Z. Wahrscheinlichkeitstheorie 3, 110—121 (1964)

Electrostatic Capacity, Heat Flow, and Brownian Motion

By

Frank Spitzer

1. Introduction

We shall point out how the electrostatic capacity of a compact set in Euclidean space governs the rate of heat flow from such a set to a surrounding conducting medium. The answers will of course be quite different in dimensions two and three. In both cases, however, we have to rely heavily on the careful probabilistic formulation of classical diffusion theory by Doob [2] and Hunt [4]. A systematic account of these matters will also be found in the book of Itô and McKean [5].

The standard Brownian motion process in dimension $s = 2$ or 3 will be denoted $x(t)$ ($= x(t, \omega)$ for ω in the appropriate probability space as defined in [5], Ch. 1 and 7). Measurable sets (events) in this space have measure $P_x[\cdot]$ when $x(0) = x$, and measurable functions (random variables) have expectation $E_x[\cdot]$. We shall always work with *compact* subsets A of Euclidean space R of dimension $s = 2$ or 3. When $s = 3$ it will be assumed that the *Newtonian capacity* $C(A)$ of A is positive, and when $s = 2$ that the *logarithmic capacity* of A is positive. In the latter case, however, we shall be concerned with the set function $K(A) = -\gamma(A)$, where $\gamma(A) =$ *Robin's constant*. Thus $K(A)$ is the natural logarithm of the logarithmic capacity of A (see [8], Ch. V).

For such a set A we define

$$T_A = \inf[t \,|\, t > 0,\ x(t) \in A] \leqq \infty .$$

A point $x \in A$ is called *regular* if

$$P_x[T_A = 0] = 1 .$$

It is known [4] that under our assumptions concerning A

$$f_A(x, t) = P_x[T_A \leqq t] > 0 \qquad \text{for all } t > 0$$

and that $f_A(x, t)$ is the unique solution of the heat conduction problem

$$\frac{\partial f}{\partial t} = \frac{1}{2} \Delta f \qquad \text{for } t > 0,\ x \in R - A$$

subject to the initial condition

$$f(x, 0) = 0 \qquad \text{for } x \in R - A$$

and boundary condition

$$\lim_{x \to y} f(x, t) = 1 \quad \text{for} \quad t > 0,\ y = \text{regular point} \in A .$$

Thus $f_A(x,t)$ may be interpreted as the temperature at time t at the point $x \in R - A$, and we proceed to investigate the integral of this temperature over $R - A$, i. e.

the *total energy flow in time t from the set A into the surrounding medium* $R - A$. Let us call it

$$E_A(t) = \int_{R-A} P_x[T_A \leqq t]\, dx = \int_{R-A} f_A(x,t)\, dx\,. \tag{1.1}$$

The asymptotic behavior of $E_A(t)$ will be shown to depend on the capacity of the set A. In dimension $s = 3$ the capacity quite naturally determines the principal term, but when $s = 2$ it enters only the second term of the asymptotic expansion of $E_A(t)$.

Theorem 1. *In dimension three*

$$E_A(t) = t\,C(A) + 4(2\pi)^{-3/2}[C(A)]^2 t^{1/2} + o(t^{1/2})\,,$$

as $t \to \infty$. *When A is a sphere*, $o(t^{1/2}) \equiv 0$, $t > 0$★.

Theorem 2. *In dimension two, let A and B be two compact sets of positive (logarithmic) capacity. Then*

$$E_A(t) - E_B(t) \sim \frac{2\pi t}{(\ln t)^2}[K(A) - K(B)]\,, \quad as \quad t \to \infty\,★★.$$

Alternatively (specializing to the case when B is the unit disc)

$$E_A(t) = \frac{2\pi t}{\ln t} + \frac{2\pi t}{(\ln t)^2}[K(A) + 1 + \gamma - \ln 2] + o(t \ln^{-2} t)$$

as $t \to \infty$. *Here* $\gamma = .5772\ldots$ *is Euler's constant.*

Remark: There are many equivalent definitions of capacity and the normalization is rather arbitrary. For the above theorems to hold the proper normalization is that which gives Newtonian capacity $2\pi\varrho$ to a sphere of radius ϱ and logarithmic capacity ϱ^2 to a disc of radius ϱ (i. e. if $A = [z|\ |z| \leqq \varrho]$, then $K(A) = 2 \ln \varrho$).

2. Proof of theorem 1

For arbitrary Borel sets $B \subset R$

$$P_x[x(t) \in B] = \int_B p(x,y,t)\, dy\,, \qquad x \in R\,,$$

where p is the transition function

$$p(x,y,t) = (2\pi t)^{-s/2} e^{-|x-y|^2/2t}\,, \qquad x, y \in R,\ t > 0\,.$$

Furthermore

$$P_x[T_A > t, x(t) \in B] = \int_B q_A(x,y,t)\, dy\,, \qquad x \in R, t > 0\,, \tag{2.1}$$

★ M. Kac has investigated the asymptotic expansion of the Laplace transform of $E_A(t)$ (private communication). His expansion is valid under slight regularity conditions concering the set A (which must have positive volume $|A|$). Formal term by term inversion yields the same terms of order t and $t^{1/2}$ as in Theorem 1. The rest, which we denoted $o(t^{1/2})$ becomes

$$o(t)^{1/2} = \frac{1}{2\pi^2}[C(A)]^3 - |A| - \frac{1}{2\pi}\int_A\int_A e(dx)\,|x-y|\,e(dy) + o(1)\,.$$

Here $e(dx)$ is the equilibrium charge of the set A, to be defined below, in the proof of Theorem 1.
★★ $f(t) \sim g(t)$ means $f(t)/g(t) \to 1$ as $t \to \infty$.

defines the density $q_A(x, y, t)$ which exists according to HUNT ([4] section 4) and is a *symmetric* function of x and y. Note that the symmetry of q_A which is not the trivial matter it might seem at first glance, is the key to the entire proof.

To study the asymptotic behavior of $E_A(t)$ we use (1.1) to write, for $0 < h < t$,

$$E_A(t) - E_A(t-h) = \int_{R-A} \{P_x[T_A > t-h] - P_x[T_A > t]\}\, dx \,. \tag{2.2}$$

Using the strong Markov property of Brownian motion (see [4] section 2; it will be taken for granted from now on) the integrand on the right in (2.2) becomes

$$\int_R \int_R q_A(x, u, t-h)\, p(u, y, h)\, du\, dy - \int_R q_A(x, y, t)\, dy \,. \tag{2.3}$$

The integrations are over R, since $q_A(x, y, t) = 0$ when $x \in A$ or $y \in A$. The symmetry of each of the densities p and q_A in the space variables permits the transformation of (2.3) into

$$\int_R \int_R p(y, u, h)\, q_A(u, x, t-h)\, du\, dy - \int_R q_A(y, x, t)\, dy \,. \tag{2.4}$$

Now we substitute (2.4) into (2.2) and carry out the integration of x over $R - A$. Thus

$$\begin{aligned} &E_A(t) - E_A(t-h) \\ &= \int_R \int_R p(y, u, h)\, P_u[T_A > t-h]\, du\, dy - \int_R P_y[T_A > t]\, dy \\ &= \int_R \{P_y[x(\tau) \in R - A \quad \text{for} \quad h < \tau \leqq t] - P_y[T_A > t]\}\, dy \\ &= \int_R P_y[T_A \leqq h; x(\tau) \in R - A \quad \text{for} \quad h < \tau \leqq t]\, dy \,. \end{aligned} \tag{2.5}$$

Equation (2.5) shows that $E_A(t) - E_A(t-h)$ is a monotone function of t, so that

$$\lim_{t \to \infty} [E_A(t) - E_A(t-h)] = \int_R P_y[T_A \leqq h; x(\tau) \in R - A \quad \text{for} \quad \tau > h]\, dy \,. \tag{2.6}$$

This limit is an additive function of h. It is obviously bounded and measurable so that it is a linear function of h. Hence we may write

$$C'(A) = \frac{1}{h} \int_R P_y[T_A \leqq h, x(\tau) \in R - A \quad \text{for} \quad \tau > h]\, dy, \quad h > 0 \,. \tag{2.7}$$

In dimension $s \leqq 2$ clearly $C'(A) = 0$. Theorem 1 however is concerned with $s = 3$, and in this case we have to identify the set function $C'(A)$ with the Newtonian capacity $C(A)$. That is done by ITÔ and MCKEAN ([5] section 7.7). They define the family of measures (charges)

$$e_h(dy) = \frac{1}{h} P_y[T_A \leqq h; x(\tau) \in R - A \quad \text{for} \quad \tau > h]\, dy \,,$$

and prove by a short calculation that the corresponding potentials

$$p_h(x) = \frac{1}{2\pi} \int_R |x - y|^{-1} e_h(dy)$$

increase to $P_x[T_A < \infty]$ as $h \searrow 0$. They further show that a subsequence of these charges therefore converges (weakly) to an equilibrium charge $e(dy)$, supported by A. The capacitory potential corresponding to $e(dy)$ is $P_x[T_A < \infty] \leqq 1$, which, as pointed out in section 1, has boundary value one at the regular points of A. Thus the total charge

$$\int_R e_h(dy) = \int_R e(dy) = \int_A e(dy) = C'(A)$$

is the usual capacity $C(A)$ associated with the Green function $(2\pi)^{-1}|x - y|^{-1}$. If A is a sphere of radius ϱ, then

$$C(A) = \sup \int_A e(dy)$$

over all charges e such that

$$\frac{1}{2\pi} \int_A \frac{e(dy)}{|x - y|} \leqq 1, \; x \in R,$$

which gives $C(A) = 2\pi\varrho$. Equations (2.6) and (2.7) therefore yield the principal term in Theorem 1.

To obtain the second term in Theorem 1 let $|A|$ denote the volume of A. In view of equation (2.7)

$$\begin{aligned} &E_A(t) - tC(A) \\ &= \int_{R-A} P_x[T_A \leqq t]\,dx - \int_R P_x[T_A \leqq t;\quad x(\tau) \in R - A \quad \text{for} \quad \tau > t]\,dx \\ &= \int_R P_x[T_A \leqq t;\quad x(\tau) \in A \quad \text{for some} \quad \tau > t]\,dx - |A| \\ &= \int_R \int_R [p(x, y, t) - q_A(x, y, t)]\, P_y[T_A < \infty]\,dy\,dx - |A|\,. \end{aligned}$$

Using the symmetry of p and q_A,

$$\begin{aligned} (2.8) \qquad &E_A(t) - tC(A) \\ &= \int_R \Big\{\int_R [p(y, x, t) - q_A(y, x, t)]\,dx\Big\}\, P_y[T_A < \infty]\,dy - |A| \\ &= \int_R P_y[T_A \leqq t]\, P_y[T_A < \infty]\,dy - |A| \\ &= \int_{R-A} P_y[T_A \leqq t]\, P_y[T_A < \infty]\,dy\,. \end{aligned}$$

By equation (2.8)

$$\begin{aligned} D_A(t, h) &= E_A(t) - E_A(t - h) - hC(A) \\ &= \int_{R-A} P_x[t - h < T_A \leqq t]\, P_x[T_A < \infty]\,dx \\ &= \int_{x \in R-A} \int_{y \in R} P_x[t - h < T_A \leqq t; x(t) \in dy]\, P_x[T_A < \infty]\,dx\,. \end{aligned}$$

Just as in going from (2.3) to (2.4) we now use the symmetry of $q_A(x, y, t)$ to reverse the direction of time. The result is

$$\begin{aligned} D_A(t,h) &= \int_{x\in R-A}\int_{y\in R} P_y[T_A \leqq h; x(\tau)\in R-A \\ &\qquad \text{for} \quad h<\tau\leqq t; x(t)\in dx]\, P_x[T_A<\infty]\, dx \\ &= \int_R P_y[T_A\leqq h; x(\tau)\in R-A \quad \text{for} \quad h<\tau\leqq t; x(\tau)\in A \\ &\qquad \text{for some} \quad \tau>t]\, dy \\ &= \int_{y\in R}\int_{z\in R} P_y[T_A\leqq h; x(h)\in dz]\, P_z[t-h<T_A<\infty]\, dy. \end{aligned} \tag{2.9}$$

Now we shall apply the following

Lemma. *In dimension three, for* $x \in R - A$,

$$\begin{aligned} f_A(x,\infty) - f_A(x,t) &= P_x[t<T_A<\infty] \\ &\sim f_A(x,\infty)\, C(A)\, 2\,(2\pi)^{-3/2}\, t^{-1/2}, \quad as \quad t\to\infty, \end{aligned}$$

uniformly for x in a compact set.

This lemma is due to JOFFE [6] who proved it for sets A with positive volume $|A|$, using the theory of additive functionals of the Wiener process of M. KAC. We shall first complete the proof of Theorem 1, and then sketch a proof of the lemma which does not require Tauberian arguments, and which applies to any compact set A with positive capacity.

The lemma applied to (2.9) gives

$$\begin{aligned} D_A(t,h) &\sim 2\,C(A)\,(2\pi)^{-3/2}\,(t-h)^{-1/2}\,\cdot \\ &\quad \cdot \int_{y\in R}\int_{z\in R-A} P_y[T_A\leqq h; x(h)\in dz]\, P_z[T_A=\infty]\, dy \\ &\sim 2\,C(A)\,(2\pi)^{-3/2}\, t^{-1/2} \int_R P_y[T_A\leqq h; x(\tau)\in R-A \quad \text{for} \quad \tau>h]\, dy. \end{aligned}$$

Comparison with equation (2.7) shows that

$$D_A(t,h) \sim 2h\,C(A)\,(2\pi)^{-3/2}\, t^{-1/2}\int_R e_h(dy),$$

where the total mass of e_h is $C(A)$. That proves Theorem 1.

To derive the Lemma observe that

$$\lim_{|x|\to\infty} |x|\, f_A(x,\infty) = \lim_{|x|\to\infty} |x|\, P_x[T_A<\infty] = \frac{C(A)}{2\pi}, \tag{2.10}$$

since $f_A(x, \infty)$ is the capacitory potential

$$f_A(x,\infty) = \frac{1}{2\pi}\int_A \frac{e(dy)}{|x-y|},$$

where $e(dy)$ is the weak limit of the measures $e_h(dy)$ in (2.7). Now we write

$$P_x[t<T_A<\infty] = \int_{R-A} q_A(x,y,t)\, P_y[T_A<\infty]\, dy,$$

and use (2.10) to conclude

$$P_x[t < T_A < \infty] \sim \frac{C(A)}{2\pi} \int_{R-A} \frac{q_A(x,y,t)}{|y|}\,dy\,.$$

Decomposing further

$$P_x[t < T_A < \infty] \sim \frac{C(A)}{2\pi} \int_{R-A} \frac{p(x,y,t)}{|y|}\,dy$$

$$- \frac{C(A)}{2\pi} \int_{R-A} \frac{P_x[x(t) \in dy;\, T_A \le t]}{|y|} = \frac{C(A)}{2\pi}\,[A_x(t) - B_x(t)]\,.$$

It is easy to check that

$$A_x(t) = \int_{R-A} \frac{p(x,y,t)}{|y|}\,dy \sim \sqrt{\frac{2}{\pi}}\,t^{-1/2}, \quad \text{as} \quad t \to \infty\,,$$

and therefore the Lemma will be proved if we show that

$$B_x(t) = \int_{R-A} \frac{P_x[x(t) \in dy;\, T_A \le t]}{|y|} \sim \sqrt{\frac{2}{\pi}}\,P_x[T_A < \infty]\,t^{-1/2} \quad \text{as} \quad t \to \infty\,.$$

That is the only delicate part of the proof. Decompose

$$B_x(t) = \int_{y \in R-A} \int_{z \in R} \int_0^t \frac{1}{|y|}\,d_\tau\, P_x[T_A \leqq \tau; x(T_A) \in dz]\,p(z,y,t-\tau)\,dy\,.$$

Interchanging the order of integration, and calculating the integral on y (in polar coordinates) gives

$$B_x(t) \sim \int_{z \in R} \int_0^t d_\tau\, P_x[T_A \leqq \tau; x(T_A) \in dz] \left\{ \int_R \frac{p(0,y,t-\tau)}{|y|}\,dy \right\}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^t \frac{d_\tau\, P_x[T_A \le \tau]}{\sqrt{t-\tau}}\,.$$

Finally, decompose

$$B_x(t) \sim \sqrt{\frac{2}{\pi}}\,\frac{P_x[T_A \le t]}{\sqrt{t}} + C_x(t)\,,$$

where $C_x(t)$ is the error term

$$C_x(t) = \sqrt{\frac{2}{\pi}} \int_0^t \left[\frac{1}{\sqrt{t-\tau}} - \frac{1}{\sqrt{t}}\right] d_\tau\, P_x[T_A \leqq \tau]\,.$$

If we can show that $\sqrt{t}\,C_x(t) \to 0$ as $t \to \infty$, then

$$B_x(t) \sim \sqrt{\frac{2}{\pi}}\,P_x(T_A < \infty]\,t^{-1/2}$$

and the proof is complete. Since the integrand in the formula for $C_x(t)$ is non-negative, that is readily accomplished by probability arguments which give the estimate

$$\int_t^{t+1} d_\tau P_x[T_A \leqq \tau] = P_x[t < T_A \leqq t+1]$$
$$\leqq \int_R p(x, y, t) P_y[T_S \leqq 1] dy \leqq k t^{-3/2}.$$

Here S is a large sphere containing A and $k > 0$, depending on x and on A but not on t. A look back over the proof shows that uniformity on compact sets is automatic.

It only remains to verify theat the error term $o(t^{-1/2})$ vanishes when the set A is a sphere. That is immediate from an explicit calculation of $f_A(x, t)$, (see [1], p. 247).

Remark: Only for diffusion in dimension $s \geqq 5$ is the limit

$$\lim_{t\to\infty} [E_A(t) - t C(A)] = \int_{R-A} (P_y[T_A < \infty])^2 dy$$

finite for sets of positive capacity.

3. Proof of theorem 2

Here it is essential that the dimension be $s = 2$, and our proof is entirely based on Hunt's investigation of the asymptotic behavior of $P_x[T_A > t]$. Following HUNT ([4] section 5) we define, for any compact $A \subset R$ with positive capacity, the Green function

$$G_A(x, y) = \int_0^\infty q_A(x, y, t) dt, \qquad x, y \in R.$$

Here q_A was defined in (2.1). HUNT showed that

$$H_A(x) = \lim_{|y|\to\infty} G_A(x, y) < \infty, \qquad x \in R$$

exists, and that also

$$\lim_{|x|\to\infty} [H_A(x) - \frac{1}{\pi} \ln |x|] = -(2\pi)^{-1} K(A) \tag{3.1}$$

exists and is finite. For the identification of this limit with $-(2\pi)^{-1} K(A)$ we refer to Nevanlinna's characterization of Robin's constant $-K(A)$ in terms of the asymptotic behavior of the Green function with pole at infinity ([8] Ch. 5, section 3). We further record from [2] or [4] that H_A is harmonic on $R - A$, so that for large enough $\varrho > 0$ the mean value theorem yields

$$\int_{|x|=\varrho} [H_A(x) - \frac{1}{\pi} \ln |x|] d\zeta(x) = -(2\pi)^{-1} K(A). \tag{3.2}$$

Here ζ denotes the constant measure of mass one on the circumference $|x| = \varrho$, and we shall use the symbol ζ for this purpose exclusively.

We need also some preliminary information concerning $E_A(t)$ when A is a disc. To simplify the notation write

$$E_{[z\,||z|\leqq r]}(t) = E_r(t), \quad T_{[z\,||z|\leqq r]} = T_r.$$

It is simplest to solve the heat conduction problem in section 1 by the method of Laplace transforms to obtain

$$\int_0^\infty P_x[T_r \leqq t]\, e^{-\lambda t}\, dt = \lambda^{-1} \frac{K_0(\sqrt{2\lambda}\,|x|)}{K_0(\sqrt{2\lambda}\, r)} \tag{3.3}$$

when $\lambda > 0$, $|x| > r$. Here $K_0(z)$ is the Bessel function of the second kind with imaginary argument, of order 0, and $K_1(z)$ is the corresponding function of order 1. It is well known that

$$z^{-1}\frac{d}{dz}[z^\nu K_\nu(z)] = -z^{\nu-1} K_{\nu-1}(z),$$

which gives

$$\begin{aligned}\int_0^\infty E_r(t)\, e^{-\lambda t}\, dt &= \int_{|x|>r} \int_0^\infty P_x[T_r \leqq t]\, e^{-\lambda t}\, dt\, dx \\ &= 2\pi r \lambda^{-3/2} \frac{K_1(r\sqrt{2\lambda})}{K_0(r\sqrt{2\lambda})}.\end{aligned}$$

For small positive z

$$K_0(z) = -\log\frac{z}{2} - \gamma + O(z),$$

$$K_1(z) = z^{-1} + O(1),$$

so that

$$\int_0^\infty E_r(t)\, e^{-\lambda t}\, dt \sim \frac{2\pi}{\lambda^2 \ln(1/\lambda)}, \quad \text{as } \lambda \to 0.$$

As $E_r(t)$ is monotone Karamata's theorem yields the conclusion

$$E_r(t) \sim \frac{2\pi t}{\ln t}, \quad \text{as } t \to \infty. \tag{3.4}$$

Observe that the first order approximation to $E_r(t)$ in (3.4) is independent of r so that we are still far from the goal of obtaining an approximation which does depend on the set A. The crucial step is again due to Hunt who used (3.1), (3.2), and a Tauberian argument similar to that which gave (3.4) to verify the conjecture of Kac that for $x \in R - A$

$$P_x[T_A > t] \sim \frac{2\pi}{\ln t} H_A(x), \quad \text{as } t \to \infty.$$

Hunt's proof of this result ([*4*], section 6) requires no essential modification to yield

$$\int_{|x|=\varrho} P_x[T_A > t]\, d\zeta(x) \sim \frac{2\pi}{\ln t} \int_{|x|=\varrho} H_A(x)\, d\zeta(x), \quad t \to \infty \tag{3.5}$$

for every $\varrho > 0$, large enough so that A lies in the open disc $|z| < \varrho$.

The rest of our proof depends on the observation that a Brownian motion starting at a point outside a large disc has to hit the circumference before visiting a set A in the interior. Given A we choose r so large that $A \subset [z \mid |z| < r]$ and write

$$E_r(t) - E_A(t) = \int_{|x|>r} \{P_x[T_r \leqq t] - P_x[T_A \leqq t]\}\, dx$$

$$- \int_{[x \mid x\in R-A, |x| \leqq r]} P_x[T_A \leqq t]\, dx = \int_{|x|>r} P_x[T_r \leqq t < T_A]\, dx + O(1),$$

$O(1)$ being bounded as $t \to \infty$. Using the strong Markov property

$$E_r(t) - E_A(t) \tag{3.6}$$
$$= \int_{|x|>r} \Big\{ \int_{|y|=r} \int_0^t d_\tau P_x[T_r \leqq \tau;\ x(T_r) \in dy]\, P_y[T_A > t-\tau] \Big\}\, dx + O(1).$$

Now let

$$g_r(t) = \int_{|y|=r} P_y[T_A > t]\, d\zeta(y).$$

In view of the rotation invariance of Brownian motion

$$\int_{|x|>r} P_x[T_r \leqq t; x(T_r) \in dy]\, dx = \int_{|x|>r} P_x[T_r \leqq t]\, dx\, d\zeta(y) = E_r(t)\, d\zeta(y),$$

so that (3.6) becomes

$$E_r(t) - E_A(t) = \int_0^t g_r(t-\tau)\, d_\tau E_r(\tau) + O(1), \quad t > 0. \tag{3.7}$$

The asymptotic behavior of $E_r(t)$ was given in (3.4) so that we proceed to estimate $g_r(t)$. By use of (3.5)

$$g_r(t) \sim \frac{2\pi}{\ln t} \int_{|x|=r} H_A(x)\, d\zeta(x)$$
$$= \frac{2\pi}{\ln t} \int_{|x|=r} \Big[H_A(x) - \frac{1}{\pi} \ln |x| \Big]\, d\zeta(x) + \frac{2 \ln r}{\ln t},$$

and equation (3.2) now yields

$$g_r(t) \sim (\ln t)^{-1} [2 \ln r - K(A)], \quad t \to \infty. \tag{3.8}$$

It only remains to substitute (3.4) and (3.8) into (3.7) to conclude that as $t \to \infty$

$$E_r(t) - E_A(t)$$
$$\sim 2\pi [2 \ln r - K(A)] \int_2^{t-2} \frac{1}{\ln(t-\tau)}\, d\Big(\frac{\tau}{\ln \tau}\Big) \tag{3.9}$$
$$\sim \frac{2\pi t}{(\ln t)^2} [2 \ln r - K(A)].$$

Given 2 compacet sets A and B, each with positive capacity, we may choose r so large that (3.9) holds both for A and for B. The difference of the resulting versions of (3.9) reads

$$E_A(t) - E_B(t) \sim \frac{2\pi t}{(\ln t)^2}[K(A) - K(B)], \quad t \to \infty, \tag{3.10}$$

which proves the first half of Theorem 2. The proof would be complete if we knew that the unit disc dissipates the energy

$$E_1(t) = \frac{2\pi t}{\ln t} + \frac{2\pi t}{(\ln t)^2}(1 + \gamma - \ln 2) + o(t \ln^{-2} t). \tag{3.11}$$

For this fact see Carslaw and Jaeger ([1] pp. 335—336), who invert the Laplace transform in (3.3) to obtain a complicated expression for $f(x, t) = P_x[T_1 \leqq t]$, and a simpler one for the temperature gradient at the circumference of the unit disc, namely

$$\Phi(t) = \left[\frac{\partial}{\partial |x|} f(x, t)\right]_{|x|=1} = -\frac{4}{\pi^2}\int_0^\infty \frac{1}{u} e^{-u^2 t/2} \frac{du}{J_0^2(u) + Y_0^2(u)}. \tag{3.12}$$

But

$$\frac{\partial E_1(t)}{\partial t} = \frac{\partial}{\partial t}\int_{|x| \geqq 1} f(x, t)\, dx = \frac{1}{2}\int_{|x| \geqq 1} \Delta f(x, t)\, dx,$$

and integration by parts yields

$$\frac{\partial E_1}{\partial t} = -\pi \Phi(t). \tag{3.13}$$

Jaeger ([1] p. 336) states that

$$-\Phi(t) = \frac{2}{\ln 2t - 2\gamma} - \frac{2\gamma}{(\ln 2t - 2\gamma)^2} + o(\ln^{-2} t), \quad t \to \infty. \tag{3.14}$$

To sketch an elementary proof (all the arguments in the literature seem to infer the behavior of $\Phi(t)$ as $t \to \infty$ from that of its Laplace transform near zero) let $\approx$ denote equality up to terms of smaller order than $(\ln t)^{-2}$, as $t \to \infty$. Then, by (3.12)

$$-\Phi(t) \approx \frac{4}{\pi^2}\int_0^A \frac{e^{-\frac{u^2}{2}t}}{u[J_0^2(u) + Y_0^2(u)]}\, du$$

for every $A > 0$, since $J_0^2(u) + Y_0^2(u)$ is bounded away from 0. From the asymptotic behavior of these Bessel functions

$$J_0^2(u) + Y_0^2(u) = 1 + \frac{4}{\pi^2}\left(\ln\frac{u}{2} + \gamma\right)^2 + o(u), \quad u \to \infty,$$

so that

$$-\Phi(t) \approx \int_0^A \frac{e^{-\frac{u^2}{2}t}}{u\left[\frac{\pi^2}{4} + \left(\ln\frac{u}{2} + \gamma\right)^2\right]}\, du = 2\int_0^{\frac{A}{2}t^2} \frac{e^{-x}}{x[\pi^2 + (\ln x - \ln \tau)^2]}\, dx,$$

where we have defined τ by

$$\ln \tau = \ln 2t - 2\gamma .$$

It follows that for every $B > 0$

$$-\Phi(t) \approx 2 \int_0^{B\tau} \frac{e^{-x}}{x[\pi^2 + (\ln x - \ln \tau)^2]} dx$$

$$\approx 2 \int_0^{B\tau} \frac{e^{-x}}{x(\ln x - \ln \tau)^2} dx, \quad \tau \to \infty .$$

Now an integration by parts yields

$$-\Phi(t) \approx 2 \int_0^{B\tau} \frac{e^{-x}}{\ln \tau - \ln x} dx .$$

Finally

$$-\Phi(t) \approx \frac{2}{\ln \tau} + 2 \int_0^{B\tau} e^{-x} \left[\frac{1}{\ln \tau - \ln x} - \frac{1}{\ln \tau}\right] dx$$

$$\approx \frac{2}{\ln \tau} + \frac{2}{(\ln \tau)^2} \int_0^{B\tau} e^{-x} \ln x \, dx$$

$$\approx \frac{2}{\ln \tau} + \frac{2}{(\ln \tau)^2} \int_0^{\infty} e^{-x} \ln x \, dx$$

$$= \frac{2}{\ln \tau} - \frac{2\gamma}{(\ln \tau)^2}, \quad \tau \to \infty .$$

Thus (3.14) is proved, and by (3.13) we may integrate (3.14) to obtain (3.11). That completes the proof of Theorem 2.

4. Related probability problems

Both theorems are valid in a more general setting, roughly speaking for stochastic processes whose potential theory resembles the classical one. In particular the result of KAC and HUNT concerning the asymptotic behavior of $P_x[T_A > t]$ has been extended to arbitrary recurrent random walk in [*7*]. (See also [*9*], Ch. 7, problem 10 for an incomplete generalization of Theorem 2.) However, Theorem 1 has the more interesting history of having suggested improvements even in the case of Brownian motion. In 1951 DVORETZKY and ERDÖS [*3*] considered a related problem for simple random walk on the lattice points of s dimensional Euclidean space. Let L_n = the number of distinct points visited by the random walk in time n. If T_x denotes the time of the first visit to the point x, then

$$E[L_n] = \sum_{x \neq 0} P[T_x \leqq n] .$$

Thus $E[L_n]$ is the discrete analogue of $E_A(t)$ with the set $A = \{0\}$ consisting of a single point. Theorem 1 then has the natural analogue that $E[L_n]/n$ tends to a limit, but DVORETZKY and ERDÖS proved more, i. e., they showed that L_n/n *converges with probability one.* Now replace L_n by $L_A(n)$ = the number of lattice points x such that the random walk $x_k \in x + A$ for some k in the interval $[1, n]$. Again one finds that $E[L_A(n)]$ is the analogue of the energy $E_A(t)$ and that the corresponding strong (probability one) statement holds. (For a proof valid for arbitrary random walk, and for related results see [*10*].) The corresponding strong law for Brownian motion (a proof based on Theorem 1 and Birkhoff's ergodic theorem is given by WHITMAN [*10*]) reads as follows:

For Brownian motion $x(t)$ in 3 space R (in any dimension s if $C(A)$ below is interpreted correctly) and for any compact set $A \subset R$ let $A_t = [x \mid x \in R, x(\tau) \in x + A$ for some $0 \leqq \tau \leqq t]$. Let $A_0 = A$, and denote the Lebesgue measure (volume) of A_t by $|A_t|$. Then

$$E[|A_t|] = E_A(t) + |A|$$

in the notation of (1.1) and

$$\lim_{t \to \infty} \frac{|A_t|}{t} = C(A)$$

with probability one on the probability space of the Brownian motion.

References

[*1*] CARSLAW, H. S., and J. C. JAEGER: Conduction of heat in solids. 2nd ed. Oxford: Clarendon Press, 1959.

[*2*] DOOB, J. L.: A probability approach to the heat equation. Trans. Amer. math. Soc. **80**, 216–280 (1955).

[*3*] DVORETZKY, A., and P. ERDÖS: Some problems on random walk in spac. Proc. 2nd Berkeley Sympos. math. Statist. Probability. 353–368 (1951).

[*4*] HUNT, G. A.: Some theorems concerning Brownian motion. Trans. Amer. math. Soc. **81**, 294–319 (1956).

[*5*] ITÔ, K., and H. P. MCKEAN: Diffusion processes and their sample paths. Berlin: Springer 1964.

[*6*] JOFFE, A.: Sojourn time for stable processes. Thesis, Cornell U., 1959.

[*7*] KESTEN, H., and F. SPITZER: Ratio theorems for random walk (I). J. Analyse math. **11**, 285–321 (1963).

[*8*] NEVANLINNA, R. H.: Eindeutige analytische Funktionen. 2nd ed. Berlin: Springer, 1953.

[*9*] SPITZER, F.: Principles of random walk. Princeton: Van Nostrand 1964.

[*10*] WHITMAN, W.: Thesis. Cornell U. 1964.

Department of Mathematics
Cornell University
Ithaca, New York, USA

(Received August 20, 1963)