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A Formal Approach to Stochastic Stability

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The characteristic function of a stable probability distribution has a very specific structure. It is the purpose of the present paper to show that the stability requirement, if stated for characteristic functions, is so strong that distinctive features of this structure can be obtained without using the positive-definiteness of the characteristic function at all, or by using it only minimally.

I. Statements

A standard characteristic function $\varphi(\alpha)$, that is a Fourier Transform

$$\varphi(\alpha)=\int_{-\infty}^{+\infty} e^{i\alpha}\, dF(x), \qquad dF\geqq 0, \qquad \int_{-\infty}^{\infty} dF=1 \tag{1}$$

of a probability distribution in $(-\infty,\infty)$, is usually envisaged on the entire line

$$-\infty<\alpha<\infty, \tag{2}$$

but, due to

$$\varphi(-\alpha)=\overline{\varphi(\alpha)}, \tag{3}$$

it is determined by its values on the closed half-line

$$0\leqq\alpha<\infty. \tag{4}$$

It will suit our purposes to envisage all our functions $\varphi(\alpha)$, whether they will be characteristic functions or not, primarily on this half-line (4), and only secondarily on the entire line (2); and whenever their occurrence on the entire line will be required by the context, it will be taken for granted that they have been extended from the half-line (4) to the entire line (2) by means of (3). The reader is asked to keep this constantly in mind.

We are now attaching the concept of stability directly to functions in the half-line (4) in the following way.

Definition 1. We call a function $\varphi(\alpha)$ *stable* if

1. it is complex-valued and continuous in (4)

2. $\varphi(0)=1,$ (5)

3. $|\varphi(\alpha)|\leqq 1,$ (6)

and, what is decisive,

4. corresponding to every integer n, $n=1,2,\ldots$, there is a positive real number c_n, and a real number γ_n,

$$c_n>0, \qquad -\infty<\gamma_n<\infty \tag{7}$$

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such that

$$\varphi(\alpha)^n = \varphi(c_n \alpha) \cdot e^{i\gamma_n \alpha}, \qquad n=1,2,\ldots. \tag{8}$$

We emphasize that this definition does not stipulate at all that $\varphi(\alpha)$ shall be positive-definite, in the sense that

$$\sum_{m,n=1}^{N} \varphi(\alpha_m - \alpha_n)\, z_m \bar{z}_n \geqq 0, \qquad N=3,4,\ldots. \tag{9}$$

that is that $\varphi(\alpha)$ shall be a Fourier Transform (1). If $\varphi(\alpha)$ happens to be such a transform, then it is stable in the sense of Definition 1, if and only of the corresponding distribution $F(x)$ is stable in the familiar stochastic meaning of the term, see Gnedenko-Kolmogorov [1], Chapter 7. We also observe that in (9) we do not begin with $N=2$ but only with $N=3$, because condition (9) for $N=2$ is equivalent with the pair of properties $\varphi(-\alpha)=\overline{\varphi(\alpha)}$, $|\varphi(\alpha)| \leqq \varphi(0)$, which however we have stipulated as basic properties of our functions in general, without reference to positive-definiteness as such.

Now, without stipulating (9) at all, the following structure theorem holds.

Theorem 1. *Any stable function $\varphi(\alpha)$ has the following two properties*

(1) *It is different from* 0 *for all α, so that*

$$\varphi(\alpha) = e^{-\psi(\alpha)}, \tag{10}$$

where $\psi(\alpha)$ is a continuous complex-valued function in $0 \leqq \alpha < \infty$ which is uniquely determined by the requirement $\psi(0)=0$.

(2) *Either $\psi(\alpha)$ is "degenerate", in the sense that*

$$\psi(\alpha) = i\,C\alpha, \qquad -\infty < C < \infty, \tag{11}$$

or there exists an exponent p,

$$0 < p < \infty \tag{12}$$

such that

$$c_n = n^{1/p}, \qquad n=1,2,\ldots \tag{13}$$

and

$$\psi(\alpha) = (A+iB)\,\alpha^p + i\,C\alpha, \qquad \text{for } 0<p<1,\ 1<p<\infty \tag{14}$$

$$\psi(\alpha) = (A+iB\log\alpha)\,\alpha + i\,C\alpha \qquad \text{for } p=1 \tag{15}$$

where

$$A \geqq 0, \qquad -\infty < B < \infty,\ -\infty < C < \infty \tag{16}$$

and

$$A+iB \neq 0. \tag{17}$$

Conversely, a function $\varphi(\alpha)$ having properties (1) *and* (2) *is stable* (*in the sense of our Definition* 1).

The converse is very easy, and we leave its verification to the reader.

With regard to "degeneracy" we wish to make the following remarks. Firstly, relation (11) also subsumes the case $\psi(\alpha)=0$, namely for $C=0$. Secondly, if we would forgo the requirement (17), that is if we would allow $A=0$, $B=0$ in (14) and (15), then the degenerate functions would fall under (14) and (15) for all p in (12). And thirdly, if $\varphi(\alpha)$ is a characteristic function then the "degenerate" case corresponds to the Bernoulli case of a distribution $F(x)$ which is concentrated in one point only.

Turning now to the substance of Theorem 1 we note as follows.

In the stochastic case, that is, if $\varphi(\alpha)$ satisfies the positivity requirement (9) for all N, then the quantities p, A, B, which occur in Theorem 1, are subject to the following limitations

$$0 < p \leqq 2, \tag{18}$$

$$|B| \leqq A \cdot \left|\tan \frac{\pi}{2} p\right|, \quad \text{for } 0<p<1,\ 1<p\leqq 2, \tag{19}$$

$$|B| \leqq A \frac{2}{\pi}, \qquad \text{for } p=1, \tag{20}$$

in which the bounds

$$\left|\tan \frac{\pi}{2} p\right|, \quad \frac{2}{\pi} \tag{21}$$

cannot be replaced by smaller ones, see, for instance Gnedenko-Kolmogorov [1], p. 164.

It would be too much to expect that these limitations, with the precise bounds (21), can be obtained without the use of the Fourier representation (1) itself, that is without using, indirectly, the positivity requirement (9) in its entirety, that is for all N. But we are going to show that a "minimal" part of requirement (9) suffices to secure the limitation (18) for p precisely, and the limitations (19) and (20) imprecisely, that is with bounds larger than (21). And even this imprecision will be relatively "harmless" inasmuch as it will not occur in the interesting case $p=2$. In this case our bound will also be 0 as it is in the stochastic relation (19). This will be stated expressly in Theorem 2 which will be formulated after the introduction of a prerequisite definition.

Definition 2. A function $\varphi(\alpha)$ is minimally positive-definite if it satisfies the inequality

$$\sum_{m,n=1}^{3} \varphi(\alpha_m - \alpha_n)\, z_m\, \bar{z}_n \geqq 0 \tag{22}$$

for all triples of points

$$\alpha_1 = 0, \qquad \alpha_2 = \alpha, \qquad \alpha_3 = 2\alpha, \quad 0 < \alpha < \infty. \tag{23}$$

That is, we require (9) only for $N=3$, and only for the triples (23).

Theorem 2. *If a stable function $\varphi(\alpha)$ is minimally positive-definite, then the exponent p of Theorem* 1 *lies in the segment* (18), *and the constants A and B of Theorem* 1 *are subject to a limitation*

$$|B| \leqq M_p A, \qquad 0 < p \leqq 2 \tag{24}$$

where

$$M_p = \frac{|1-2^{2-p}|^{\frac{1}{2}}}{|1-2^{1-p}|} \qquad \text{for } 0<p<1,\ 1<p\leqq 2 \tag{25}$$

and

$$M_1 = \frac{1}{\log 2}. \tag{26}$$

For $p=2$, $M_p=0$, and the consequence of this is worth stating, for once, for our functions $\varphi(\alpha)$ as given on the entire line $-\infty<\alpha<\infty$ by means of (3).

Corollary to Theorem 2. *If a function*

$$\varphi(\alpha) = \exp\left[-\left(A + iB\frac{|\alpha|}{\alpha}\right)\alpha^2 - iC\alpha\right], \qquad -\infty<\alpha<\infty \tag{27}$$

is minimally positive-definite then $B=0$ for whatever A. For all other p in (18) *we can only state that $B=0$ if $A=0$.*

Before turning to the proof of Theorems 1 and 2 in the next two sections we will terminate this section with a contribution, a rather secondary one, to the analysis which, in the fully stochastic case brings about the sharp necessary and sufficient conditions (19) and (20). This analysis hinges (see for instance Gnedenko-Kolmogorov [1], § 34) on the actual computation of the integral

$$I_p(\alpha) = \int_0^\infty \left(1 + i\alpha\lambda(x) - e^{i\alpha x}\right)\frac{dx}{x^{1+p}}, \qquad 0<p<2, \tag{28}$$

in which $\lambda(x)$ is any real-valued measurable function in $0<x<\infty$ for which

$$|\lambda(x)| \leqq M < \infty, \qquad 0<x<\infty,$$

$$\lambda(x) = x + 0(x^2), \quad \text{at} \quad x=0;$$

the value of the integral for $0<\alpha<\infty$ being

$$I_p(\alpha) = \frac{\Gamma(1-p)}{p}\alpha^p e^{-i\frac{\pi}{2}p} + iC_p\alpha, \qquad 0<p<1,\ 1<p<2, \tag{29}$$

$$I_p(\alpha) = \alpha\left[\frac{\pi}{2} + i\log\alpha\right] + iC_1\alpha, \qquad p=1 \tag{30}$$

where C_p is a real-valued constant whose value depends on the choice of the auxiliary function $\lambda(x)$.

Now, we are going to give our own computation of $I_p(\alpha)$, for $0\leqq\alpha<\infty$, which is more systematic and polished than known ones. We introduce the open complex half-plane

$$z=\sigma-i\alpha, \qquad 0<\sigma<\infty,\ -\infty<\alpha<\infty, \tag{31}$$

and in it the function

$$\Phi(z) = \int_0^\infty \left(1 - z\lambda(x) - e^{-zx}\right)\frac{dx}{x^{1+p}}. \tag{32}$$

This integral is boundedly convergent in the neighborhood of every point of (31), so that $\Phi(z)$ is holomorphic, and it can be easily seen that for all points of $0\leqq\alpha<\infty$, we have

$$I_p(\alpha) = \lim \Phi(\sigma - i\alpha), \qquad \sigma\downarrow 0, \tag{33}$$

where $I_p(0)=0$.

By Vitali, we can differentiate (32) under the integral arbitrarily often. In particular,

$$\Phi''(z) = -\int_0^\infty e^{-zx} x^{1-p} dx$$
$$= -\Gamma(2-p)\, z^{p-2},$$

where $z^{p-2} = e^{(p-2)\log z}$ and $\log z$ is determined uniquely in (31) by the convention that it shall be real for real z. Therefore

$$\Phi(z) = \frac{\Gamma(1-p)}{p} z^p - Cz + E, \qquad 0<p<1,\ 1<p<2,$$
$$\Phi(z) = -z\log z - Cz + E, \qquad p=1.$$

Now, C is real because $\Phi(z)$ is real for positive real z, and $E=0$ because $\Phi(z)$ tends to 0, as z tends to 0 along the positive real axis. But for $z=-i\alpha$, $\alpha>0$,

$$z^p = (-i\alpha)^p = \alpha^p e^{-i\frac{\pi}{2}p}$$
$$-z\log z = i\alpha \log(-i\alpha) = i\alpha\left[\log\alpha - i\frac{\pi}{2}\right],$$

whence the formulas (29) and (30).

II. Proof of Theorem 1

If contrary to our assertion there are points in $0<\alpha<\infty$ where $\varphi(\alpha)=0$ then consider the set S of all such points. Since relation (8) implies

$$|\varphi(\alpha)|^n = |\varphi(c_n\alpha)|, \qquad n=1,2,\dots$$

and hence, also

$$\left|\varphi\left(\frac{\alpha}{c_n}\right)\right|^n = |\varphi(\alpha)|, \qquad n=1,2,\dots,$$

it follows that $c_n S \subset S$ and $\frac{1}{c_n} S \subset S$. But, $\varphi(\alpha)$ being continuous, S is closed. Therefore $c_n=1$, so that

$$|\varphi(\alpha)| = |\varphi(\alpha)|^n, \qquad n=1,\dots.$$

Letting $n\to\infty$ it follows that $|\varphi(\alpha)|$ assumes values 1 and 0 and no others, which is incompatible with continuity.

Thus we have a representation (10). The exponent $\psi(\alpha)$ is a continuous complex-valued function in (4) with the properties

$$\psi(0)=0, \qquad \operatorname{Re}\psi(\alpha) \geqq 0, \tag{34}$$

$$n\psi(\alpha) = \psi(c_n\alpha) + i\gamma_n\alpha, \tag{35}$$

and from these we have to deduce part (2) of Theorem 1.

Step 1. From (35) we obtain

$$\frac{1}{m}\psi(\alpha) = \psi\left(\frac{\alpha}{c_m}\right) - i\frac{\gamma_m}{m c_m}\alpha,$$

so that

$$\frac{n}{m}\psi(\alpha)=n\psi\left(\frac{\alpha}{c_m}\right)-i\frac{n\gamma_m}{m c_m}\alpha$$
$$=\psi\left(\frac{c_n}{c_m}\alpha\right)+i\frac{\gamma_n}{c_m}\alpha-i\frac{n\gamma_m}{m c_m}\alpha.$$

Thus for every *positive rational* number r we have

$$r\psi(\alpha)=\psi(c(r)\,\alpha)+i\gamma(r)\,\alpha \tag{36}$$

where $c(r)>0$ and $\gamma(r)$ is real.

Step 2. Barring (11), these numbers $c(r)$, $\gamma(r)$ are unique.

In general, given an identity

$$\psi(c'\alpha)=i\gamma'\alpha=\psi(c''\alpha)+i\gamma''\alpha \tag{37}$$

for positive c', c'', and real γ', γ'', then, barring (11), $c'=c''$, and thus also $\gamma'=\gamma''$.

In fact, (37) implies

$$\psi(\alpha)-\psi(q\alpha)=i\delta\alpha \tag{38}$$

where $q=\frac{c''}{c'}$, $\delta=\frac{\gamma''-\gamma'}{c'}$. Now, assume for instance $c''<c'$, so that $0<q<1$. From (38) we obtain

$$\psi(q^m\alpha)-\psi(q^{m+1}\alpha)=i\delta q^m\alpha,$$

and summation over m leads to

$$\psi(\alpha)-\psi(q^n\alpha)=i\delta\frac{1-q^n}{1-q}\alpha.$$

Letting $n\to\infty$, we obtain, by (34), $\psi(\alpha)=\frac{i\delta}{1-q}\alpha$, and this is (11).

Step 3. If $\psi(\alpha)$ is non-degenerate, then for bounded $\{r\}$ the $\{c(r)\}$ are bounded.

Otherwise there would exist a sequence $\{r_n\}$ such that

$$r_n\leqq\zeta<\infty,\qquad c(r_n)\to\infty. \tag{39}$$

Now, (36) would lead to

$$r_n\psi\left(\frac{\alpha}{c(r_n)}\right)=\psi(\alpha)+iC_n\alpha \tag{40}$$

where

$$C_n=\frac{\gamma_n}{c(r_n)}. \tag{41}$$

But, by (39) and (34) the left side in (40) is convergent, namely to 0, therefore C_n must be convergent, to a constant C_0. Thus (40) leads to $\psi(\alpha)=-iC_0\alpha$, with is (11).

Step 4. Barring (11), if a positive real number ζ is the limit of a sequence of rational numbers $\{r_n\}$, then the corresponding $c(r_n)$ are convergent.

In fact, by Step 3. the $c(r_n)$ are bounded, and if they would not converge, there would be two subsequences $\{r'_m\}$, $\{r''_m\}$ of $\{r_n\}$ such that

$$c(r'_m)\to c',\qquad c(r''_m)\to c''$$

with $c' \neq c''$. The relation

$$r_n \psi(\alpha) = \psi(c_n \alpha) + i \gamma_n \alpha \tag{42}$$

would then lead to two separate limit relations

$$\zeta \psi(\alpha) = \psi(c' \alpha) + i \gamma' \alpha,$$
$$\zeta \psi(\alpha) = \psi(c'' \alpha) + i \gamma'' \alpha.$$

Now, apply Step 2.

Step 5. For a non-degenerate $\psi(\alpha)$, Step 4. extends relation (36) from all positive rational numbers to all positive real number r, the coefficients $c(r)$ being strictly positive and uniquely determined. Furthermore, $c(r)$ is continuous for all positive real r.

In fact, if $\{\zeta_n\}$ and ζ_0 are positive real numbers with $\zeta_n \to \zeta_0$, then there are positive rational numbers $\{r_n\}$ with

$$|r_n - \zeta_n| < \frac{1}{n}, \qquad |c(r_n) - c(\zeta_n)| < \frac{1}{n},$$

and the continuity of $c(r)$ follows from preceding steps.

Step 6. Barring (11), $c(r) \to 0$ as $r \downarrow 0$, and $c(r) \to \infty$ as $r \to \infty$.

In fact, if $r_n \downarrow 0$, then $c(r_n)$ and $\gamma(r_n)$ are bounded by Step 3. For a subsequence $\{r'_m\}$ there exist limits

$$c(r'_m) \to c(+0), \qquad \gamma(r'_m) \to \gamma(+0).$$

Now, (42) gives

$$\psi\big(c(+0)\alpha\big) + i\gamma(+0)\,\alpha = 0,$$

and if $c(+0) > 0$ then this is relation (11).

Similarly, if for $r_n \to \infty$ we had $c(r_n) \to l < \infty$ then

$$\psi(\alpha) = \frac{1}{r_n} \psi\big(c(r_n)\alpha\big) + i \frac{\gamma(r_n)}{r_n} \alpha$$

would result in $\psi(\alpha) = i C \alpha$ which is a relation (11).

Gathering up all the information about $c(r)$ we find that, barring (11), the function

$$c(r) = \beta, \qquad 0 \leqq r < \infty$$

has an inverse

$$r = \chi(\beta), \qquad 0 \leqq \beta < \infty,$$

so that (36) can be put into the form

$$\chi(\beta)\, \psi(\alpha) = \psi(\alpha\beta) + i\delta(\beta)\,\alpha, \qquad [\delta(\beta) = \gamma(\chi(\beta))] \tag{43}$$

with which we will deal henceforth.

For $\alpha = 1$ we obtain $i\delta(\beta) = \chi(\beta)\,\psi(1) - \psi(\beta)$, and substituting this back into (43) gives

$$\chi(\beta)\,\psi(\alpha) = \psi(\beta\alpha) + \big(\chi(\beta)\,\psi(1) - \psi(\beta)\big)\,\alpha.$$

Next, if we interchange α, β and subtract, so as to eliminate $\psi(\beta\alpha)$, we are led to

$$\big(\chi(\beta) - \beta\big)\big(\psi(\alpha) - \psi(1)\alpha\big) = \big(\chi(\alpha) - \alpha\big)\big(\psi(\beta) - \psi(1)\beta\big). \tag{44}$$

Now, either

$$\psi(\alpha)-\psi(1)\alpha=0, \qquad \text{for all } \alpha \tag{45}$$

or,

$$\psi(\alpha_0)-\psi(1)\,\alpha_0 \neq 0, \qquad \text{for some } \alpha_0. \tag{46}$$

Since, $\psi(1)=A+iC$ with $A \geqq 0$ by (34), relation (45) falls under (15) with $B=0$, and we need only examine (46). But under (46), relation (44) becomes

$$\chi(\beta)=D\psi(\beta)+E\beta, \tag{47}$$

where

$$D=\frac{\chi(\alpha_0)-\alpha_0}{\psi(\alpha_0)-\psi(1)\alpha_0}, \qquad E=1-D\psi(1),$$

and we now have to separate the cases $D=0$ and $D \neq 0$.

If $D=0$, then $\chi(\beta)=\beta$ for all β, and (43) turns into

$$\psi(\alpha\beta)=\beta\psi(\alpha)-i\delta(\beta)\alpha.$$

For $\alpha=1$, we obtain $\psi(\beta)=\beta\psi(1)-i\delta(\beta)$; thus

$$\psi(\alpha\beta)=\beta\psi(\alpha)+\alpha\psi(\beta)-\alpha\beta\psi(1),$$

and this means that the function

$$\rho(\alpha)=\frac{\psi(\alpha)}{\alpha}-\psi(1)$$

satisfies the functional equation

$$\rho(\alpha\beta)=\rho(\alpha)+\rho(\beta).$$

Therefore

$$\rho(\alpha)=(R+iB)\log\alpha,$$

or

$$\begin{aligned}\psi(\alpha)&=(R+iB)\,\alpha\log\alpha+\psi(1)\,\alpha\\&=(R+iB)\,\alpha\log\alpha+(A+iC)\,\alpha\end{aligned} \tag{48}$$

and $\operatorname{Re}\psi(\alpha)=\alpha(R\log\alpha+A)$. Now, this can be $\geqq 0$ for all positive α only if $R=0$, and, by (48), $\psi(\alpha)$ falls again under (15).

If $D \neq 0$, the insertion of (47) into (43) gives

$$D\psi(\beta)\,\psi(\alpha)+E\beta\psi(\alpha)=\psi(\beta\alpha)+i\delta(\beta)\alpha. \tag{49}$$

Finding $i\delta(\beta)$ by putting $\alpha=1$, and reinserting into (49) leads to

$$D\psi(\beta)\,\psi(\alpha)+E\beta\psi(\alpha)+E\alpha\psi(\beta)-E\psi(1)\,\alpha\beta=\psi(\alpha\beta),$$

and the function

$$\sigma(\alpha)=D\,\frac{\psi(\alpha)}{\alpha}+E \tag{50}$$

satisfies the functional equation

$$\sigma(\alpha\beta)=\sigma(\alpha)\cdot\sigma(\beta).$$

The latter has, for $\alpha>0$, and exceptional solution

$$\sigma(\alpha)=0 \tag{51}$$

and a general solution

$$\sigma(\alpha)=\alpha^z=e^{z\log\alpha} \tag{52}$$

in which z is any fixed complex number whatsoever. The solution (51) leads to

$$\psi(\alpha)=(A+iC)\,\alpha$$

which falls again under (15), but the solution (52) leads to

$$\psi(\alpha)=(A+iB)\,\alpha^{z+1}+(P+iC)\,\alpha, \qquad (A+iB)\neq 0, \tag{53}$$

which finally points towards (14).

Replacing α by $\alpha\beta$ in (53) gives

$$\psi(\alpha\beta)=\beta^{z+1}(A+iB)\,\alpha^{z+1}+\beta(P+iC)\,\alpha,$$

and, by (43), we also have

$$\psi(\alpha\beta)=\chi(\beta)\,(A+iB)\,\alpha^{z+1}+\chi(\beta)\,(P+iC)\,\alpha-i\delta(\beta)\,\alpha,$$

where $\chi(\beta)$ and $\delta(\beta)$ are both real valued. Therefore $z+1$ must be real valued, so that

$$\psi(\alpha)=(A+iB)\,\alpha^p+(P+iC)\,\alpha \qquad (A+iB)\neq 0.$$

where p is real valued, and $\chi(\beta)=\beta^p$. But we must also have $\psi(\alpha)\downarrow 0$ for $\alpha\downarrow 0$, so that $0<p<\infty$. For $p=1$, this $\psi(\alpha)$ is still of the form (15). But for $p\neq 1$ it assumes the form (14), except that we still have to establish $P=0$. This, however, follows from the fact that $\psi(\alpha\beta)$ is both

$$\beta^p(A+iB)\,\alpha^p+\beta(P+iC)\,\alpha$$

and

$$\beta^p(A+iB)\,\alpha^p+\beta^p(P+iC)\,\alpha-i\delta(\beta)\,\alpha.$$

The proof of Theorem 1 is finally completed.

III. Proof of Theorem 2

As heretofore, every function $\varphi(\alpha)$ will be subject, first of all, to (5), (6), and (3).

If a square matrix, whether real or complex-valued is positive (semi-)definite then its determinant is non-negative. Thus, assumption (22) implies

$$\det|\varphi(\alpha_m-\alpha_n)|_{m,n=1,2,3}\geqq 0. \tag{54}$$

We will evaluate this not only for the special triples (23), but, somewhat more generally, for triples

$$\alpha_1=0, \qquad \alpha_2=\alpha, \qquad \alpha_3=\alpha+\beta. \tag{55}$$

If we put

$$u=\varphi(\alpha), \quad v=\varphi(\beta), \quad w=\varphi(\alpha+\beta)$$

then (54) is

$$\begin{vmatrix} 1 & \bar{u} & \bar{w} \\ u & 1 & \bar{v} \\ w & v & 1 \end{vmatrix} \geqq 0.$$

and this is

$$1-u\bar{u}-v\bar{v} \geqq w\bar{w}-uv\bar{w}-\bar{u}\bar{v}w.$$

Therefore

$$|w-uv|^2 \leqq (1-u\bar{u})(1-v\bar{v}) \tag{56}$$

or

$$|\varphi(\alpha+\beta)-\varphi(\alpha)\,\varphi(\beta)|^2 \leqq (1-|\varphi(\alpha)|^2)(1-|\varphi(\beta)|^2); \tag{57}$$

compare Hewitt-Ross, [2], p. 255.

For Theorem 2 we need (56) only for

$$\beta=\alpha, \quad \text{that is } v=u,$$

in which case (56) becomes

$$|w-u^2|^2 \leqq (1-|u|^2)^2. \tag{58}$$

By (6), $|u| \leqq 1$, so that (58) leads to

$$|w-u^2| \leqq 1-|u|^2. \tag{59}$$

Combining this with

$$|w-u^2| \geqq |u|^2-|w|$$

we obtain

$$|w| \geqq 2|u|^2-1,$$

and by squaring

$$|w|^2 \geqq 4|u|^4-4|u|^2+1.$$

Thus

$$1-|w|^2 \leqq 4|u|^2(1-|u|^2) \leqq 4(1-|u|^2),$$

that is

$$1-|\varphi(2\alpha)|^2 \leqq 4(1-|\varphi(\alpha)|^2). \tag{60}$$

For characteristic functions this inequality was introduced and used in Gnedenko-Kolmogorov, [1], § 14.

If in (60) we insert a function

$$\varphi(\alpha)=\exp\left[-(A+iB)\,\alpha^p-iC\alpha\right], \quad p>0,\ A>0 \tag{61}$$

we obtain

$$1-\exp(-2A\,2^p\alpha^p) \leqq 4(1-\exp(-2A\alpha^p)).$$

Dividing both sides by α^p and letting $\alpha\downarrow 0$ we obtain

$$2A\,2^p=4\cdot 2A,$$

that is $2^p \leqq 4$, or $p \leqq 2$, which proves the first assertion of Theorem 2.

For the further assertions of the theorem we must use (59) itself, that is

$$|\varphi(2\alpha)-\varphi(\alpha)^2|\leqq 1-|\varphi(\alpha)|^2. \tag{62}$$

If $\varphi(\alpha)$ is (61), this gives

$$e^{-2A\alpha^p}|\exp[-(A+iB)(2^p-2)\alpha^p]-1|\leqq 1-e^{-2A\alpha^p},$$

and if we divide by α^p and let $\alpha\downarrow 0$ we obtain

$$|A+iB|\cdot|2^p-2|\leqq 2A,$$

and hence

$$B^2(2^p-2)^2\leqq A^2(2^{2p}-2^{p+2}).$$

Now, if $p\neq 1$, then $2^p-2\neq 0$, and we obtain the inequality (24) with the factor (25).

For $p=1$ our function $\varphi(\alpha)$ is

$$\varphi(\alpha)=\exp[-(A+iB\log\alpha)\alpha-iC\alpha] \tag{63}$$

and (62) leads to

$$e^{-2A\alpha}|e^{-B\cdot 2\log 2\cdot\alpha}-1|\leqq 1-e^{-2A\alpha}.$$

If we divide by α and let $\alpha\downarrow 0$ we obtain

$$|B|\cdot 2\log 2\leqq 2A$$

which completes the proof of Theorem 2.

Remark 1. The statement in the corollary to Theorem 2 that $A=0$ implies $B=0$ can be stated as follows.

If a function

$$\varphi(\alpha)=e^{i\chi(\alpha)},\qquad -\infty<\alpha<\infty \tag{64}$$

$$\chi(\alpha)\ \text{real},\qquad \chi(-\alpha)=-\chi(\alpha) \tag{65}$$

is stable and minimally positive definite then it is degenerate, that is, $\chi(\alpha)=C\alpha$.

Now, in this conclusion, stability can be dispensed with entirely, provided we make the positive definiteness of $\varphi(\alpha)$ a little more than minimal, in the sense that we stipulate (54) not only for the triples (23), but also for triples (55), that is, if we may apply (57) for general α, β. In fact, (64), (65) imply

$$|\varphi(\alpha)|=|\varphi(\beta)|=1, \tag{66}$$

so that, by (57), we have

$$\varphi(\alpha+\beta)=\varphi(\alpha)\,\varphi(\beta).$$

Thus $\varphi(\alpha)$ is an almost periodic character on the line, and, as such it is indeed a function of the form $e^{iC\alpha}$.

Remark 2. We can even make a statement for any function $\varphi(\alpha)$ which satisfies the inequality (57) for general α, β, and which is such that $|\varphi(\beta_0)|=1$ for some particular $\beta_0\neq 0$.

We then obtain

$$\varphi(\alpha+\beta_0)=\varphi(\alpha)\,\varphi(\beta_0),$$

and if we put $\varphi(\beta_0)=e^{iC\beta_0}$ then

$$\varphi(\alpha)=e^{iC\alpha}\chi(\alpha)$$

where $\chi(\alpha)$ is periodic with period β_0.

For characteristic functions this is a familiar fact, though usually stated somewhat differently. See Lukacs [3], p. 25.

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