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Application of Pseudo-Boolean Programming to the Theory of Graphs

By

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Abstract. The method of pseudo-Boolean programming (given in [5], [7] and briefly described in § 2 of this paper) is used for the systematic determination of the chromatic number, of the number of internal stability, of the number of external stability and of the kernels of a finite graph.

§ 1. Introduction

The aim of the present paper is to apply the method of pseudo-Boolean programming (given in [5], [7], and briefly described in § 2 of this article) to the solution of the following problems of the theory of graphs:

— determining the number of internal stability of a graph (§ 3),
— determining the number of external stability of a graph (§ 4),
— determining the kernel of a graph (§ 5),
— determining the chromatic number of a graph (§ 6).

Throughout this paper by a graph we shall mean a finite one.

The same method of pseudo-Boolean programming was applied for finding the minimal number of rows and columns of a matrix covering its zero elements in the Hungarian method of solving transportation problems [3], [4], as well as to the minimization of Boolean functions [6], a problem arising in switching algebra.

This manuscript was ready for print when KHALED MAGHOUT's very interesting work [8] on the application of Boolean Algebra to the theory of graphs reached to us; the approaches of that paper and of the present one are of different types.

By a graph $G = (V, \varrho)$ we shall mean a finite non-empty set $V = \{v_1, \ldots, v_n\}$ of elements called vertices, and a multivalued application ϱ of V into itself. An ordered pair (v_i, v_j) of elements of V is called an edge if $v_j \in \varrho v_i$. We shall suppose that for any i, $v_i \notin \varrho v_i$.

We define for any graph $G = (V, \varrho)$ a $n \times n$ matrix $C_G = ((c_{ij}))$ by setting

$$c_{ij} = \begin{cases} 1 & \text{if} \quad v_j \in \varrho v_i \\ 0 & \text{if} \quad v_j \notin \varrho v_i . \end{cases}$$

For any set $M \subseteqq V$, the characteristic function $\chi_M(V)$ is defined by:

$$x_i^M = \chi_M(v_i) = \begin{cases} 1 & \text{if} \quad v_i \in M \\ 0 & \text{if} \quad v_i \notin M . \end{cases}$$

Thus, any set $M \subseteqq V$ is characterised with an n-tuple $(x_1^M, \ldots, x_n^M)$ of zeroes and ones.

By $|A|$ we mean the power of the set A.

The concepts of the theory of graphs are defined accordingly to [1].

For the illustration of the methods given in this paper, we shall compute in §§ 3, 4, 5 the basic numbers and in § 6 the kernels of the graph with

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and

$$\varrho v_1 = \{v_2, v_6\}, \quad \varrho v_2 = \{v_1, v_6\}, \quad \varrho v_3 = \{v_6\},$$
$$\varrho v_4 = \{v_5, v_6\}, \quad \varrho v_5 = \{v_4, v_6\}, \quad \varrho v_6 = \{v_1, v_2, v_3, v_4, v_5\}$$

(see Fig. 1).

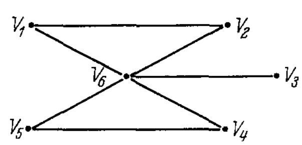

Fig. 1

The matrix $((c_{ij}))$ of this graph is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

§ 2. Pseudo-Boolean programming*

Let L_2 be the Boolean Algebra with two elements 0 and 1, its operations (the disjunction) "$\cup$", (the multiplication) "$\cdot$" and (the negation) "$-$" being defined by:

$\cup$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

a	0	1
$\bar{a}$	1	0.

We see that

$$a \cup b = a + b - ab, \tag{1}$$

and

$$\bar{a} = 1 - a$$

where addition, subtraction and multiplication are ordinary arithmetical operations.

We put as usually $x^1 = x$, $x^0 = \bar{x} = 1 - x$.

The disjunction of more variables is defined by

$$\bigcup_{i \in I} y_i = 0 \Longleftrightarrow \forall_{i \in I} (y_i = 0).$$

If $I = \emptyset$ we put $\bigcup_{i \in \emptyset} y_i = 0$.

A function
$$F : L_2^n \to R$$
is called a pseudo-Boolean function; here L_2^n is the chartesian product

$$\underbrace{L_2 \times L_2 \times \ldots \times L_2}_{n},$$

while R is the field of real numbers.

* For proofs and details see [5] and [7].

We have

$$F(x_1, x_2, \ldots, x_n) = x_1 F(1, x_2, \ldots, x_n) + \bar{x}_1 F(0, x_2, \ldots, x_n)$$
$$= x_1 F(1, x_2, \ldots, x_n) + (1 - x_1) F(0, x_2, \ldots, x_n) \quad (2)$$
$$= x_1 [F(1, x_2, \ldots, x_n) - F(0, x_2, \ldots, x_n)] + F(0, x_2, \ldots, x_n).$$

Let us denote

$$g_1(x_2, \ldots, x_n) = F(1, x_2, \ldots, x_n) - F(0, x_2, \ldots, x_n)$$
$$h_1(x_2, \ldots, x_n) = F(0, x_2, \ldots, x_n);$$

g_1 and h_1 are pseudo-Boolean functions of $x_2, x_3, \ldots, x_n$. Thus, we have the following decomposition:

$$F = x_1 g_1 + h_1. \quad (3)$$

By induction it follows that any pseudo-Boolean function may be written as a polynomial with real coefficients, linear in each variable.

The following procedure is given for the minimization of a pseudo-Boolean function F. Let us put $F_1 = F(x_1, \ldots, x_n)$.

If $F_i(x_i, x_{i+1}, \ldots, x_n)$ is defined $(1 \leqq i < n)$ we have:

$$F_i(x_i, x_{i+1}, \ldots, x_n) = x_i g_i(x_{i+1}, \ldots, x_n) + h_i(x_{i+1}, \ldots, x_n).$$

Let us denote

$$M_i = \{(\alpha_{i+1}, \ldots, \alpha_n) \in L_2^{n-i} \mid g_i(\alpha_{i+1}, \ldots, \alpha_n) < 0\},$$
$$N_i = \{(\beta_{i+1}, \ldots, \beta_n) \in L_2^{n-i} \mid g_i(\beta_{i+1}, \ldots, \beta_n) = 0\}.$$

We put

$$x_i = \bigcup_{(\alpha_{i+1}, \ldots, \alpha_n) \in M_i} x_{i+1}^{\alpha_{i+1}} \ldots x_n^{\alpha_n} \cup u_i \bigcup_{(\beta_{i+1}, \ldots, \beta_n) \in N_i} x_{i+1}^{\beta_{i+1}} \ldots x_n^{\beta_n}, \quad (4.\text{i.})$$

where u_i is an arbitrary parameter in L_2.

Let x_i^+ be the expression of x_i obtained by taking $u_i = 0$ and replacing the operation "$\cup$" with aid of formula (1).

We put $F_{i+1}(x_{i+1}, \ldots, x_n) = F_i(x_i^+, x_{i+1}, \ldots, x_n)$ and continue the above procedure until we get

$$F_n = x_n g_n + h_n.$$

We put now

$$x_n = \begin{cases} 1 & \text{if the constant } g_n < 0 \\ 0 & \text{if the constant } g_n > 0 \\ u_n & \text{if the constant } g_n = 0. \end{cases} \quad (4.\text{n.})$$

Introducing the values of x_n given by (4.n.) in (4.n—1) we obtain x_{n-1}; introducing these values of x_n and x_{n-1} in (4.n—2) we obtain x_{n-2}, etc. In this way, we obtain

$$x_i = x_i(u_i, u_{i+1}, \ldots, u_n) \qquad (i = 1, \ldots, n) \quad (5)$$

where $u_1, \ldots, u_n$ are arbitrary parameters in L_2.

It is proved that for each system of values of the parameters $u_1, u_2, \ldots, u_n$ the system (5) yields a minimum of F_1, and conversely any minimum of F_1 can be obtained in this way.

Note. The computation of $x_1, x_2, \ldots, x_n$ may be carried out in an order different from the above, if this seems to be more convenient.

Examples of application of the above procedure of minimizing a pseudo-Boolean function will be given in the following paragraphs.

§ 3. The number of internal stability

A set $R \subseteq V$ is called an internally stable set if $\varrho R \cap R = \emptyset$, i.e. if

$$v_i \in R, \quad c_{ij} = 1 \Rightarrow v_j \notin R.$$

Let $\mathfrak{R}$ be the family of all internally stable sets of a graph G; by the number of internal stability of G we mean

$$\alpha(G) = \max_{R \in \mathfrak{R}} |R|.$$

Let us denote with P_{31} *the problem of determining the number of internal stability of a graph.*

Denoting with $\chi_R(v_i) = x_i$ $(i = 1, \ldots, n)$ the values of the characteristic function of a set R, we can easily prove the following

Lemma 1. *R is an internally stable set of G if and only if*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j = 0.$$

As

$$|R| = \sum_{i=1}^{n} x_i$$

we see that problem 3.1. is equivalent with

Problem 3.2. Find values $x_i \in L_2$ $(i = 1, \ldots, n)$, subject to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j = 0 \tag{6}$$

and so that

$$-\sum_{k=1}^{n} x_k \tag{7}$$

would be minimal.

Now, let us consider

Problem 3.3. Minimize the expression

$$E = (n+1) \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j - \sum_{k=1}^{n} x_k. \tag{8}$$

with $x_i \in L_2$ $(i = 1, \ldots, n)$.

Any set $(x_1^0, \ldots, x_n^0)$ with $x_i^0 \in L_2$ subject to (6) and minimizing (7), also minimizes (8). Indeed, if

$$(n+1) \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j - \sum_{k=1}^{n} x_k < (n+1) \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i^0 x_j^0 - \sum_{k=1}^{n} x_k^0$$

then

$$(n+1)\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_i x_j < \sum_{k=1}^{n} x_k - \sum_{k=1}^{n} x_k^0$$

and as

$$\sum_{k=1}^{n} x_k - \sum_{k=1}^{n} x_k^0 < n,$$

it means that,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_i x_j = 0.$$

Therefore,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_i x_j = 0$$

and

$$-\sum_{k=1}^{n} x_k < -\sum_{k=1}^{n} x_k^0,$$

thus contradicting the definition of $(x_1^0, \ldots, x_n^0)$.

Conversely if the set $(x_1^*, \ldots, x_n^*)$ minimizes (8) then it is subject to (6). If not,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_i^* x_j^* \geqq 1$$

and $E(x_1^*, \ldots, x_n^*) \geqq 1$, contradicting $E(0, \ldots, 0) = 0 < 1 \leqq E(x_1^*, \ldots, x_n^*)$.

Thus problem 3.3. is equivalent with problem 3.2.

From the above lemma we have

Theorem I. *For any* $(x_1^0, \ldots, x_n^0)$ *minimizing*

$$E = (n+1)\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_i x_j - \sum_{k=1}^{n} x_k$$

and for

$$R^0 = \{v_i \mid v_i \in V, x_i^0 = 1\},$$

we have

$$\alpha(G) = \sum_{i=1}^{n} x_i^0 = |R^0|,$$

and any maximal internally stable set may be obtained in this way.

Thus, the problem of determining the number of internal stability of a graph is reduced to one of pseudo-Boolean programming.

Example. In our example $n = 6$, $n + 1 = 7$ and

$$E = 7(2x_1x_2 + 2x_1x_6 + 2x_2x_6 + 2x_3x_6 + 2x_4x_5 + 2x_4x_6 + 2x_5x_6) - \\ - x_1 - x_2 - x_3 - x_4 - x_5 - x_6;$$

$$g_1 = 14x_2 + 14x_6 - 1$$

and

$$x_1 = \bar{x}_2\bar{x}_6 = 1 - x_2 - x_6 + x_2x_6. \tag{9.1}$$

$$E_2 = 13x_2x_6 + 14x_3x_6 + 14x_4x_5 + 14x_4x_6 + 14x_5x_6 - x_3 - x_4 - x_5 - 1;$$

$$g_2 = 13x_6$$

and

$$x_2 = u_2 \cdot \bar{x}_6, \tag{9.2}$$

$$x_2^+ = 0. \tag{9.2+}$$

$$E_3 = -x_3 + 14x_3x_6 + 14x_4x_5 + 14x_4x_6 + 14x_5x_6 - x_4 - x_5 - 1;$$

$$g_3 = -1 + 14x_6$$

and

$$x_3 = \bar{x}_6 = 1 - x_6. \tag{9.3}$$

$$E_4 = -x_4 + 14x_4x_5 + 14x_4x_6 + 14x_5x_6 - x_5 + x_6 - 2;$$

$$g_4 = -1 + 14x_5 + 14x_6$$

and

$$x_4 = \bar{x}_5\bar{x}_6 = 1 - x_5 - x_6 + x_5x_6. \tag{9.4}$$

$$E_5 = 13x_5x_6 + 2x_6 - 3;$$

$$g_5 = 13x_6$$

and

$$x_5 = u_5\bar{x}_6, \tag{9.5}$$

$$x_5^+ = 0. \tag{9.5+}$$

$$E_6 = 2x_6 - 3;$$

$$g_6 = 2$$

and

$$x_6 = 0. \tag{9.6}$$

From (9.6), (9.5), (9.4), (9.3), (9.2), (9.1) we have

$$x_1 = \bar{u}_2, \quad x_2 = u_2, \quad x_3 = 1, \quad x_4 = \bar{u}_5, \quad x_5 = u_5, \quad x_6 = 0.$$

Hence

$$\alpha(G) = 3$$

and its maximal internally stable sets are

$$R_1 = \{v_1, v_3, v_4\}, \quad R_2 = \{v_1, v_3, v_5\},$$
$$R_3 = \{v_2, v_3, v_4\}, \quad R_4 = \{v_2, v_3, v_5\}.$$

§ 4. The number of external stability

A set $S \subseteqq V$ is called an externally stable set, if for any $s \notin S$, $\varrho s \cap S \neq \emptyset$, i.e. if

$$v_i \in S \Rightarrow (\exists)\, v_j \in S, \quad c_{ij} = 1.$$

Let $\mathfrak{S}$ be the family of all externally stable sets of a graph G; by the number of external stability of G we mean

$$\beta(G) = \min_{S \in \mathfrak{S}} |S|.$$

Denoting with $\chi_S(v_i) = x_i$ $(i = 1, \ldots, n)$ the values of the characteristic function of a set S, we can easily prove the following

Lemma 2. *S is an externally stable set of G if and only if*

$$\sum_{i=1}^{n} \prod_{j=1}^{n} (1 - c'_{ij} x_j) = 0 \tag{10}$$

where

$$c'_{ij} = c_{ij} + \delta_i^j,$$

δ_i^j being the Kronecker symbol.

An analogous reasoning as that of the previous paragraph proves

Theorem II. *For any* $(x_1^0, \ldots, x_n^0)$ *minimizing*

$$H = (n+1) \sum_{i=1}^{n} \prod_{j=1}^{n} (1 - c'_{ij} x_j) + \sum_{k=1}^{n} x_k \tag{11}$$

and for

$$S^0 = \{v_i \,|\, v_i \in V,\, x_i^0 = 1\}.$$

we have

$$\beta(G) = \sum_{i=1}^{n} x_i^0$$

and any minimal externally stable set may be obtained in this way.

Thus the problem of determing the number of external stability of a graph is reduced to one of pseudo-Boolean programming.

Note. In the computation of the minimum, it seems convenient to replace $\bar{x}_i$ with y_i.

Example.

$$H_1 = 7(2y_1y_2y_6 + y_3y_6 + 2y_4y_5y_6 + y_1y_2y_3y_4y_5y_6) - \\ - y_1 - y_2 - y_3 - y_4 - y_5 - y_6 + 6;$$

$$g_1 = 14y_2y_6 + 7y_2y_3y_4y_5y_6 - 1$$

and

$$y_1 = 1 - y_2y_6. \tag{12.1}$$

$$H_2 = 7y_3y_6 + 14y_4y_5y_6 + y_2y_6 - y_2 - y_3 - y_4 - y_5 - y_6 + 5;$$

$$g_2 = -1 + y_6$$

and

$$y_2 = \bar{y}_6 + u_2y_6, \tag{12.2}$$

$$y_2^+ = 1 - y_6. \tag{12.2$^+$}$$

$$H_3 = 7y_3y_6 + 14y_4y_5y_6 - y_3 - y_4 - y_5 + 4;$$

$$g_3 = 7y_6 - 1$$

and

$$y_3 = \bar{y}_6. \tag{12.3}$$

$$H_4 = 14y_4y_5y_6 - y_4 - y_5 + y_6 + 3;$$

$$g_4 = 14y_5y_6 - 1$$

and

$$y_4 = 1 - y_5y_6. \tag{12.4}$$

$$H_5 = y_5y_6 - y_5 + y_6 + 2;$$

$$g_5 = -1 + y_6$$

and

$$y_5 = \bar{y}_6 + u_5y_6, \tag{12.5}$$

$$y_5^+ = 1 - y_6. \tag{12.5$^+$}$$

$$H_6 = 2y_6 + 1;$$

$$g_6 = 2,$$

and

$$y_6 = 0. \tag{12.6}$$

From (12.6), (12.5), (12.4), (12.3), (12.2), (12.1) we find $y_6 = 0$, $y_5 = y_4 = y_3 = y_2 = y_1 = 1$ or, $x_1 = x_2 = x_3 = x_4 = x_5 = 0$, $x_6 = 1$. Thus,

$$\beta(G) = 1$$

and its only minimal externally stable set is $S_1 = \{v_6\}$.

§ 5. The kernel of a graph

A set $T \subseteqq V$ which is both internally and externally stable is called a kernel of the graph. It is shown that a necessary and sufficient condition for T to be a kernel is that

$$\chi_T(v_i) = 1 - \max_{v_j \in \varrho v_i} \chi_T(v_j)$$

where, χ_T is the characteristic function of T ([1], theorem 3, chapter 5). As usually, we put $x_i = \chi_T(v_i)$ and $y_i = \bar{x}_i$. We have

$$x_i = 1 - \max_j c_{ij} x_j$$

or,

$$y_i = \max c_{ij} x_j = \bigcup_{j=1}^{n} c_{ij} x_j \qquad (i = 1, \ldots, n). \tag{13}$$

If α and β are elements of L_2 then $\alpha = \beta$ is equivalent with

$$\alpha \bar{\beta} + \bar{\alpha} \beta = 0.$$

Thus, from (13) we have,

$$y_i \overline{\bigcup_{j=1}^{n} c_{ij} x_j} + x_i \bigcup_{j=1}^{n} c_{ij} x_j = 0 \qquad (i = 1, \ldots, n)$$

or,

$$y_i \prod_{j=1}^{n} (1 - c_{ij} x_j) + \bigcup_{j=1}^{n} c_{ij} x_i x_j = 0 \qquad (i = 1, \ldots, n).$$

As $c_{ij} x_i x_j \geqq 0$ $(i, j = 1, \ldots, n)$ we may put

$$\bar{x}_i \prod_{j=1}^{n} (1 - c_{ij} x_j) + \sum_{j=1}^{n} c_{ij} x_i x_j = 0 \qquad (i = 1, \ldots, n).$$

Since the left-hand part of the above expression is non-negative for any i, the condition may be written in the following form:

$$J = \sum_{i=1}^{n} \bar{x}_i \prod_{j=1}^{n} (1 - c_{ij} x_j) + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j = 0. \tag{14}$$

As $J(x_1, \ldots, x_n) \geqq 0$ for any $(x_1, \ldots, x_n) \in L_2^n$ it is obvious that for finding those $x_1, \ldots, x_n$ which fulfill the condition (14) we have to minimize the pseudo-Boolean function J; if $J_{\min} = J(x_1^0, \ldots, x_n^0) > 0$ then the graph has no kernel; if $J_{\min} = J(x_1^0, \ldots, x_n^0) = 0$ then the graph has the kernel

$$T = \{v_k \mid v_k \in V, x_k^0 = 1\}$$

and any kernel can be obtained in this way.

We have thus proved

Theorem III. *The graph $G = (V, \varrho)$ has a kernel if and only if the minimum of the pseudo-Boolean function*

$$J = \sum_{i=1}^{n} \bar{x}_i \prod_{j=1}^{n} (1 - c_{ij} x_j) + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j \tag{15}$$

is zero. In this case, if $J_{\min} = J(x_1^0, \ldots, x_n^0) = 0$ *then* $T = \{v_i \mid v_i \in V,\ x_i^0 = 1\}$ *is a kernel and any kernel may be obtained in this way.*

Note. The same problem may be solved by computing the solutions of the Boolean equation

$$\bigcup_{i=1}^{n} \bar{x}_i \prod_{j=1}^{n} (\bar{c}_{ij} \cup \bar{x}_j) \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} c_{ij} x_i x_j = 0. \tag{16}$$

This can be carried out in various ways; see for instance [9].

Example. We have

$$\begin{aligned} J_1 &= 2y_1y_2y_6 + y_3y_6 + 2y_4y_5y_6 + y_1y_2y_3y_4y_5y_6 + 2(1-y_1)(1-y_2) + \\ &+ 2(1-y_1)(1-y_6) + 2(1-y_2)(1-y_6) + 2(1-y_3)(1-y_6) + \\ &+ 2(1-y_4)(1-y_5) + 2(1-y_4)(1-y_6) + 2(1-y_5)(1-y_6) \\ &= 14 - 4y_1 - 4y_2 - 2y_3 - 4y_4 - 4y_5 - 10y_6 + 2y_1y_2 + 2y_1y_6 + 2y_2y_6 + \\ &+ 3y_3y_6 + 2y_4y_5 + 2y_4y_6 + 2y_5y_6 + 2y_1y_2y_6 + 2y_4y_5y_6 + y_1y_2y_3y_4y_5y_6; \end{aligned}$$

$$g_1 = -4 + 2y_2 + 2y_6 + 2y_2y_6 + y_2y_3y_4y_5y_6$$

and

$$y_1 = 1 - y_2y_6. \tag{17.1}$$

$$\begin{aligned} J_2 &= 10 - 2y_2 - 2y_3 - 4y_4 - 4y_5 - 8y_6 + 2y_2y_6 + 3y_3y_6 + \\ &+ 2y_4y_5 + 2y_4y_6 + 2y_5y_6 + 2y_4y_5y_6; \end{aligned}$$

$$g_2 = 2y_6 - 2$$

and

$$y_2 = \bar{y}_6 + u_2y_6, \tag{17.2}$$

$$y_2^+ = 1 - y_6. \tag{17.2$^+$}$$

$$\begin{aligned} J_3 &= 8 - 2y_3 - 4y_4 - 4y_5 - 6y_6 + 3y_3y_6 + 2y_4y_5 + 2y_4y_6 + \\ &+ 2y_5y_6 + 2y_4y_5y_6; \end{aligned}$$

$$g_3 = -2 + 3y_6$$

and

$$y_3 = \bar{y}_6. \tag{17.3}$$

$$J_4 = 6 - 4y_4 - 4y_5 - 4y_6 + 2y_4y_5 + 2y_4y_6 + 2y_5y_6 + 2y_4y_5y_6;$$

$$g_4 = -4 + 2y_5 + 2y_6 + 2y_5y_6$$

and

$$y_4 = 1 - y_5y_6. \tag{17.4}$$

$$J_5 = 2 - 2y_5 - 2y_6 + 2y_5y_6;$$

$$g_5 = -2 + 2y_6$$

and

$$y_5 = \bar{y}_6 + u_5y_6, \tag{17.5}$$

$$y_5^+ = 1 - y_6. \tag{17.5$^+$}$$

Now, $J_6 = 0$, $g_6 = 0$ and hence

$$y_6 = u_6. \tag{17.6}$$

From (17.6), (17.5), (17.4), (17.3), (17.2), (17.1) we find

$$\left.\begin{aligned} x_1 &= 1 - y_1 = u_2u_6 \\ x_2 &= 1 - y_2 = u_6 - u_2u_6 = \bar{u}_2u_6 \\ x_3 &= 1 - y_3 = u_6 \\ x_4 &= 1 - y_4 = u_5u_6 \\ x_5 &= 1 - y_5 = u_6 - u_5u_6 = \bar{u}_5u_6 \\ x_6 &= 1 - y_6 = \bar{u}_6 \end{aligned}\right\} \tag{18}$$

For the various values of the parameters u_2, u_5 and u_6 we find the kernels

$$T_1 = \{v_1, v_3, v_4\},$$
$$T_2 = \{v_1, v_3, v_5\},$$
$$T_3 = \{v_2, v_3, v_4\},$$
$$T_4 = \{v_2, v_3, v_5\},$$
$$T_5 = \{v_6\}.$$

Note. It is proved (theorem 5 of chapter V of [1]) that if each subgraph of a graph has a kernel then the graph has a function of Grundy. It may be determined then with a procedure based on pseudo-Boolean programming.

§ 6. The chromatic number

Given a finite graph $G = (V, \varrho)$ by a chromatic decomposition of it we mean a family of disjoint internally stable subsets $M_1, M_2, \ldots, M_k$ of V so that $\bigcup_{h=1}^{k} M_h = V$. The chromatic decomposition with the smallest number $\gamma(G)$ of subsets is called a minimal chromatic decomposition and $\gamma(G)$ is called the *chromatic number* of the graph.

An internally stable set M of $G = (V, \varrho)$ is called superior if for any internally stable set N of G, $M \subseteqq N$ implies $M = N$.

Let γ_1 be the minimal number of internally stable (not obviously disjoint) sets covering V. Let γ_2 be the minimal number of superior internally stable sets covering V.

Lemma 3. $\gamma = \gamma_1$.

Proof. Let $M_1, \ldots, M_{\gamma_1}$ be internally stable subsets of V, covering it and let us put

$$\begin{aligned} P_1 &= M_1, \\ P_2 &= M_2 - M_1, \\ &\cdots\cdots\cdots \\ P_i &= M_i - \bigcup_{j=1}^{i-1} M_j, \\ &\cdots\cdots\cdots \\ P_{\gamma_1} &= M_{\gamma_1} - \bigcup_{j=1}^{\gamma_1 - 1} M_j. \end{aligned} \tag{19}$$

If $P_i = \emptyset$, then $M_i \subseteqq \bigcup_{j=1}^{i-1} M_j$ and the family $M_1, \ldots, M_{i-1}, M_{i+1}, \ldots, M_{\gamma_1}$ would be a covering of V with only $\gamma_1 - 1$ internally stable subsets, in contradiction with the definition of γ_1. Therefore $P_i \neq \emptyset$ $(i = 1, \ldots, \gamma_1)$. Thus, we have obtained a covering of V with γ_1 disjoint internally stable subsets of it. Hence $\gamma \leqq \gamma_1$.

As the converse relation does obviously hold, the lemma is proved.

Lemma 4. $\gamma_1 = \gamma_2$.

Any internally stable set can be imbedded in a superior internally stable one. Hence, for any family $M_1, \ldots, M_{\gamma_1}$ of internally stable sets covering V, there exists a family $N_1, \ldots, N_k$ of superior internally stable sets covering V with $k \leqq \gamma_1$. Thus we see that $\gamma_2 \leqq \gamma_1$. As the converse relation does obviously hold, the lemma is proved.

Theorem IV. *The chromatic number γ of a graph $G = (V, \varrho)$ is equal to the minimal number of superior internally stable sets covering V.*

Let $\mathfrak{M} = \{M_1, \ldots, M_q\}$ be the family of all superior internally stable sets of G; they may be determined (see lemma 1) by minimizing the expression

$$W = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j .$$

Let us put $d_{ij} = 1$ if the vertex v_i belongs to the superior internally stable set M_j and $d_{ij} = 0$ in the other case. For any subfamily $\mathfrak{M}'$ of $\mathfrak{M}$ we shall denote with x_j the values of the characteristic function $\chi_{\mathfrak{M}'}(M_j)$.

Lemma 5. *$\mathfrak{M}'$ is a covering of V if and only if*

$$\sum_{i=1}^{n} \prod_{j=1}^{q} (\bar{d}_{ij} + d_{ij} \bar{x}_j) = 0 . \tag{20}$$

Proof. The vertex v_i belongs to an element of $\mathfrak{M}'$ if and only if

$$\bigcup_{j=1}^{q} d_{ij} x_j = 1 \tag{21}$$

i.e.

$$\prod_{j=1}^{q} (\bar{d}_{ij} \cup \bar{x}_j) = 0 . \tag{22}$$

That means that $\mathfrak{M}'$ is a covering of V if and only if for any i, we have

$$\prod_{j=1}^{q} (\bar{d}_{ij} + d_{ij} \bar{x}_j) = 0 \tag{23}$$

or

$$\sum_{i=1}^{n} \prod_{j=1}^{q} (\bar{d}_{ij} + d_{ij} \bar{x}_j) = 0 \tag{24}$$

q.e.d.

An absolutely analogous reasoning with that of the previous paragraph shows that we have

Theorem V. *The chromatic number γ of a graph $G = (V, \varrho)$ is equal to the minimum of the pseudo-Boolean expression*

$$K = (q+1) \sum_{i=1}^{n} \prod_{j=1}^{q} (\bar{d}_{ij} + d_{ij} \bar{x}_j) + \sum_{j=1}^{q} x_j ; \tag{25}$$

and a minimal chromatic decomposition $P_1, \ldots, P_\gamma$ of G may be obtained from that family of superior internally stable subsets M_j of V for which $x_j = 1$, by formulas (19).

S i m p l i f i c a t i o n 1. If there exists a vertex v_i of G, covered only by a single superior internally stable set M_j, then obviously $x_j = 1$ and we can examine only the subgraph generated by the vertices $V - M_j$.

Simplification 2. If there exist vertices v_{i_1} and v_{i_2} so that for any superior internally stable set M_j

$$v_{i_1} \in M_j \Rightarrow v_{i_2} \in M_j,$$

then we can simplify the problem by examining only the subgraph generated by $V - \{v_{i_2}\}$.

Repeat simplifications 1 and 2 as many times as possible.

Note. It seems convinient to work with the unknowns $y_j = 1 - x_j$.

Example. To determine all the internally stable sets let us minimize:

$$W_1 = 2(x_1x_2 + x_1x_6 + x_2x_6 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6)$$

$$g_1 = 2(x_2 + x_6),$$

$$x_1 = u_1\bar{x}_2\bar{x}_6, \tag{26.1}$$

$$x_1^+ = 0. \tag{26.1$^+$}$$

$$W_2 = 2(x_2x_6 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6),$$

$$g_2 = 2x_6,$$

$$x_2 = u_2\bar{x}_6, \tag{26.2}$$

$$x_2^+ = 0. \tag{26.2$^+$}$$

$$W_3 = 2(x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6)$$

$$g_3 = 2x_6,$$

$$x_3 = u_3\bar{x}_6, \tag{26.3}$$

$$x_3^+ = 0. \tag{26.3$^+$}$$

$$W_4 = 2(x_4x_5 + x_4x_6 + x_5x_6)$$

$$g_4 = 2(x_5 + x_6),$$

$$x_4 = u_4\bar{x}_5\bar{x}_6, \tag{26.4}$$

$$x_4^+ = 0. \tag{26.4$^+$}$$

$$W_5 = 2x_5x_6$$

$$g_5 = 2x_6$$

$$x_5 = u_5\bar{x}_6, \tag{26.5}$$

$$x_5^+ = 0. \tag{26.5$^+$}$$

$$W_6 = 0$$

$$g_6 = 0,$$

$$x_6 = u_6. \tag{26.6}$$

We have thus

$$\left.\begin{aligned} x_1 &= u_1\bar{u}_2u_6 \\ x_2 &= u_2\bar{u}_6 \\ x_3 &= u_3\bar{u}_6 \\ x_4 &= u_4\bar{u}_5\bar{u}_6 \\ x_5 &= u_5\bar{u}_6 \\ x_6 &= u_6 \end{aligned}\right\} \tag{27}$$

and this gives us all the internally stable sets of the graph. It is easy to verify that the superior ones one:

$$M_1 = \{v_1, v_3, v_4\},$$
$$M_2 = \{v_1, v_3, v_5\},$$
$$M_3 = \{v_2, v_3, v_4\},$$
$$M_4 = \{v_2, v_3, v_5\},$$
$$M_5 = \{v_6\}.$$

As v_6 is contained only in M_5 we can confine ourselves (simplification 1) to the subgraph generated by $\{v_1, v_2, v_3, v_4, v_5\}$.

As v_3 is contained in all M_j's we may confine ourselves (simplification 3) to the subgraph generated by $\{v_1, v_2, v_4, v_5\}$ and its superior internally stable sets

$$M_1' = \{v_1, v_4\},$$
$$M_2' = \{v_1, v_5\},$$
$$M_3' = \{v_2, v_4\},$$
$$M_4' = \{v_2, v_5\}.$$

The matrix $((d_{ij}))$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and

$$K_1 = 5(y_1 y_2 + y_3 y_4 + y_1 y_3 + y_2 y_4) + 4 - y_1 - y_2 - y_3 - y_4$$
$$g_1 = 5 y_2 + 5 y_3 - 1$$

and

$$y_1 = \bar{y}_2 \bar{y}_3. \tag{28.1}$$
$$K_2 = 5 y_2 y_4 - y_2 y_3 + 5 y_3 y_4 - y_4 + 3,$$
$$g_2 = 5 y_4 - y_3$$

and

$$y_2 = y_3 \bar{y}_4 \cup u_2 \bar{y}_3 \bar{y}_4, \tag{28.2}$$
$$y_2^+ = y_3 - y_3 y_4. \tag{28.2+}$$
$$K_3 = - y_3 + 6 y_3 y_4 - y_4 + 3$$
$$g_3 = - 1 + 6 y_4$$

and

$$y_3 = \bar{y}_4. \tag{28.3}$$
$$K_4 = 2$$
$$g_4 = 0$$

and

$$y_4 = u_4. \tag{28.4}$$

Hence γ' of the simplified problems is 2; γ will be equal to γ' plus one (corresponding to the eliminated set M_5), i.e.

$$\gamma = 3. \tag{29}$$

The coverings of V with superior internally stable subsets of it may be obtained by giving various values to the parameters.

The solution of (28.1), (28.2), (28.3), (28.4) being $(u_4, \bar{u}_4, \bar{u}_4, u_4)$ we obtain the coverings $\{M_1', M_4'\}$ and $\{M_2', M_3'\}$. Thus the coverings of the initial problem will be:

$$\{M_1, M_4, M_5\} \tag{30}$$

and

$$\{M_2, M_3, M_5\}. \tag{31}$$

Applying formulas (19) in various orders we obtain the minimal chromatic decompositions:

$$\left.\begin{aligned} P_1^I &= \{v_1, v_3, v_4\} \\ P_2^I &= \{v_2, v_5\} \\ P_3^I &= \{v_6\} \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} P_1^{II} &= \{v_2, v_3, v_5\} \\ P_2^{II} &= \{v_1, v_4\} \\ P_3^{II} &= \{v_6\} \end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} P_1^{III} &= \{v_1, v_3, v_5\} \\ P_2^{III} &= \{v_2, v_4\} \\ P_3^{III} &= \{v_6\} \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} P_1^{IV} &= \{v_2, v_3, v_4\} \\ P_2^{IV} &= \{v_1, v_5\} \\ P_3^{IV} &= \{v_6\} \end{aligned}\right\} \tag{35}$$

Note. The above method of determining the minimal chromatic decompositions of a graph combined with the method given in theorem 4 of chapter 4 of [1] permits us to determine a function of GRUNDY on G.

ČULIK recently defined in a paper to appear in "Problemy Kybernetiki" the number of completeness of a graph G as being the minimal number of complete subgraphs of G containing all its vertices and edges. This problem may also be treated by pseudo-Boolean programming.

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