mu \in \mathscr{H}$.* (Here, $(\mathscr{X}^n, \mathscr{A}^n)$ is the measurable space generated by n replicates of $(\mathscr{X}, \mathscr{A})$.)

This result can be extended to include probability measures ν containing a countably infinite number of components. One simply must combine closure properties 4′ and 5 (the latter given below).

Before we state closure property 5, we must introduce some new notation and a convention. For two probability measures $\mu=\mu_1\times\mu_2\times\cdots$ and $\nu=\nu_1\times\nu_2\times\cdots$ in $\mathcal{M}$, we shall write $\mu\subseteq\nu$ if every component of μ is a component of ν with the frequencies of the repetitions in ν as great as those in μ. Expressed alternatively, μ can be produced from ν by (possibly) deleting some of the components of ν and rearranging those which remain. If $\mu\subseteq\nu$ and $\nu\subseteq\mu$, then μ can be produced from ν by permuting the components of ν. Then closure property 2 says that $\mu\in\mathcal{H}$ iff $\nu\in\mathcal{H}$. Since membership or nonmembership in $\mathcal{H}$ is our sole focus of attention, there is no harm done by treating μ and ν as equivalent. With this convention, $\mu\subseteq\nu$ and $\nu\subseteq\mu\Rightarrow\mu=\nu$. Now suppose $\mu^{(i)}\in\mathcal{M}$ and $\mu^{(i)}\subseteq\nu$ for some $\nu\in\mathcal{M}$, $i\in I$. If μ is the smallest such member ν of $\mathcal{M}$, i.e., if μ is such a ν and $\mu\subseteq\nu$ for every such ν, then we shall write $\mu=\bigcup_{i=I}\mu^{(i)}$. It is easily seen that such a μ always exists when I is nonempty and countable, and that it is unique up to an equivalence.

Closure Property 5. *If $\mu^{(n)}\in\mathcal{H}$ and $\mu^{(n)}\subseteq\mu^{(n+1)}$ for $n\geqq 0$, then $\mu=\bigcup_{n\geqq 0}\mu^{(n)}\in\mathcal{H}$.*

Proof. The essence of the general argument can be seen in the proof of the following special case: Let $\mu^{(0)}=\nu=\nu_1\times\nu_2\times\cdots$ and, for $n\geqq 1$, $\mu^{(n)}=\nu_1\times\tau_1\times\nu_2\times\tau_2\times\cdots\times\nu_n\times\tau_n\times\nu_{n+1}\times\nu_{n+2}\times\cdots$, where the ν_i's and τ_i's are probability measures on $(\mathcal{X},\mathcal{A})$. Then $\mu=\bigtimes_{n=1}^{\infty}(\nu_n\times\tau_n)$. Let A be any invariant event. For each $n\geqq 0$, Fubini's theorem yields

$$\mu(A)=\int_{\mathcal{X}^\infty}\mu^{(n)}(A_{z_{n+1},z_{n+2},\ldots})\bigtimes_{k=n+1}^{\infty}\tau_k(d(z_{n+1},z_{n+2},\ldots)),\tag{6}$$

where each section

$$A_{z_{n+1},z_{n+2},\ldots}=\{(y_1,z_1,\ldots,y_n,z_n,y_{n+1},y_{n+2},\ldots)\in\mathcal{X}^\infty\colon (y_1,z_1,y_2,z_2,\ldots)\in A\}$$

is always an invariant event.

Let $E_n=\{(z_{n+1},z_{n+2},\ldots)\in\mathcal{X}^\infty\colon \mu^{(n)}(A_{z_{n+1},z_{n+2},\ldots})=1\}$, $n\geqq 0$. On E_n^c (the complement of E_n relative to $\mathcal{X}^\infty$),

$$0=\mu^{(n)}(A_{z_{n+1},z_{n+2},\ldots})=\int_{\mathcal{X}^n}\nu(A_z)\bigtimes_{k=1}^{n}\tau_k(d(z_1,\ldots,z_n)),$$

where $z=(z_1,z_2,\ldots)$. Thus, $\int_{\mathcal{X}^n\times E_n^c}\nu(A_z)\,\tau(dz)=0$, $n\geqq 0$, where $\tau=\tau_1\times\tau_2\times\cdots$. Similarly, $\int_{\mathcal{X}^n\times E_n}(1-\nu(A_z))\,\tau(dz)=0$, $n\geqq 0$. Hence, for $n\geqq 0$,

$$\begin{aligned}\nu(A_z)&=0\quad\text{on }\mathcal{X}^n\times E_n^c\text{ a.e. }\tau\\&=1\quad\text{on }\mathcal{X}^n\times E_n\text{ a.e. }\tau.\end{aligned}$$

Therefore $\tau(E_0 \Delta(\mathscr{X}^n \times E_n))=0$ for $n \geqq 0$. This says that E_0 is τ-tail-approximable. (See Blum and Pathak (1972) for the meaning of tail approximability.) Thus E_0 is τ-equivalent to a tail event and, hence, $\tau(E_0)=0$ or 1. Then for the case $n=0$ in (6),

$$\mu(A)=\int_{E_0} \nu(A_z)\,\tau(dz)=\tau(E_0)=0 \quad \text{or } 1.$$

3. An Application

We are now in a position to say much more about independent Bernoulli random variables than earlier descriptions of the Hewitt-Savage zero-one property permitted. The study of such random variables is equivalent to the study of probability measures $\mu=\mu_1 \times \mu_2 \times \cdots \in \mathscr{M}$ when $\mathscr{X}=\{0,1\}$ and $\mathscr{A}$ is its power set. Clearly the membership or nonmembership of μ in $\mathscr{H}$ depends completely upon the sequence of parameters

$$p_n=\mu_n(\{1\}), \qquad n \geqq 1.$$

We shall demonstrate the following: *If* $\{p_n\}$ *possesses a limit point* $p \in (0,1)$, *then* $\mu \in \mathscr{H}$.

Proof. Let $\nu=\nu_1 \times \nu_2 \times \cdots \in \mathscr{M}$ be such that $p=\nu_n(\{1\})$ for all n. By the classical Hewitt-Savage zero-one law, $\nu \in \mathscr{H}$. Now, by assumption, some subsequence $p_{n'}$ of p_n converges to p. There exists a further subsequence $p_{n''}$ which converges to p so rapidly that the corresponding product probability $\tau=\tau_1 \times \tau_2 \times \cdots$ consisting of components $\mu_{n''}$ taken from μ, is absolutely continuous with respect to ν. (It is sufficient that $\sum_{n''}(p-p_{n''})^2<\infty$. This follows from Kakutani's theorem, referred to previously.) According to closure property 1, $\tau \in \mathscr{H}$. In turn, it follows from closure properties 4′ and 5 that $\mu \in \mathscr{H}$.

The following is a partial converse: *If* $0<\sum_{n=1} \min(p_n, 1-p_n)<\infty$, *then* $\mu \notin \mathscr{H}$.

Proof. If $\sum_{n=1}^{\infty} \min(p_n,(1-p_n))<\infty$, then it follow from the Borel-Cantelli lemma that μ has a countable support. Specifically, every point $x=(x_1, x_2, \ldots)$ in the support is such that for some large N, depending on the point, and every $n \geqq N$, $x_n=0$ if $p_n \leqq 1/2$, and $x_n=1$ if $p_n>1/2$. Let x be a point in the support and $A=o(x)$ (the orbit of x). A point $y \in \mathscr{X}^\infty$ belongs to A iff the sum $\sum_{n=1}^{\infty}(y_n-x_n)$ has a finite number of nonzero terms and equals zero. If $\sum_{n=1}^{\infty} \min(p_n,(1-p_n))>0$, there exists an index m for which p_m is neither zero nor one. In such a case, the point $y=(x_1, x_2, \ldots, x_{m-1}, (1-x_m), x_{m+1}, x_{m+2}, \ldots)$ must be in the support of μ but outside the set A. Then $0<\mu(\{x\}) \leqq \mu(A) \leqq 1-\mu(\{y\})<1$. Thus, $\mu \notin \mathscr{H}$.

Our results do not determine whether μ belongs to $\mathscr{H}$ when, for instance, $p_n=1/n$, $n \geqq 1$. By using an approach that is very different from ours, Aldous and

Pitman (1977)[1] have recently settled the membership issue for every sequence $\{p_n\}$. When $\mathscr{X}$ is finite, they have shown that $\mu \in \mathscr{H}$, if and only if, the sum $\sum_{n=1}^{\infty} \min(\mu_n(A), \mu_n(A^c))$ is 0 or ∞ for each set $A \in \mathscr{A}$. Thus, in the present context, $\mu \in \mathscr{H}$ if, and only if, $\sum_{n=1}^{\infty} \min(p_n, 1-p_n)$ is 0 or ∞.

4. Conclusions

We view the collection of results in this paper as a modest beginning. There remain many other issues that need to be settled before a very clear picture of $\mathscr{H}$ can emerge. For instance, it is (implicitly) suggested by all previous papers concerned with the Hewitt-Savage zero-one law that the following should be true: If $\mu = \mu_1 \times \mu_2 \times \cdots \in \mathscr{H}$, then $\mu^1 = \mu_2 \times \mu_3 \times \cdots \in \mathscr{H}$. We have no idea whether this is true. Nor do we know whether the conditions $\mu^{(i)} \in \mathscr{H}$, $i=1,2$, imply $\bigcup_{i=1}^{2} \mu^{(i)} \in \mathscr{H}$. In spite of the many unanswered questions, our results demonstrate conclusively that the Hewitt-Savage zero-one property holds much more widely than other literature[2] on the subject has suggested.

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[1] The author was unaware of their manuscript when this paper appeared in its original unpublished form.

[2] An exception must be made for the recent manuscript by Aldous and Pitman (1977) referred to earlier.