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On the Asymptotic Behaviour of $\sum f(n_k x)$

Main Theorems

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1. § Introduction

Let $f(x)(-\infty<x<\infty)$ be a measurable function such that

$$f(x+1)=f(x), \qquad \int_0^1 f(x)\,dx=0, \qquad \|f\|^2=\int_0^1 f^2(x)\,dx<+\infty. \tag{1.1}$$

The asymptotic properties of the sequence $f(n_k x)$ for rapidly increasing sequences n_k of integers have been investigated by many authors. In particular, it has been proved (see [3]) that for any fixed f satisfying (1.1) there exists a sequence n_k such that the sequence $f(n_k x)$ imitates the properties of independent random variables in a very strong sense. If f satisfies certain smoothness conditions, one can also give estimates for the rate of growth of the sequence n_k implying this "independent-like" behaviour. For instance, if f satisfies the Lipschitz condition then $n_{k+1}/n_k\to\infty$ guarantees that $f(n_k x)$ obeys the central limit theorem and the law of the iterated logarithm (see [7, 17]). Here $n_{k+1}/n_k\to\infty$ cannot be replaced by the weaker condition

$$n_{k+1}/n_k\geqq q>1 \qquad (k=1,2,\ldots) \tag{1.2}$$

even for very smooth functions f: a simple example shows that for any given q (arbitrary large) there exists a trigonometric polynomial f and a sequence n_k satisfying (1.2) such that the sequence $f(n_k x)$ fails to satisfy the central limit theorem. A closer look at the problem shows that in the case when only (1.2) is assumed, the asymptotic behaviour of $f(n_k x)$ is strongly influenced by the arithmetical properties of the sequence n_k. For instance, the independent-like behaviour holds if $n_k=a^k$ ($a\geqq 2$ is integer) but can fail if $n_k=a^k-1$. Similarly, we have the independent-like behaviour if $n_{k+1}/n_k\to\alpha$ where α^r is irrational for every integer $r\geqq 1$. This phenomenon has been investigated profoundly by Gapoškin (see [4] and also [2]) who has given a necessary and sufficient condition for $f(n_k x)$ to obey the central limit theorem. Let us say that a sequence $m_1<m_2<\cdots$ of positive integers satisfies condition B_2 if the number of solutions of $m_k\pm m_l=v$ $(k>l)$ does not exceed a constant C for any $v>0$. Gapoškin's theorem states that if n_k satisfies (1.2) then the sequence $f(n_k x)$ obeys the central limit theorem for all sufficiently smooth functions f (satisfying (1.1)) if and only if, for any $m \geqq 1$, the set-theoretic union of the sequences $\{n_k\}, \{2n_k\}, \ldots, \{mn_k\}$ satisfies condition B_2.

The purpose of the present paper is to investigate the asymptotic properties of the sequence $f(n_k x)$ when only (1.2) is assumed. Though the central limit theorem does not necessarily hold for such sequences, we shall formulate positive results without imposing any arithmetical restrictions on n_k. To state the results qualitatively, let us note that the validity of the central limit theorem and the law of the iterated logarithm for the sequence $f(n_k x)$ mean that the asymptotic distribution and asymptotic order of magnitude of $\sum_{k=1}^{N} f(n_k x)$, $N \to \infty$ are the same as those of $\zeta(N)$, $N \to \infty$ where ζ is a standard Wiener-process. Let us introduce the quantities

$$v_{M,N}^{i,k} = 2^k \int_{i2^{-k}}^{(i+1)2^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx.$$

Now, the main result of our paper states that if f is smooth enough, the sequence n_k satisfies (1.2) and $C_1 N \leqq v_{M,N}^{i,k} \leqq C_2 N$ hold for certain values of M, N, i, k (C_1, C_2 are positive constants) then the asymptotic behaviour of $\sum_{k=1}^{N} f(n_k x)$, $N \to \infty$ is the same as that of $\zeta(\tau_N)$, $N \to \infty$ where ζ is a standard Wiener process and τ_N is a sequence of random variables which is closely related to the quantities $v_{M,N}^{i,k}$. Hence the asymptotic behaviour of $\sum_{k=1}^{N} f(n_k x)$ is intimately connected with that of the $v_{M,N}^{i,k}$'s. Using this fact, we can derive many limit theorems for the sequence $f(n_k x)$ both in the case when the sequence exhibits the independent-like behaviour and in the case when this behaviour does not hold. As a first application let us consider the case when the sequence n_k satisfies a certain arithmetical condition of Gapoškin type. Then, as one can see easily, the quantities $v_{M,N}^{i,k}$ become asymptotically independent of i, k and thus τ_N become asymptotically constant. Hence, in this case the asymptotic behaviour of $\sum_{k=1}^{N} f(n_k x)$ is the same as that of $\zeta(a_N)$ for a certain numerical sequence a_N. Some typical corrollaries of this fact are Donsker's invariance principle, Strassen's law of the iterated logarithm and Kolmogorov-Erdös-Petrovski type upper and lower class criteria for the sequence $f(n_k x)$. These results unify and extend several limit theorems obtained earlier in the literature. As a second application we shall show that though the central limit theorem and the law of the iterated logarithm do not necessarily hold under condition (1.2), they are "nearly" satisfied if q is large. More exactly, if f is smooth enough and (1.2) holds then we have

$$\overline{\lim_{N \to \infty}} \sup_{-\infty < t < +\infty} \left| P\left(\sum_{k=1}^{N} f(n_k x) < \sigma t \sqrt{N} \right) - \Phi(t) \right| \leqq \varepsilon(q)^{1}$$

[1] In probabilistic statements concerning the sequence $f(n_k x)$ the probability space is the interval $[0, 1)$ with Lebesgue measure.

where $\sigma = \|f\|$ and $\varepsilon(q) \to 0$ if $q \to \infty$. Similarly,

$$1-\varepsilon(q) \leqq \overline{\lim_{N\to\infty}} (2\sigma^2 N \operatorname{loglog} N)^{-1/2} \sum_{k=1}^{N} f(n_k x) \leqq 1+\varepsilon(q) \qquad \text{a.e.}$$

The functional versions of these results are also valid. Thirdly, our results lead to interesting consequences even in the classical case $f(x) = \cos 2\pi x$ when we get an a.s. invariance principle under the mere condition (1.2).

It is of some interest to note that our results hold without the assumption that n_k are integers. It seems that for non-integral n_k even the central limit theorems and laws of the iterated logarithm implied by our results are new. We get, e.g., the interesting result that the sequence $f(q^k x)$ obeys the central limit theorem (in the sense of footnote 1) for any real $q > 1$. (For a related result see [15].)

The idea of the proofs is to split the partial sums of $f(n_k x)$ into disjoint blocks and apply Strassen's well-known martingale invariance principle (Theorem (4.4) of [14]) for these blocks. In the paper [1] we used the same method to get a.s. invariance principles for mixing processes. Independently and at the same time, Philipp and Stout used a similar approach to get a.s. invariance principles for many classes of weakly dependent random variables; see their nice and exhaustive paper [12].

Our paper consists of two parts. In the present, first part we establish some a.s. invariance principles for the sequence $f(n_k x)$ under general conditions on $f(x)$ and $\{n_k\}$. In the second part (see the next paper in this journal) we give some applications of these theorems.

2. § Main Results

Before formulating our theorems we make a few preliminary remarks.

Let $X_1, X_2, \ldots$ be a sequence of independent random variables on the probability space $(\Omega, \mathscr{F}, P)$ and put $S_n = \sum_{i=1}^{n} X_i$ $(S_0 = 0)$. The investigation of the asymptotic properties of the sequence S_n is a classical problem of probability theory. In [13, 14] Strassen developed a new and powerful method for approaching this problem. Namely, he proved that in certain cases it is possible to construct a Wiener-process $\zeta(t)$ such that the sequences S_n and $\zeta(n)$ are "close" to each other with probability one. Such an approximation theorem was called by him an "almost sure invariance principle" because a theorem of this type enables us to carry over many asymptotic properties of the Wiener-process in an unchanged form for the partial sums S_n. For instance, it is easy to see that if $S_n = \zeta(n) + o(n^{1/2-\eta})$ almost surely with a suitable Wiener process $\zeta(t)$ and a constant $\eta > 0$ then the sequence $X_1, X_2, \ldots$ obeys not only the central limit theorem and the law of the iterated logarithm but also a larger class of stronger limit theorems including Donsker's invariance principle, the functional form of the law of the iterated logarithm, the Kolmogorov-Erdös-Petrovski integral test for functions of upper and lower classes etc. Now, a typical result of Strassen states that if $X_1, X_2, \ldots$ are independent, identically distributed with $E X_1 = 0$,

$EX_1^2=1, E|X_1|^{2+\delta}<\infty$ $(\delta>0)$ then there exists a new probability space $(\Omega', \mathscr{F}', P')$, a sequence $X_1', X_2', \ldots$ of i.i.d. random variables and a Wiener-process $\zeta'(t)$ (all defined on $(\Omega', \mathscr{F}', P')$) such that X_1 and X_1' have the same distribution and putting $S_n'=\sum_{i=1}^{n} X_i'$ we have $S_n'=\zeta'(n)+o(n^{1/2-\eta})$ for $n\to\infty$ with probability one where η is a positive constant depending on δ. The fact that here we approximate not the partial sums of the sequence $X_1, X_2, \ldots$ but a "copy" of it, makes no trouble in applications since if a sequence $Y_1, Y_2, \ldots$ (of arbitrary random variables) obeys, e.g., the law of the iterated logarithm then the same holds for every sequence $Y_1', Y_2', \ldots$ having the same finite dimensional distributions. Strassen also proved that similar results hold in the case when $X_1, X_2, \ldots$ is a martingale difference sequence. In the latter case, however, the approximation theorem has the slightly modified form $S_n=\zeta(\tau_n)+o(n^{1/2-\eta})$ a.s. where ζ is a Wiener-process and τ_n is a certain increasing sequence of random variables such that $\tau_n\to\infty$ a.s. In the present paper we shall prove results of this type for the sequence $f(n_k x)$ which can be considered as a sequence of (dependent) random variables on the probability space $(\Omega_0, \mathscr{F}_0, P_0)$ where $\Omega_0=[0,1)$, $\mathscr{F}_0$ is the class of measurable subsets of $[0,1)$ and P_0 is the Lebesgue measure on $\mathscr{F}_0$.

Throughout our paper we shall assume the standard condition

$$|f(x)|\leqq M \quad \text{and} \quad \|f-s_n\|\leqq An^{-\alpha} \qquad (\alpha>0, n=1,2,\ldots) \tag{2.1}$$

where s_n denotes the n-th partial sum of the Fourier-series of f. The second relation of (2.1) can also be written as

$$\frac{1}{2}\sum_{k=n+1}^{\infty}(a_k^2+b_k^2)\leqq A^2 n^{-2\alpha} \tag{2.2}$$

where

$$f\sim a_0+\sum_{k=1}^{\infty}(a_k\cos 2\pi kx+b_k\sin 2\pi kx)$$

is the Fourier-expansion of f. Condition (2.1) is satisfied, e.g., if f satisfies the Lipschitz α condition (see [18] p. 241, formula (3.3)) or it is of bounded variation. (In the latter case we have $a_k=O(1/k)$, $b_k=O(1/k)$ and thus (2.2) is valid with $\alpha=1/2$.)

Definition 1. Let $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ be two sequences of random variables defined on possibly different probability spaces. We say that the two sequences are equivalent if their finite dimensional distributions are the same.

Definition 2. Let $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ be sequences of random variables on the probability spaces $(\Omega, \mathscr{F}, P)$ and $(\tilde\Omega, \tilde{\mathscr{F}}, \tilde P)$, respectively. We say that the two sequences are quasi-equivalent if there exist sequences $\hat Y_1, \hat Y_2, \ldots$ and $\hat Z_1, \hat Z_2, \ldots$ (defined on $(\Omega, \mathscr{F}, P)$ and $(\tilde\Omega, \tilde{\mathscr{F}}, \tilde P)$, respectively) such that $\sum_{k=1}^{\infty}|Y_k-\hat Y_k|<\infty$ a.s., $\sum_{k=1}^{\infty}|Z_k-\hat Z_k|<\infty$ a.s. and the sequences $\hat Y_1, \hat Y_2, \ldots$ and $\hat Z_1, \hat Z_2, \ldots$ are equivalent.

We can now formulate our results.

Theorem 1. *Let us assume that*

a) $f(x)$ *satisfies* (1.1) *and* (2.1).

b) The sequence n_k *of positive numbers satisfies* (1.2).

c) There exist constants $\sigma_2 > \sigma_1 > 0$ *such that for* $k \geqq 1$, $0 \leqq i \leqq 2^k - 1$ *we have*

$$\sigma_1 N \leqq 2^k \int_{i2^{-k}}^{(i+1)2^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx \leqq \sigma_2 N \tag{2.3}$$

provided that M, N *and* $n_M/N \cdot 2^k$ *are large enough.*

Then there exists a probability space $(\Omega, \mathscr{F}, P)$ *and a sequence* $X_1, X_2, \ldots$ *of random variables (defined on* $(\Omega, \mathscr{F}, P)$*) such that the sequences* $\{f(n_k x)\}$ *and* $\{X_k\}$ *are quasi-equivalent and*

$$X_1 + \cdots + X_n = \zeta(\tau_n) + o(n^{1/2 - \eta}) \qquad \text{a.s. as } n \to \infty \tag{2.4}$$

where $\eta > 0$ *is an absolute constant,* $\zeta(t)$ *is a Wiener-process on* $(\Omega, \mathscr{F}, P)$ *and* τ_n *is a strictly increasing sequence of positive random variables (also defined on* $(\Omega, \mathscr{F}, P)$*) such that* $\tau_n - \tau_{n-1} = O(1)$ a.s. *as* $n \to \infty$ *and*

$$\sigma_1 \leqq \liminf_{n \to \infty} \frac{\tau_n}{n} \leqq \limsup_{n \to \infty} \frac{\tau_n}{n} \leqq \sigma_2 \qquad \text{a.s.} \tag{2.5}$$

In applications we shall need also the following, somewhat more general form of Theorem 1:

Theorem 2. *Let us replace condition* c) *of Theorem* 1 *by the following condition* c*):

c*) *There exist constants* $\sigma_2 > \sigma_1 > 0$ *such that for* $k \geqq 1$, $0 \leqq i \leqq 2^k - 1$ *we have*

$$\sigma_1 a_{M,N} \leqq 2^k \int_{i2^{-k}}^{(i+1)2^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx \leqq \sigma_2 a_{M,N} \tag{2.6}$$

provided that M, N *and* $n_M/N \cdot 2^k$ *are large enough. Here* $a_{M,N}$ $(M \geqq 0,\ N \geqq 1)$ *are positive numbers such that* $A_1 N \leqq a_{M,N} \leqq A_2 N$ $(M \geqq M_0,\ N \geqq N_0)$ *with positive constants* A_1, A_2.

Then the conclusion of Theorem 1 *remains valid but instead of* (2.5) *we have now*

$$\sigma_1 \leqq \liminf_{n \to \infty} \frac{\tau_n}{b_n} \leqq \limsup_{n \to \infty} \frac{\tau_n}{b_n} \leqq \sigma_2 \qquad \text{a.s.} \tag{2.7}$$

where b_n *is a strictly increasing numerical sequence such that*[2] $b_n \asymp n$.

We formulate one more theorem which states that replacing (2.3) by a stronger assumption, we shall have (2.4) with $\tau_n = \sigma n$.

Theorem 3. *Let us assume that*

a) $f(x)$ *satisfies* (1.1) *and* (2.1).

b) *The sequence* n_k *of positive numbers satisfies* (1.2).

c) *There exists a constant* $\sigma > 0$ *such that for* $k \geqq 1$, $0 \leqq i \leqq 2^k - 1$, $M \geqq M_0$,

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[2] The symbols $a_n \sim b_n$ and $a_n \asymp b_n$ mean $\lim_{n\to\infty} a_n/b_n = 1$ and $0 < \liminf_{n\to\infty} a_n/b_n \leqq \limsup_{n\to\infty} a_n/b_n < \infty$, respectively.

$N \geqq N_0$ we have

$$2^k \int_{i2^{-k}}^{(i+1)2^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx = \sigma N + O\left(\frac{N \cdot 2^k}{n_M} \right) \tag{2.8}$$

where the constant in O depends only on $f(x)$ and q.

Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a sequence $X_1, X_2, \ldots$ of random variables (defined on $(\Omega, \mathcal{F}, P)$) such that the sequences $\{f(n_k x)\}$ and $\{X_k\}$ are quasi-equivalent and

$$X_1 + \cdots + X_n = \zeta(\sigma n) + o(n^{1/2-\eta}) \quad \text{a.s. } as\ n \to \infty \tag{2.9}$$

where $\zeta(t)$ is a Wiener-process on $(\Omega, \mathcal{F}, P)$ and $\eta > 0$ is an absolute constant.

Remarks. 1. Conditions c) of Theorem 1 and Theorem 3 are of the same nature: both require an estimate for the quantity

$$2^k \int_{i2^{-k}}^{(i+1)2^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx$$

provided that M, N and $n_M/N \cdot 2^k$ are large enough. These conditions can slightly be weakened. In Theorem 1 it is sufficient to require that (2.3) holds if M, N and $n_M/N^\gamma \cdot 2^k$ are large enough where $\gamma > 0$ is a fixed constant. Similarly, the remainder term $O(N \cdot 2^k/n_M)$ in (2.8) (and also in (2.10) below) can be replaced by $O(N^\gamma 2^k/n_M)$. The proofs of Theorems 1–3 (given in § 4) apply also under these conditions without change.

2. If the sequence $f(n_k x)$ satisfies condition c) of Theorem 1 with some pairs (σ_1, σ_2) then we can choose a universal $\{X_n\}$, $\{\tau_n\}$ and ζ satisfying (2.4), (2.5) (and $\tau_n - \tau_{n-1} = O(1)$ a.s.) with all the pairs (σ_1, σ_2). A similar remark holds for Theorem 2 (when also b_n can be chosen universal).

3. The proofs of our theorems will yield an explicit estimate for the absolute constant η in (2.4) and (2.9). Actually, we shall see that (2.4) and (2.9) are valid with any constant $0 < \eta < 1/40$. We chould get slightly better estimates by more precise calculations and by some simple modifications of the argument but to find the best constant seems to be very difficult. In view of a recent result of Komlós, Major and Tusnády (see [10]) it is even possible that the remainder term $o(n^{1/2-\eta})$ in (2.4) and (2.9) can be replaced by $o(n^\varepsilon)$ for any $\varepsilon > 0$.

4. For the sequence b_n in Theorem 2 we have

$$A_1 \leqq \liminf_{n \to \infty} b_n/n \leqq \limsup_{n \to \infty} b_n/n \leqq A_2.$$

5. Let us replace condition (2.8) by

$$2^k \int_{i2^{-k}}^{(i+1)2^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx = a_{M,N} + O\left(\frac{N \cdot 2^k}{n_M} \right) \tag{2.10}$$

where $a_{M,N}$ ($M \geqq 0$, $N \geqq 1$) are positive numbers such that $A_1 N \leqq a_{M,N} \leqq A_2 N$ ($M \geqq M_0$, $N \geqq N_0$) with positive constants A_1, A_2. Then the conclusion of Theorem 3 remains valid with the modification that $\zeta(\sigma n)$ in (2.9) is to be replaced by $\zeta(b_n)$ where b_n is a strictly increasing numerical sequence such that $b_n \asymp n$. (Actually $A_1 \leqq \liminf_{n \to \infty} b_n/n \leqq \limsup_{n \to \infty} b_n/n \leqq A_2$.)

6. Theorem 1 remains valid if (2.3) is replaced by

$$\sigma_1 N \leqq a^k \int_{ia^{-k}}^{(i+1)a^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx \leqq \sigma_2 N$$

for a fixed integer $a \geqq 2$. A similar remark applies to Theorems 2 and 3.

3. § Some Standard Inequalities

In this section we shall formulate some inequalities, each of standard type, which will be used in the proofs of Theorems 1–3. Their proofs will be given in Section 5.

Throughout this paper, the following notation will be useful. Given an integrable function $g(t)$ $(0 \leqq t < 1)$ and an integer $m \geqq 1$, let $[g]_m$ denote the function in $0 \leqq t < 1$ which takes the constant value

$$m \int_{k/m}^{(k+1)/m} g(s)\, ds$$

in the interval $[k/m, (k+1)/m)$ $(k=0, 1, \ldots, m-1)$ [3]

Lemma (3.1). *Let $f(x)$ satisfy* (1.1) *and the second relation of* (2.1). *Put $\psi(x) = f(\lambda x)$ $(0 \leqq x < 1)$ where $\lambda \geqq 1$ is an arbitrary real number. Then we have for any integer $m \geqq \lambda$*

$$\|\psi - [\psi]_m\| \leqq C_1 \left(\frac{m}{\lambda}\right)^{-\alpha/3}$$

where C_1 is a positive constant depending only on f.

Lemma (3.2). *Let $g(x)$ $(-\infty < x < \infty)$ be a function such that*

$$g(x+1) = g(x), \qquad \int_0^1 g(x)\, dx = 0. \tag{3.1}$$

Then we have

$$\left| \int_a^b g(\lambda x)\, dx \right| \leqq \frac{2}{\lambda} \int_0^1 |g(x)|\, dx$$

for any real numbers $a < b$ and any $\lambda > 0$.

Lemma (3.3). *Let $f(x)$ satisfy* (1.1) *and the second relation of* (2.1) *with $A > 1$ and $0 < \alpha \leqq 1$. Let $1 \leqq m_1 < m_2 < \cdots < m_n$ be arbitrary real numbers such that $m_{k+1}/m_k \geqq q > 1$ for $1 \leqq k \leqq n-1$. Then we have for any real a*

$$\int_a^{a+1} (f(m_1 x) + \cdots + f(m_n x))^2\, dx = n \|f\|^2 + T$$

where

$$|T| \leqq \frac{4q}{q-1} \|f\|^2 + C_2 A \left(\frac{1}{q^{1/2}-1} + \frac{1}{q^\alpha - 1} + \frac{\|f\|}{\log q} \right) (\|f\|^2 + \|f\|)\, n$$

with an absolute constant C_2.

[3] This notation is a modification of a notation used in [6].

Corollary 1. *Assume the conditions of Lemma* (3.3) *and also* $\|f\| \leqq 1$. *Then we have for any real a*

$$\int_a^{a+1} (f(m_1 x) + \cdots + f(m_n x))^2 \, dx \leqq C_3 A \|f\| n$$

where the constant C_3 *depends only on* q.

Corollary 2. *Let* $f(x)$ *satisfy* (1.1) *and the second relation of* (2.1). *Let* $1 \leqq m_1 < m_2 < \cdots$ *be a sequence of real numbers such that* $m_{k+1}/m_k \to \infty$. *Then we have*

$$\int_a^{a+1} \left(\sum_{\nu=l+1}^{l+n} f(m_\nu x) \right)^2 dx \sim n \|f\|^2 \quad \text{as } n \to \infty$$

uniformly in l *and* a.

Lemma (3.4). *Let* $f(t)$ *satisfy* (1.1) *and* (2.1) *and let* $n_1 < n_2 < \cdots$ *be a sequence of positive numbers such that* $n_{k+1}/n_k \geqq q > 1$ *for* $k \geqq 1$. *Then for any* $N \geqq N_0$ (*where* N_0 *depends on* $f(t)$ *and* q) *we have*

$$\sum_{\nu=1}^{N} f(n_\nu t) = \xi_1 + \xi_2$$

where the random variables ξ_1, ξ_2 *satisfy*

$$P(|\xi_1| \geqq y \sqrt{N}) \leqq C_4 e^{-C_5 y} \quad (y \geqq 0)^4 \tag{3.2}$$

and

$$\|\xi_2\| \leqq 1. \tag{3.3}$$

The constants C_4, C_5 *depend on* $f(t)$ *and* q.

This lemma is a variant of similar lemmas of Takahashi [16] and Philipp [11]. Its proof (which is given in § 5) depends on the same ideas.

Lemma (3.5). *Let* η_1, η_2 *and* $\eta = \eta_1 + \eta_2$ *be square integrable random variables with distribution functions* $F_1(x)$, $F_2(x)$ *and* $G(x)$, *respectively. Then we have for any* $a \geqq 0$

$$\int_{x^2 \geqq a} x^2 \, dG(x) \leqq 4 \int_{x^2 \geqq a/4} x^2 \, dF_1(x) + 4 \int_{x^2 \geqq a/4} x^2 \, dF_2(x).$$

Lemma (3.6). *Let* $\zeta(t)$ *be a (separable) Wiener-process and let* B, B', r, s *be positive numbers. Then we have*

$$\sup_{\substack{0 \leqq t_1 < t_2 \leqq B k^r \\ |t_2 - t_1| \leqq B' k^s}} |\zeta(t_2) - \zeta(t_1)| = O(k^{s/2} \log k) \quad \text{a.s. as } k \to \infty.$$

Finally we formulate a simple lemma which is not an inequality but will be useful in the proof of our main theorems.

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[4] We shall use the symbols E, P also in the probability space $(\Omega_0, \mathscr{F}_0, P_0)$, they denote expectation and probability with respect to Lebesgue measure. When it is more convenient, we shall use also the symbols $\int_A$ and $|A|$ for Lebesgue integral and measure.

Lemma (3.7). *Let $Y_1, Y_2, \ldots$ be a sequence of random variables on the probability space $(\Omega, \mathscr{F}, P)$ such that every Y_i takes only finitely many values. Let $1=a_1<b_1<a_2<b_2<\cdots$ be a sequence of integers and put $U_k=\sum_{i=a_k}^{b_k} Y_i$. Let $(\Omega', \mathscr{F}', P')$ be an other, atomless*[5] *probability space and let $U_1', U_2', \ldots$ be a sequence of random variables on $(\Omega', \mathscr{F}', P')$ which is equivalent to $U_1, U_2, \ldots$. Then there exists a sequence $Y_1', Y_2', \ldots$ of random variables on $(\Omega', \mathscr{F}', P')$ which is equivalent to $Y_1, Y_2, \ldots$ and $U_k'=\sum_{i=a_k}^{b_k} Y_i'$ $(k=1,2,\ldots)$.*

4. § Proof of the Main Theorems

We begin with the proof of Theorem 1. As the first step of the proof we approximate $f(n_k x)$ by a step-function $\varphi_k(x)$ as follows. Let $2^l \leqq n_k < 2^{l+1}$, put $m=\left[l+\frac{60}{\alpha}\log k\right]$ and define $\varphi_k(x)$ by

$$\varphi_k(x)=[f(n_k x)]_{2^m}.\text{[6]} \tag{4.1}$$

Using Lemma (3.1) we get

$$\begin{aligned}\|f(n_k x)-\varphi_k(x)\| &\leqq C\left(\frac{2^m}{n_k}\right)^{-\alpha/3} \leqq C\left(\frac{2^{l+\frac{60}{\alpha}\log k-1}}{2^{l+1}}\right)^{-\alpha/3}\\ &\leqq C\cdot 2^{-20\log k} \leqq C k^{-10} \qquad (k=2,3,\ldots).\end{aligned} \tag{4.2}$$

(In this section C will denote positive constants, not always the same, depending only on $f(t)$ and q.) Relation (4.2) implies

$$\sum_{k=1}^{\infty} |f(n_k x)-\varphi_k(x)|<\infty \qquad \text{for almost all } x\in[0,1). \tag{4.3}$$

We now divide the set of positive integers into consecutive blocks

$$\Delta_1, \Delta_1', \Delta_2, \Delta_2', \ldots, \Delta_k, \Delta_k', \ldots$$

(without gaps) in such a way that Δ_k contains $[k^{1/2}]$ consecutive integers, Δ_k' contains $[k^{1/4}]$ consecutive integers $(k=1,2,\ldots)$. We shall call Δ_k long blocks and Δ_k' short blocks. Put

$$T_k=\sum_{\nu\in\Delta_k} f(n_\nu x), \qquad D_k=\sum_{\nu\in\Delta_k}\varphi_\nu(x), \tag{4.4}$$

$$\bar{D}_k=D_k-E(D_k\,|\,D_1,\ldots,D_{k-1}).\text{[7]} \tag{4.5}$$

Let further $\mathscr{F}_{k-1}$ and $\bar{\mathscr{F}}_{k-1}$ denote the σ-fields generated by $D_1,\ldots,D_{k-1}$ and $\bar{D}_1,\ldots,\bar{D}_{k-1}$, respectively. (Evidently $\bar{\mathscr{F}}_{k-1}\subset\mathscr{F}_{k-1}$.)

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[5] A probability space $(\Omega, \mathscr{F}, P)$ is called atomless if for every $A\in\mathscr{F}$ with $P(A)>0$ and for every $0<p<P(A)$ there exists a $B\in\mathscr{F}$, $B\subset A$ such that $P(B)=p$.

[6] The right-hand side of (4.1) means $[g]_{2^m}$ where g is the function defined by $g(x)=f(n_k x)$ $(0\leqq x<1)$.

[7] See footnote 4.

Lemma (4.1). *We have (as $k \to \infty$)*

$$E(D_k|\mathscr{F}_{k-1}) = O(k^{-2}) \quad \text{a.e.} \tag{4.6}$$

$$\sigma_1[k^{1/2}] + O(k^{-2}) \leqq E(D_k^2|\mathscr{F}_{k-1}) \leqq \sigma_2[k^{1/2}] + O(k^{-2}) \quad \text{a.e.} \tag{4.7}$$

These relations remain valid if the conditioning σ-field $\mathscr{F}_{k-1}$ is replaced by $\overline{\mathscr{F}_{k-1}}$. (We note that the constants implied by O can now depend also on the element x of the probability space.)

Proof. We first show that

$$|E(T_k|\mathscr{F}_{k-1})| \leqq C k^{-2} \quad \text{everywhere} \tag{4.8}$$

$$\sigma_1[k^{1/2}] \leqq E(T_k^2|\mathscr{F}_{k-1}) \leqq \sigma_2[k^{1/2}] \quad \text{everywhere } (k \geqq k_0) \tag{4.9}$$

$$\|T_k - D_k\| \leqq C k^{-4}. \tag{4.10}$$

From these relations (4.6), (4.7) will follow easily.

Let $b = b(k)$ denote the largest integer of the block Δ_{k-1}, let l be an integer such that $2^l \leqq n_b < 2^{l+1}$ and put $w = \left[l + \frac{60}{\alpha}\log b\right]$. From the definition of φ_k it follows that every φ_v, $1 \leqq v \leqq b$ takes a constant value on each interval A of the form

$$A = [i2^{-w}, (i+1)2^{-w}) \quad (i = 0, 1, \ldots, 2^w - 1) \tag{4.11}$$

and thus every set $\{D_1 = a_1, \ldots, D_{k-1} = a_{k-1}\}$ where $a_1, \ldots, a_{k-1}$ are constants, can be obtained as a union of intervals of the form (4.11). In other words, the σ-field $\mathscr{F}_{k-1}$ is purely atomic and each of its atoms is a union of intervals of the form (4.11). Hence to prove (4.8) and (4.9) it is sufficient to show that

$$\left| |A|^{-1} \int_A T_k \, dx \right| \leqq C k^{-2} \tag{4.12}$$

and

$$\sigma_1[k^{1/2}] \leqq |A|^{-1} \int_A T_k^2 \, dx \leqq \sigma_2[k^{1/2}] \quad (k \geqq k_0) \tag{4.13}$$

hold for any A in (4.11) ($|A|$ denotes the Lebesgue-measure of A). To get (4.12) let $c = c(k)$ denote the smallest integer of the block Δ_k. By (1.2) we have

$$\sum_{v \in \Delta_k} \frac{1}{n_v} \leqq \sum_{j=c}^{\infty} \frac{1}{n_j} \leqq \frac{1}{n_c}(1 + q^{-1} + q^{-2} + \cdots) = \frac{q}{q-1} \frac{1}{n_c}$$

and

$$\frac{n_b}{n_c} \leqq q^{-(c-b)} = q^{-[(k-1)^{1/4}]-1} \leqq q^{-(k-1)^{1/4}}$$

Hence applying Lemma (3.2) and using the trivial relation $b \leqq 2k^{3/2}$ we get (A is the set in (4.11))

$$\begin{aligned} \left| |A|^{-1} \int_A T_k\,dx \right| &= \left| 2^w \int_{i2^{-w}}^{(i+1)2^{-w}} \sum_{v\in\Delta_k} f(n_v x)\,dx \right| \\ &\leqq 2^w \cdot C \sum_{v\in\Delta_k} \frac{2}{n_v} \leqq C\frac{2^w}{n_c} \\ &\leqq C\frac{2^{l+\frac{60}{\alpha}\log b}}{n_c} \leqq C\frac{n_b}{n_c} b^{\frac{60}{\alpha}} \leqq C q^{-(k-1)^{1/4}} k^{\frac{90}{\alpha}} \leqq C k^{-2} \end{aligned} \tag{4.14}$$

and thus (4.12) is valid. As to (4.13), this follows immediately from condition c) of Theorem 1 provided that $k, c(k)$ and $n_c/2^w k^{1/2}$ are sufficiently large. But this is valid for $k \geqq k_0$ since by a part of estimate (4.14) we have $2^w/n_c = O(k^{-2})$ and thus $n_c/2^w k^{1/2} \to \infty$. Hence (4.8) and (4.9) are proved. Finally, (4.10) follows from (4.2) and the Minkowski inequality:

$$\|D_k - T_k\| \leqq C \sum_{v\in\Delta_k} v^{-10} \leqq C \sum_{v=[(k-1)^{1/2}]}^{\infty} v^{-10} \leqq C k^{-4}.$$

We can now easily prove relations (4.6) and (4.7). In fact, the expectation of the k-th term of the series

$$\sum_{k=1}^{\infty} k^6 E(|T_k - D_k|^2 \mid \mathscr{F}_{k-1})$$

is $k^6 E(|T_k - D_k|^2)$ which is $O(k^{-2})$ by (4.10). Hence the series is almost everywhere convergent by the Beppo Levi theorem and consequently

$$E(|T_k - D_k|^2 \mid \mathscr{F}_{k-1}) = O(k^{-6}) \qquad \text{a.e.} \tag{4.15}$$

(4.15) implies, via the (conditional) Cauchy-Schwarz inequality, that

$$E(|T_k - D_k| \mid \mathscr{F}_{k-1}) = O(k^{-3})$$

which, together with (4.8), gives (4.6). Furthermore, by the conditional Minkowski inequality and (4.15) we have

$$|E(D_k^2 \mid \mathscr{F}_{k-1})^{1/2} - E(T_k^2 \mid \mathscr{F}_{k-1})^{1/2}| \leqq E(|T_k - D_k|^2 \mid \mathscr{F}_{k-1})^{1/2} = O(k^{-3}) \qquad \text{a.e.}$$

Hence, using (4.9) we get the upper half of (4.7):

$$\begin{aligned} E(D_k^2 \mid \mathscr{F}_{k-1}) &\leqq ((\sigma_2[k^{1/2}])^{1/2} + O(k^{-3}))^2 = \sigma_2[k^{1/2}] + O(k^{1/4}k^{-3}) + O(k^{-6}) \\ &= \sigma_2[k^{1/2}] + O(k^{-2}) \qquad \text{a.e.} \end{aligned}$$

The lower part of (4.7) can be proved similarly.

That relations (4.6), (4.7) remain valid if the conditioning σ-field $\mathscr{F}_{k-1}$ is replaced by $\overline{\mathscr{F}}_{k-1}$ can be proved exactly in the same way as above. We only have to remark that, by $\overline{\mathscr{F}}_{k-1} \subset \mathscr{F}_{k-1}$, the σ-field $\overline{\mathscr{F}}_{k-1}$ is also purely atomic and each of its atoms is a union of intervals of the form (4.11).

The following lemma is a consequence of Lemma (4.1).

Lemma (4.2). *We have* (*as* $k\to\infty$)

$$\sigma_1[k^{1/2}]+O(k^{-2})\leqq E(\bar{D}_k^2\,|\,\bar{\mathscr{F}}_{k-1})\leqq\sigma_2[k^{1/2}]+O(k^{-2})\quad\text{a.e.}$$

Proof. Put $U_k=E(D_k\,|\,\mathscr{F}_{k-1})$. Since U_k is $\mathscr{F}_{k-1}$ measurable, we have by a simple calculation

$$E(\bar{D}_k^2\,|\,\mathscr{F}_{k-1})=E((D_k-U_k)^2\,|\,\mathscr{F}_{k-1})=E(D_k^2\,|\,\mathscr{F}_{k-1})-U_k^2.$$

Taking expected values with respect to $\bar{\mathscr{F}}_{k-1}$ and using $\bar{\mathscr{F}}_{k-1}\subset\mathscr{F}_{k-1}$, we get

$$E(\bar{D}_k^2\,|\,\bar{\mathscr{F}}_{k-1})=E(D_k^2\,|\,\bar{\mathscr{F}}_{k-1})-E(U_k^2\,|\,\bar{\mathscr{F}}_{k-1}).$$

Thus, in view of Lemma (4.1), it suffices to show that

$$E(U_k^2\,|\,\bar{\mathscr{F}}_{k-1})^{1/2}=O(k^{-1})\quad\text{a.e.}\tag{4.16}$$

To prove (4.16) we first note that (4.8) implies

$$E(U_k^{*2}\,|\,\bar{\mathscr{F}}_{k-1})^{1/2}=O(k^{-1})\quad\text{a.e.}\tag{4.17}$$

where $U_k^*=E(T_k\,|\,\mathscr{F}_{k-1})$. (In fact, by (4.8) we have $|U_k^*|\leqq Ck^{-2}$ on the whole probability space.) To deduce (4.16) from (4.17) it is sufficient to show that the left hand sides of (4.16) and (4.17) differ only by $O(k^{-1})$ and this will follow if we show that

$$\sum_{k=1}^{\infty}k^2\,|E(U_k^{*2}\,|\,\bar{\mathscr{F}}_{k-1})^{1/2}-E(U_k^2\,|\,\bar{\mathscr{F}}_{k-1})^{1/2}|^2<\infty\quad\text{a.e.}\tag{4.18}$$

Now, by the (conditional) Minkowski inequality, the k-th term of the series in (4.18) can be majorized by $k^2E((U_k^*-U_k)^2\,|\,\bar{\mathscr{F}}_{k-1})$ the expectation of which is

$$k^2E((U_k^*-U_k)^2)=k^2E(E((T_k-D_k)\,|\,\mathscr{F}_{k-1})^2)\leqq k^2E((T_k-D_k)^2)=O(k^{-6})$$

by (4.10). Hence (4.18) follows from the Beppo Levi theorem.

Lemma (4.3). *Let* $\bar{F}_k(x)$ *denote the distribution function of the random variable* $\bar{D}_k$, *i.e. put* $\bar{F}_k(x)=P(\bar{D}_k<x)$. *Then we have for any* $a\geqq 1$, $k\geqq k_0$

$$\int_{x^2\geqq a}x^2\,d\bar{F}_k(x)\leqq 16+C'a\,k^{1/2}e^{-C''\sqrt{a}/k^{1/4}}$$

where C', C'' (*and also* k_0) *depend only on* $f(t)$ *and* q.

Proof. We evidently have

$$\bar{D}_k-T_k=(D_k-T_k)+E((T_k-D_k)\,|\,\mathscr{F}_{k-1})-E(T_k\,|\,\mathscr{F}_{k-1}).\tag{4.19}$$

From (4.8) and (4.10) it follows that the L_2 norm of the three summands on the right hand side of (4.19) cannot exceed Ck^{-4}, Ck^{-4}, Ck^{-2}, respectively. Hence (4.19) implies $\|\bar{D}_k-T_k\|\leqq 1$ for $k\geqq k_1$ or equivalently,

$$\bar{D}_k=T_k+\xi,\qquad\|\xi\|\leqq 1\quad(k\geqq k_1).\tag{4.20}$$

Using Lemma (3.4) we get that

$$T_k = \xi_1 + \xi_2 \qquad (k \geqq k_2)$$

where $\|\xi_2\| \leqq 1$ and

$$P(|\xi_1| \geqq y k^{1/4}) \leqq C_4 e^{-C_5 y} \qquad \text{for } y \geqq 0.$$

This fact, together with (4.20), implies that

$$D_k = \eta_1 + \eta_2 \qquad (k \geqq k_3)$$

where $\|\eta_1\| \leqq 2$ and

$$P(|\eta_2| \geqq x) \leqq C_4 e^{-C_5 x/k^{1/4}} \qquad \text{for } x \geqq 0. \tag{4.21}$$

Using Lemma (3.5) we get for any $a>0$, $k \geqq k_3$

$$\int_{x^2 \geqq a} x^2 \, d\bar{F}_k(x) \leqq 4 \int_{x^2 \geqq a/4} x^2 \, dG(x) + 4 \int_{x^2 \geqq a/4} x^2 \, dJ(x) \tag{4.22}$$

where $G(x)$ and $J(x)$ denote the distribution functions of η_1 and η_2, respectively. Since we have $\|\eta_1\| \leqq 2$, the first summand on the right hand side of (4.22) cannot exceed 16. On the other hand, an integration by parts yields

$$\int_{x \geqq \sqrt{a}/2} x^2 \, dJ(x) = \frac{a}{4} \left(1 - J\left(\frac{\sqrt{a}}{2}\right)\right) + \int_{\sqrt{a}/2}^{\infty} 2(1 - J(x))\, x \, dx. \tag{4.23}$$

By (4.21) we have $1 - J(x) \leqq C_4 \exp(-C_5 x/k^{1/4})$ for $x \geqq 0$. Using this fact and calculating the integral $\int_{\sqrt{a}/2}^{\infty} x \exp(-C_5 x/k^{1/4}) dx$ exactly, we get from (4.23) by an easy calculation

$$\int_{x \geqq \sqrt{a}/2} x^2 \, dJ(x) \leqq C^* a k^{1/2} e^{-C'' \sqrt{a}/k^{1/4}} \qquad (a \geqq 1)$$

where C^*, C'' are positive constants depending only on $f(t)$ and q. A similar estimate holds for the integral $\int_{x \leqq -\sqrt{a}/2} x^2 \, dJ(x)$ and thus Lemma (4.3) is proved.

Remark. So for we have considered only the long block sums $D_k = \sum_{\nu \in \Delta_k} \varphi_\nu$. All the statements proved above, however, have their exact counterparts for the short block sums $H_k = \sum_{\nu \in \Delta'_k} \varphi_\nu$. Put $\bar{H}_k = H_k - E(H_k | H_1, \ldots, H_{k-1})$ and let $\mathscr{H}_{k-1}$ and $\bar{\mathscr{H}}_{k-1}$ denote the σ-fields generated by $H_1, \ldots, H_{k-1}$ and $\bar{H}_1, \ldots, \bar{H}_{k-1}$, respectively. Then we have (as $k \to \infty$)

$$E(H_k | \mathscr{H}_{k-1}) = O(k^{-2}) \qquad \text{a.e.} \tag{4.24}$$

$$\sigma_1 [k^{1/4}] + O(k^{-2}) \leqq E(\bar{H}_k^2 | \bar{\mathscr{H}}_{k-1}) \leqq \sigma_2 [k^{1/4}] + O(k^{-2}) \qquad \text{a.e.} \tag{4.25}$$

$$\int_{x^2 \geqq a} x^2 \, d\hat{F}_k(x) \leqq 16 + \hat{C} a k^{1/4} e^{-\tilde{C} \sqrt{a}/k^{1/8}} \qquad (a \geqq 1,\ k \geqq k_0) \tag{4.26}$$

where $\hat{F}_k(x) = P(\bar{H}_k < x)$ and $\hat{C}$, $\tilde{C}$ (and k_0) are positive constants depending only on $f(t)$ and q. The proofs of the above statements are exactly the same as those of Lemmas (4.1)–(4.3).

Making use of Lemmas (4.1)–(4.3), the proof of Theorem 1 can be completed easily. The idea is simply to apply Theorem (4.4) of Strassen's paper [14] to the long block sums $T_k = \sum_{\nu\in\Delta_k} f(n_\nu x)$. We cannot do this directly, however, because $\{T_k\}$ is not necessarily a martingale difference sequence (though it is approximately such a sequence by the above considerations). Therefore we first replace the functions $f(n_\nu x)$ by functions $\tilde{\varphi}_\nu(x)$ which are very close to them and for which the long block sums $\sum_{\nu\in\Delta_k}\tilde{\varphi}_\nu(x)$ (and actually also the short block sums $\sum_{\nu\in\Delta'_k}\tilde{\varphi}_\nu(x)$) constitute a martingale difference sequence. For instance, we can take

$$\tilde{\varphi}_\nu(x) = \begin{cases} \varphi_\nu - [k^{1/2}]^{-1} E(D_k \mid D_1, \ldots, D_{k-1}) & \text{if } \nu\in\Delta_k \\ \varphi_\nu - [k^{1/4}]^{-1} E(H_k \mid H_1, \ldots, H_{k-1}) & \text{if } \nu\in\Delta'_k \end{cases} \qquad (k=1,2,\ldots)$$

In this case we have

$$\sum_{\nu\in\Delta_k}\tilde{\varphi}_\nu(x) = D_k - E(D_k \mid D_1, \ldots, D_{k-1}) = \bar{D}_k,$$

$$\sum_{\nu\in\Delta'_k}\tilde{\varphi}_\nu(x) = H_k - E(H_k \mid H_1, \ldots, H_{k-1}) = \bar{H}_k$$

which are indeed martingale difference sequences. As to the discrepancy $|f(n_\nu x) - \tilde{\varphi}_\nu(x)|$, we have by (4.3), (4.6) and (4.24),

$$\begin{aligned} &\sum_{\nu=1}^{\infty} |f(n_\nu x) - \tilde{\varphi}_\nu(x)| \\ &\quad \leqq \sum_{\nu=1}^{\infty} |f(n_\nu x) - \varphi_\nu(x)| + \sum_{\nu=1}^{\infty} |\varphi_\nu(x) - \tilde{\varphi}_\nu(x)| \\ &\quad = \sum_{\nu=1}^{\infty} |f(n_\nu x) - \varphi_\nu(x)| + \sum_{k=1}^{\infty} |E(D_k \mid D_1, \ldots, D_{k-1})| \\ &\qquad + \sum_{k=1}^{\infty} E(H_k \mid H_1, \ldots, H_{k-1})| < \infty \end{aligned} \tag{4.27}$$

for almost all $x\in[0,1)$. From now on, therefore, we can focus our attention to the sequence $\tilde{\varphi}_\nu(x)$ (which is, moreover, a sequence of r.v.-s taking only finitely many values) and the task becomes to show that the conclusion of Theorem 1 holds for the sequence $\tilde{\varphi}_\nu(x)$ instead of $f(n_\nu x)$.

Let us apply Theorem (4.4) of Strassen's paper [14] to the long block sums $\bar{D}_k = \sum_{\nu\in\Delta_k}\tilde{\varphi}_\nu$ with $f(x) = x^{9/10}$. Of course, we have to verify

$$V_k \to \infty, \quad \sum_{k=1}^{\infty} V_k^{-9/10} \int_{x^2 > V_k^{9/10}} x^2 \, dP(\bar{D}_k < x \mid \bar{\mathscr{F}}_{k-1}) < \infty \quad \text{a.s.} \tag{4.28}$$

where $V_k = \sum_{i=1}^{k} E(\bar{D}_i^2 \mid \bar{\mathscr{F}}_{i-1})$. Let us take (4.28) temporarily granted. Then, by Strassen's theorem, there is a sequence $\hat{D}_1, \hat{D}_2, \ldots$ on a new probability space $(\Omega, \mathscr{F}, P)$ which is equivalent to $\bar{D}_1, \bar{D}_2, \ldots$ and

$$\hat{D}_1 + \cdots + \hat{D}_k = \zeta(\hat{V}_k) + o(\hat{V}_k^{19/40} \log \hat{V}_k) \quad \text{a.s. as } k\to\infty \tag{4.29}$$

where ζ is a Wiener-process on $(\Omega, \mathscr{F}, P)$ and $\hat{V}_k = \sum_{i=1}^{k} E(\hat{D}_i^2 | \hat{D}_1, \ldots, \hat{D}_{i-1})$. We can also assume, without loss of generality, that $(\Omega, \mathscr{F}, P)$ is an atomless probability space. By Lemma (3.7) there exists a sequence $\{X_\nu\}$ of r.v.-s – equivalent to $\{\tilde{\varphi}_\nu\}$ – on $(\Omega, \mathscr{F}, P)$ such that the $\hat{D}_k$'s are just the long block sums obtained from this sequence:

$$\hat{D}_k = \sum_{\nu \in \Delta_k} X_\nu \quad (k=1,2,\ldots). \tag{4.30}$$

It is easy to see that $X_1, X_2, \ldots$ satisfy the requirements of Theorem 1 i.e. we have

$$X_1 + \cdots + X_n = \zeta(\tau_n) + o(n^{19/40} \log n) \quad \text{a.s. as } n \to \infty \tag{4.31}$$

with a certain strictly increasing sequence τ_n of random variables satisfying (2.5) and $\tau_n - \tau_{n-1} = O(1)$ a.s. as $n \to \infty$. For this purpose we first note that

$$\sigma_1 f_k + O(1) \leqq \hat{V}_k \leqq \sigma_2 f_k + O(1) \quad \text{a.s.} \tag{4.32}$$

and

$$\hat{V}_k - \hat{V}_{k-1} = O(k^{1/2}) \quad \text{a.s.} \tag{4.33}$$

where $f_k = \sum_{i=1}^{k} [i^{1/2}] \sim C k^{3/2}$. Indeed, the corresponding relations for V_k instead of $\hat{V}_k$ follow immediately from Lemma (4.2) and hence (4.32) and (4.33) are valid by the equivalence of $\{\bar{D}_k\}$ and $\{\hat{D}_k\}$. We also note the following simple

Lemma (4.4). *We have (as $k \to \infty$)*

$$\sum_{j=1}^{k} \sum_{\nu \in \Delta_j} X_\nu = \zeta(\hat{V}_k) + o(f_k^{19/40} \log f_k) \quad \text{a.s.} \tag{4.34}$$

$$\sum_{j=1}^{k} \sum_{\nu \in \Delta_j'} X_\nu = o(f_k^{19/40} \log f_k) \quad \text{a.s.} \tag{4.35}$$

Proof. Relation (4.34) is immediate from (4.29), (4.30) and (4.32). To see (4.35), we note the relations

$$\bar{D}_1 + \cdots + \bar{D}_k = o(f_k^{1/2} \log f_k) \quad \text{a.s.} \tag{4.36}$$

$$\bar{H}_1 + \cdots + \bar{H}_k = o(e_k^{1/2} \log e_k) \quad \text{a.s.} \tag{4.37}$$

where $e_k = \sum_{i=1}^{k} [i^{1/4}] \sim C k^{5/4}$. For the sequence $\hat{D}_1, \hat{D}_2, \ldots$ instead of $\bar{D}_1, \bar{D}_2, \ldots$ relation (4.36) follows from (4.29) by using the estimate $\zeta(t) = o(t^{1/2} \log t)$ $(t \to \infty)$ and (4.32). Hence (4.36) is valid by the equivalence of $\{\bar{D}_k\}$ and $\{\hat{D}_k\}$. (4.37) is the dual of (4.36) which can be proved in the same way as (4.36), applying Strassen's theorem to the short block sums $\bar{H}_k = \sum_{\nu \in \Delta_k'} \tilde{\varphi}_\nu$. (Here we have to verify the analogue of (4.28) for $\bar{H}_k$:

$$W_k \to \infty, \quad \sum_{k=1}^{\infty} W_k^{-9/10} \int_{x^2 > W_k^{9/10}} x^2 \, dP(\bar{H}_k < x | \bar{\mathscr{H}}_{k-1}) < \infty \quad \text{a.s.} \tag{4.38}$$

where $W_k = \sum_{i=1}^{k} E(\bar{H}_i^2 | \bar{\mathscr{H}}_{i-1})$. For the moment, let this relation be taken granted.) Relation (4.37) can be written as

$$\sum_{j=1}^{k} \sum_{v \in \Delta'_j} \tilde{\varphi}_v = o(e_k^{1/2} \log e_k) \qquad \text{a.s.} \tag{4.39}$$

which implies (4.35) by the equivalence of $\{X_v\}$ and $\{\tilde{\varphi}_v\}$ and the fact that $e_k \sim C f_k^{5/6}$. (As a matter of fact, relation (4.36) was not used, it served only to explain (4.37).) Hence the proof of Lemma (4.4) is completed.

Summing (4.34) and (4.35) we get

$$\sum_{v=1}^{r_k} X_v = \zeta(\hat{V}_k) + o(r_k^{19/40} \log r_k) \qquad \text{a.s. as } k \to \infty \tag{4.40}$$

where $r_k = \sum_{i=1}^{k} ([i^{1/2}] + [i^{1/4}]) \sim f_k \sim C k^{3/2}$. Let us define the sequence τ_n of random variables in such a way that $\tau_{r_k} = \hat{V}_k$ $(k=1,2,\ldots)$ and τ_n varies linearly between $n = r_k$ and $n = r_{k+1}$ for any $k \geqq 1$. (We define $\tau_0 = 0$.) With this choice of τ_n relation (4.40) immediately gives (4.31) for the indices $n = r_k$. To prove (4.31) for general n it suffices to show

$$\max_{r_k < n < r_{k+1}} \left| \sum_{v=r_k+1}^{n} X_v \right| = o(r_k^{19/40}) \qquad \text{a.s. as } k \to \infty \tag{4.41}$$

and

$$\max_{r_k < n < r_{k+1}} |\zeta(\tau_n) - \zeta(\tau_{r_k})| = o(r_k^{19/40}) \qquad \text{a.s. as } k \to \infty. \tag{4.42}$$

The first relation is trivial since by (4.27) and the first relation of (2.1) the sequence $\{\tilde{\varphi}_v(x)\}$ remains bounded for almost all $x \in [0, 1)$ hence $\{X_v\}$ also remains bounded with probability one (by equivalence reasons) and thus the left hand side of (4.41) is $O(r_{k+1} - r_k) = O(k^{1/2}) = O(r_k^{19/40})$ a.s. To see (4.42) let us note that

$$\tau_{r_k} = \hat{V}_k = O(f_k) = O(k^{3/2}) \qquad \text{a.s.}$$
$$\max_{r_k < n \leqq r_{k+1}} |\tau_n - \tau_{r_k}| \leqq \tau_{r_{k+1}} - \tau_{r_k} = \hat{V}_{k+1} - \hat{V}_k = O(k^{1/2}) \qquad \text{a.s.} \tag{4.43}$$

by (4.32) and (4.33). Hence (4.42) follows from Lemma (3.6) (with $r = 3/2$, $s = 1/2$). It is also evident that $\hat{V}_k$ is non-decreasing and so is τ_n, furthermore by (4.43) we have $\tau_{r_{k+1}} - \tau_{r_k} = O(k^{1/2}) = O(r_{k+1} - r_k)$ whence

$$\tau_n - \tau_{n-1} = O(1) \qquad \text{a.s.}$$

Finally, $\tau_{r_k} = \hat{V}_k$, (4.32) and $f_k \sim r_k$ imply

$$\sigma_1 \leqq \liminf_{k \to \infty} \frac{\tau_{r_k}}{r_k} \leqq \limsup_{k \to \infty} \frac{\tau_{r_k}}{r_k} \leqq \sigma_2 \qquad \text{a.s.} \tag{4.44}$$

whence (2.5) follows in view of the piecewise linearity of τ_n.

The arguments above complete the proof of Theorem 1, only two simple points remain to prove. The first is that the sequence τ_n defined above is non-decreasing but not necessarily strictly increasing. This difficulty is easy to overcome since we can find a sequence τ_n' of random variables which is strictly increasing and $|\tau_n-\tau_n'|\leq 2^{-n}$ for $n\geq 1$. It is evident that $|\zeta(\tau_n)-\zeta(\tau_n')|=O(n^{1/4})$ a.s. (this follows, e.g., from $\tau_n=O(n)$ and Lemma (3.6)) and thus (2.4), (2.5) and $\tau_n-\tau_{n-1}=O(1)$ are valid also for τ_n' instead of τ_n. The second point is that in the proof above we did not verify relations (4.28) and (4.38). This, however, is an easy task by means of Lemma (4.3). In fact, as we noted there, relation (4.32) is valid also for V_k instead of $\hat{V}_k$ and thus (4.28) will follow if we show that

$$\sum_{k=1}^{\infty} f_k^{-9/10} \int_{x^2>Bf_k^{9/10}} x^2\, dP(\bar{D}_k<x\,|\,\bar{\mathscr{F}}_{k-1})<\infty \qquad \text{a.s.} \tag{4.45}$$

for any constant $B>0$. Now, the expectation of the k-th term of the series in (4.45) is

$$f_k^{-9/10} \int_{x^2>Bf_k^{9/10}} x^2\, dP(\bar{D}_k<x)$$

which, by Lemma (4.3) and $f_k\sim Ck^{3/2}$, is $O(k^{-27/20})$. Hence (4.45) is valid by the Beppo Levi theorem. (4.38) can be proved in the same way, using (4.25) and (4.26).

Remark. It is evident that the construction of $\{X_n\}$, $\{\tau_n\}$ and $\zeta(t)$ is independent of the values of σ_1, σ_2. This proves Remark 2 after Theorem 3.

The proof of Theorem 2 is the same as that of Theorem 1; we only have to use relation (2.6) at those places where, in the proof above, (2.3) was used. Instead of Lemmas (4.1), (4.2) we shall have

Lemma (4.5). *Let $c=c(k)$ denote the smallest integer of the block Δ_k and put $p_k=a_{c(k)-1,[k^{1/2}]}$ ($a_{M,N}$ are the numbers occurring in (2.6)). Then we have (as $k\to\infty$)*

$$E(D_k\,|\,\mathscr{F}_{k-1})=O(k^{-2}) \qquad \text{a.e.}$$

$$\sigma_1 p_k+O(k^{-2})\leq E(D_k^2\,|\,\mathscr{F}_{k-1})\leq \sigma_2 p_k+O(k^{-2}) \qquad \text{a.e.}$$

These relations remain valid if the conditioning σ-field $\mathscr{F}_{k-1}$ is replaced by $\bar{\mathscr{F}}_{k-1}$. Furthermore we have

$$\sigma_1 p_k+O(k^{-2})\leq E(\bar{D}_k^2\,|\,\bar{\mathscr{F}}_{k-1})\leq \sigma_2 p_k+O(k^{-2}) \qquad \text{a.e.}$$

Similarly, instead of (4.25) we shall have

$$\sigma_1 p_k^*+O(k^{-2})\leq E(\bar{H}_k^2\,|\,\bar{\mathscr{H}}_{k-1})\leq \sigma_2 p_k^*+O(k^{-2}) \qquad \text{a.e.}$$

where $p_k^*=a_{c^*(k)-1,[k^{1/4}]}$ ($c^*(k)$ is the smallest integer of the block Δ_k'). By the assumption on $a_{M,N}$ we have $p_k\asymp k^{1/2}$, $p_k^*\asymp k^{1/4}$. The definition of f_k and e_k should be modified to $f_k=\sum_{i=1}^{k} p_i$, $e_k=\sum_{i=1}^{k} p_i^*$. Some relations $\sim$ are to be replaced by $\asymp$. Finally, instead of (4.44) we have

$$\sigma_1\leq \liminf_{k\to\infty}\frac{\tau_{r_k}}{f_k}\leq \limsup_{k\to\infty}\frac{\tau_{r_k}}{f_k}\leq\sigma_2 \qquad \text{a.s.}$$

and thus (2.7) will hold if b_n denotes the sequence for which $b_{r_k}=f_k$ $(k=1,2,\dots)$ and which varies linearly between $n=r_k$ and $n=r_{k+1}$ for any $k\geq 1$.

The above construction of b_n shows also that

$$A_1 \leqq \liminf_{n\to\infty} b_n/n \leqq \limsup_{n\to\infty} b_n/n \leqq A_2.$$

In fact, by the assumption made on $a_{M,N}$ we have

$$A_1 \sum_{i=1}^{k} [i^{1/2}] + O(1) \leqq f_k \leqq A_2 \sum_{i=1}^{k} [i^{1/2}] + O(1)$$

whence the above statement follows since $\sum_{i=1}^{k} [i^{1/2}] \sim r_k$ and b_k is piecewise linear. Hence Remark 4 after Theorem 3 is also proved.

The proof of Theorem 3 is also almost identical with that of Theorem 1. Let us observe that by (4.14) we have $2^w/n_c = O(q^{-k^{1/5}})$ and thus, using (2.8), we now get

$$|A|^{-1} \int_A T_k^2\, dx = \sigma[k^{1/2}] + O(k^{-2})$$

instead of (4.13). (The constant in O depends only on $f(t)$ and q.) Using this fact, the proofs of Lemmas (4.1) and (4.2) give the following

Lemma (4.6). *We have (as $k \to \infty$)*

$$E(D_k \mid \mathscr{F}_{k-1}) = O(k^{-2}) \qquad \text{a.e.}$$
$$E(D_k^2 \mid \mathscr{F}_{k-1}) = \sigma[k^{1/2}] + O(k^{-2}) \qquad \text{a.e.}$$

These relations remain valid if the conditioning σ-field $\mathscr{F}_{k-1}$ is replaced by $\bar{\mathscr{F}}_{k-1}$. Furthermore we have

$$E(\bar{D}_k^2 \mid \bar{\mathscr{F}}_{k-1}) = \sigma[k^{1/2}] + O(k^{-2}) \qquad \text{a.e.}$$

(The constants in O can depend also on the element x of the probability space.)

Similarly, instead of (4.25) we have

$$E(\bar{H}_k^2 \mid \bar{\mathscr{H}}_{k-1}) = \sigma[k^{1/4}] + O(k^{-2}) \qquad \text{a.e.}$$

The above relations show that in the present case Lemmas (4.1) and (4.2) and the Remark after Lemma (4.3) are valid with $\sigma_1 = \sigma_2 = \sigma$. Hence the rest of the proof of Theorem 1 applies without change and we can take $\sigma_1 = \sigma_2 = \sigma$. In particular, (4.32) yields $\hat{V}_k = \sigma f_k + O(1)$ a.s., whence

$$\tau_{r_k} = \hat{V}_k = \sigma f_k + O(1) = \sigma \left(r_k - \sum_{i=1}^{k} [i^{1/4}] \right) + O(1)$$
$$= \sigma r_k + O(k^{5/4}) = \sigma r_k + O(r_k^{5/6}) \qquad \text{a.s.}$$

and thus $\tau_n = \sigma n + O(n^{5/6})$ a.s. Using Lemma (3.6) we get

$$\zeta(\tau_n) = \zeta(\sigma n) + o(n^{5/12} \log n) \qquad \text{a.s.}$$

and this completes the proof. Remark 5 after Theorem 3 can be proved in the same way (see the proof of Theorem 2).

5. § Proof of the Standard Inequalities

Lemma (3.2) follows immediately from condition (3.1) and the relation

$$\int_a^b g(\lambda x)\,dx = \frac{1}{\lambda}\int_{a\lambda}^{b\lambda} g(t)\,dt.$$

Lemma (3.6) follows from Lemma 1 of [9] by means of the Borel-Cantelli lemma. The proof of Lemma (3.7) is also simple routine and can be omitted. Hence it suffices to prove Lemmas (3.1), (3.3), (3.4) and (3.5).

Proof of Lemma (3.1). We follow Ibragimov [6]. Let us first remark the following obvious relations:

$$[g_1+g_2]_m=[g_1]_m+[g_2]_m, \qquad [c\,g]_m=c\,[g]_m, \qquad \|[g]_m\| \leqq \|g\| \tag{5.1}$$

(c is constant). Let us now consider a function f satisfying (1.1) and the second relation of (2.1), let

$$f \sim \sum_{k=1}^{\infty} (a_k \cos 2\pi k x + b_k \sin 2\pi k x)$$

be the Fourier-expansion of f and write

$$f = f_1 + f_2 \tag{5.2}$$

where

$$f_1 = s_N = \sum_{k=1}^{N} (a_k \cos 2\pi k x + b_k \sin 2\pi k x), \qquad f_2 = f - s_N.$$

N is an integer to be specified later. If $\psi(x)=f(\lambda x)$ then by (5.2) we have

$$\psi = \psi_1 + \psi_2 \tag{5.3}$$

where $\psi_1(x)=f_1(\lambda x)$, $\psi_2(x)=f_2(\lambda x)$. Evidently

$$|\cos\beta x - [\cos\beta x]_m| \leqq \beta/m, \qquad |\sin\beta x - [\sin\beta x]_m| \leqq \beta/m$$

for any $\beta>0$ and thus by

$$\psi_1(x) = \sum_{k=1}^{N} (a_k \cos 2\pi k \lambda x + b_k \sin 2\pi k \lambda x)$$

and by the first two relations of (5.1) we have

$$|\psi_1 - [\psi_1]_m| \leqq \sum_{k=1}^{N} \frac{2\pi k \lambda(|a_k|+|b_k|)}{m}$$

$$\leqq \frac{2\pi\lambda}{m}\left(\sum_{k=1}^{N} k^2\right)^{1/2}\left[\left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{\infty} b_k^2\right)^{1/2}\right] \leqq C_6 \frac{\lambda}{m} N^{3/2} \tag{5.4}$$

where C_6 depends on f. Furthermore, by the second relation of (2.1), the third relation of (5.1) and the periodicity of f and s_N we have

$$\begin{aligned}\|\psi_2-[\psi_2]_m\|^2 &\leqq 4\|\psi_2\|^2=4\int_0^1 f_2(\lambda x)^2\,dx=\frac{4}{\lambda}\int_0^\lambda f_2(t)^2\,dt\\ &\leqq \frac{4}{\lambda}\int_0^{[\lambda]+1} f_2(t)^2\,dt=\frac{4}{\lambda}([\lambda]+1)\int_0^1 f_2(t)^2\,dt\\ &\leqq 8\|f-s_N\|^2\leqq C_7 N^{-2\alpha}\end{aligned} \tag{5.5}$$

where C_7 depends only on f. (5.3), (5.4), (5.5) and the first relation of (5.1) imply

$$\|\psi-[\psi]_m\|\leqq C_8\left(\frac{\lambda}{m}N^{3/2}+N^{-\alpha}\right)$$

whence the statement of the lemma follows by choosing $N=[(m/\lambda)^{1/3}]$.

Proof of Lemma (3.3). We shall need the following

Lemma (5.1). *Let* $f(x)$ *satisfy* (1.1), *let*

$$f\sim\sum_{k=1}^\infty (a_k\cos 2\pi kx+b_k\sin 2\pi kx)$$

be its Fourier-expansion and define

$$R(t)=\tfrac{1}{2}\sum_{k=[t]+1}^\infty (a_k^2+b_k^2)\qquad (t\geqq 0). \tag{5.6}$$

Then we have for any $\lambda_2>\lambda_1\geqq 1$ *and any real* a

$$\left|\int_a^{a+1} f(\lambda_1 x)f(\lambda_2 x)\,dx\right|\leqq\left(C_9\theta^{-1/2}+C_{10}R\left(\frac{\theta}{2}\right)^{1/2}\right)(\|f\|^2+\|f\|). \tag{5.7}$$

where $\theta=\lambda_2/\lambda_1$ *and* C_9, C_{10} *are absolute constants.*

In [8], pp. 239–240 it is shown that the left-hand side of (5.7) is at most $C_{11}\theta^{-1/2}+C_{12}R(\theta/2)^{1/2}$ where C_{11}, C_{12} are positive constants depending on $f(x)$ and a. The proof given there yields also the little more precise inequality (5.7).

Turning to the proof of Lemma (3.3), let us observe that

$$\int_a^{a+1}(f(m_1x)+\dots+f(m_nx))^2\,dx=\sum_{\nu=1}^n\int_a^{a+1} f^2(m_\nu x)\,dx + W_1+W_2+\dots+W_{n-1} \tag{5.8}$$

where

$$W_k=2\sum_{\mu=1}^{n-k}\int_a^{a+1} f(m_\mu x)f(m_{\mu+k})\,dx.$$

Applying Lemma (3.2) to the function $g(x)=f^2(x)-\|f\|^2$ we get

$$\left|\int_a^{a+1} f^2(m_\nu x)\,dx-\|f\|^2\right|=\left|\int_a^{a+1} g(m_\nu x)\,dx\right|\leqq\frac{2}{m_\nu}\int_0^1|f^2(x)-\|f\|^2|\,dx\leqq\frac{4}{m_\nu}\|f\|^2$$

and thus we have

$$\left|\sum_{v=1}^{n}\int_{a}^{a+1} f^2(m_v x)\,dx - n\|f\|^2\right| \leqq 4\|f\|^2\left(\frac{1}{m_1}+\cdots+\frac{1}{m_n}\right) \leqq 4\|f\|^2\frac{1}{m_1}\sum_{r=0}^{t-1} q^{-r}$$
$$< \frac{4\|f\|^2}{m_1}\,\frac{q}{q-1} < \frac{4q}{q-1}\|f\|^2. \tag{5.9}$$

On the other hand, $m_{l+1}/m_l \geqq q > 1$ and Lemma (5.1) imply for $1 \leqq k \leqq n-1$

$$|W_k| \leqq 2n\left(C_9 q^{-k/2} + C_{10} R\left(\frac{q^k}{2}\right)^{1/2}\right)(\|f\|^2 + \|f\|)$$

whence we get

$$|W_1+\cdots+W_{n-1}|$$
$$\leqq 2n(\|f\|^2+\|f\|)\left(C_9\sum_{k=1}^{\infty} q^{-k/2} + C_{10}\sum_{k=1}^{\infty} R\left(\frac{q^k}{2}\right)^{1/2}\right). \tag{5.10}$$

Since f satisfies the second relation of (2.1), for the function $R(t)$ we have (using $0<\alpha\leqq 1$)

$$R(t)^{1/2} = \|f - s_{[t]}\| \leqq A[t]^{-\alpha} \leqq 2At^{-\alpha} \quad \text{if } t \geqq 1$$
$$R(t)^{1/2} = \|f\| \quad \text{if } t<1.$$

Thus

$$\sum_{k=1}^{\infty} R\left(\frac{q^k}{2}\right)^{1/2} \leqq \sum_{q^k<2}\|f\| + \sum_{q^k \geqq 2} 4Aq^{-k\alpha} < \frac{\|f\|}{\log q} + \frac{4A}{q^\alpha - 1}$$

and hence by (5.10) we have (using $A>1$)

$$|W_1+\cdots+W_{n-1}| \leqq 2n(\|f\|^2+\|f\|)\,C_{13}A\left(\frac{1}{q^{1/2}-1}+\frac{1}{q^\alpha-1}+\frac{\|f\|}{\log q}\right) \tag{5.11}$$

with an absolute constant C_{13}. Relations (5.8), (5.9) and (5.11) yield the statement of Lemma (3.3).

Proof of Lemma (3.4). We shall need the following

Lemma (5.2). *Let $f(t)$ $(0\leqq t<1)$ be a square integrable function and let $s_n(t)$ and $\sigma_n(t)$ denote, respectively, the n-th partial sum and n-th $(C, 1)$ (Fejér) mean of the partial sums of the Fourier series of f. Then the relation*

$$\|f - s_n\| = O(n^{-\alpha}) \quad (0<\alpha<1)$$

implies

$$\|f-\sigma_n\| = O(n^{-\alpha}).$$

Proof. Let $f \sim a_0 + \sum_{k=1}^{\infty}(a_k\cos 2\pi kt + b_k \sin 2\pi kt)$ be the Fourier-expansion of f and put

$$R_n = \tfrac{1}{2}\sum_{k=n+1}^{\infty}(a_k^2+b_k^2).$$

Then we have

$$\|f-s_n\|^2=R_n, \qquad \|f-\sigma_n\|^2=\frac{1}{(n+1)^2}\sum_{k=1}^{n}\tfrac{1}{2}k^2(a_k^2+b_k^2)+R_n.$$

By the assumption we have $R_n=O(n^{-2\alpha})$ hence, using Abel's transformation, we get

$$\begin{aligned}\|f-\sigma_n\|^2&=\frac{1}{(n+1)^2}\sum_{k=1}^{n}k^2(R_{k-1}-R_k)+R_n\\&=\frac{1}{(n+1)^2}\left[1^2R_0-n^2R_n+\sum_{k=1}^{n-1}\left((k+1)^2-k^2\right)R_k\right]+R_n\\&\leqq\frac{R_0}{n^2}+R_n+\frac{1}{n^2}\sum_{k=1}^{n-1}3kR_k+R_n\\&=O\left(\frac{1}{n^2}\right)+O\left(\frac{1}{n^{2\alpha}}\right)+O\left(\frac{1}{n^2}\sum_{k=1}^{n-1}k^{1-2\alpha}\right)=O\left(\frac{1}{n^{2\alpha}}\right)\end{aligned}$$

proving the statement of the lemma.

We can now turn to the proof of Lemma (3.4). For the sake of simplicity we consider only the case when the Fourier series of f is a purely cosine series:

$$f\sim\sum_{k=1}^{\infty}c_k\cos 2\pi kt.$$

The general case can be treated similarly.

a) Let us first make the additional assumption that n_k are integers and the partial sums of the Fourier series of f are uniformly bounded:

$$|s_n(t)|\leqq K \quad (n\geqq 1,\ 0\leqq t\leqq 1). \tag{5.12}$$

In this case the proof will be a little simpler. (The case when the above conditions are not satisfied will be considered later.) We carry out the proof in three steps. In what follows, C denote positive constants, not always the same, depending only on $f(t)$ and q.

1. Let H be an integer such that

$$q^H>3H^\beta \tag{5.13}$$

where β is a positive integer such that $\alpha\beta\geqq 12$. Put

$$g(t)=\sum_{k=1}^{H^\beta}c_k\cos 2\pi kt \quad\text{and}\quad U_m(t)=\sum_{l=Hm+1}^{H(m+1)}g(n_l t). \tag{5.14}$$

Then we have for any real λ and $k\geqq 1$

$$\int_0^1\exp\left\{\lambda\sum_{m=0}^{k-1}U_{2m}(t)\right\}dt\leqq e^{C\lambda^2Hk+C|\lambda|^3H^3k} \tag{5.15}$$

and

$$\int_0^1\exp\left\{\lambda\sum_{m=1}^{k}U_{2m-1}(t)\right\}dt\leqq e^{C\lambda^2Hk+C|\lambda|^3H^3k} \tag{5.16}$$

Moreover, (5.15) and (5.16) remain valid if the blocks $U_{2k-2}(t)$ and $U_{2k-1}(t)$ (the last blocks in the sums in (5.15) and (5.16)) contain less than H terms.

This statement is a slight generalization of Lemma 1 of [16] (it reduces to that lemma if f is a Lip α function and $0<2\lambda H^2<1$) and can be proved in the same way. The only difference is that instead of the inequality $e^z<(1+z+z^2/2)e^{2|z|^3}$ used in [16] we now use the inequality $e^z \leqq (1+z+z^2)e^{|z|^3}$ (valid for any real z) and observe that by (5.12) we have

$$\sum_{m=0}^{k-1} |\lambda U_{2m}(t)|^3 \leqq \sum_{m=0}^{k-1} |\lambda|^3 (HK)^3 \leqq CH^3 |\lambda|^3 k \tag{5.17}$$

for any real λ. Note also that in this step of the proof we made an essential use of the fact that n_k are integers. Indeed, in the proof in [16] §2 one needs the fact that

$$\int_0^1 \prod_{i=0}^{l} \cos 2\pi u_i t \, dt = 0$$

holds provided that $u_l-(u_0+\cdots+u_{l-1})>0$. If u_i are integers, this statement is valid, as one can see easily by successive applications of the identity

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta).$$

For non-integral u_i, however, the above relation fails to hold even for $l=0$.

2. With the notations of the preceding point we have

$$\int_0^1 \exp\left\{\lambda \sum_{j=1}^{Hp+r} g(n_j t)\right\} dt \leqq e^{C\lambda^2 Hp + C|\lambda|^3 H^3 p} \tag{5.18}$$

for any integers $p \geqq 1$, $0 \leqq r < H$ and any real λ.

To prove this, let us assume, e.g., that p is even: $p=2k$. Then we have

$$\lambda \sum_{j=1}^{Hp+r} g(n_j t) = \lambda \sum_{m=0}^{k} U_{2m}(t) + \lambda \sum_{m=1}^{k} U_{2m-1}(t) \tag{5.19}$$

where $U_0, U_1, \ldots, U_{2k-1}$ are full blocks but U_{2k} contains only r terms. From (5.19) we get, using the Cauchy-Schwarz inequality,

$$\int_0^1 \exp\left\{\lambda \sum_{j=1}^{Hp+r} g(n_j t)\right\} dt$$
$$\leqq \left[\int_0^1 \exp\left\{2\lambda \sum_{m=0}^{k} U_{2m}(t)\right\} dt \cdot \int_0^1 \exp\left\{2\lambda \sum_{m=1}^{k} U_{2m-1}(t)\right\} dt\right]^{1/2}$$

whence (5.18) follows by using (5.15) and (5.16). For odd p the proof is similar.

3. Let now $N \geqq N_0$ be given and put $H=[N^{1/6}]$. If N_0 is sufficiently large then (5.13) is satisfied. With this choice of H we define the function $g(t)$ by (5.14) and write

$$\sum_{\nu=1}^{N} f(n_\nu t) = \xi_1 + \xi_2$$

where

$$\xi_1=\sum_{\nu=1}^{N} g(n_\nu t), \quad \xi_2=\sum_{\nu=1}^{N} h(n_\nu t), \qquad h(t)=f(t)-g(t). \tag{5.20}$$

We show that this decomposition satisfies the requirements of Lemma (3.4), i.e. (3.2), (3.3) hold. To prove (3.2) let us write N in the form $N=[N^{1/6}]\,p+r$ where $p\geqq 1$ and $0\leqq r<[N^{1/6}]$ are integers. With this choice of r and p we have $[N^{1/6}]\,p\leqq N$ and $[N^{1/6}]^3\,p\leqq N^{1/2}\cdot 2N^{5/6}=2N^{4/3}$ and thus (5.18) implies

$$\int_0^1 \exp\left\{\lambda \sum_{\nu=1}^{N} g(n_\nu t)\right\} dt \leqq e^{C_0\lambda^2 N+C_0|\lambda|^3 N^{4/3}} \tag{5.21}$$

for any λ where C_0 is a constant depending on $f(t)$ and q. Without loss of generality we may assume here $C_0\geqq 1$. From (5.21) we easily get

$$P(|\xi_1|\geqq y\sqrt{N})\leqq\begin{cases}2e^{-y^2/8C_0} & \text{if } 0\leqq y\leqq C_0N^{1/6}\\ 2e^{-y^{3/2}/8C_0} & \text{if } y>C_0N^{1/6}.\end{cases} \tag{5.22}$$

In fact, (5.21) and Markov's inequality imply

$$P(|\xi_1|\geqq y\sqrt{N})\leqq 2\exp\{-\lambda y\sqrt{N}+C_0\lambda^2N+C_0\lambda^3N^{4/3}\} \tag{5.23}$$

for any positive λ and y. Choosing

$$\lambda=\frac{y}{2C_0\sqrt{N}} \quad \text{and} \quad \lambda=\frac{y^{1/2}}{2\sqrt{C_0}\,N^{5/12}}$$

we get the following two estimates (valid for any $y>0$)

$$P(|\xi_1|\geqq y\sqrt{N})\leqq 2\exp\left\{-\frac{y^2}{4C_0}\left(1-\frac{y}{2C_0N^{1/6}}\right)\right\}, \tag{5.24}$$

$$P(|\xi_1|\geqq y\sqrt{N})\leqq 2\exp\left\{-\frac{y^{3/2}}{8\sqrt{C_0}}N^{1/12}\left(3-\sqrt{\frac{4C_0N^{1/6}}{y}}\right)\right\}. \tag{5.25}$$

Using (5.24) for $0<y\leqq C_0N^{1/6}$ and (5.25) for $y>C_0N^{1/6}$, we get (5.22). Evidently (5.22) implies (3.2).

To get (3.3) let us observe that the Fourier-series of h is $h\sim\sum_{k=1}^{\infty}\tilde{c}_k\cos 2\pi kt$ where $\tilde{c}_k=0$ for $1\leqq k\leqq[N^{1/6}]^\beta$ and $\tilde{c}_k=c_k$ for $k>[N^{1/6}]^\beta$. This shows that ((1.1) and) the second relation of (2.1) are valid also for h instead of f with the same A, α. (Remind that the second relation of (2.1) is equivalent to (2.2).) Hence applying Corollary 1. after Lemma (3.3) we get

$$\|\xi_2\|^2=\int_0^1\left(\sum_{\nu=1}^{N} h(n_\nu t)\right)^2 dt\leqq C_3A\|h\|N \tag{5.26}$$

(provided that $\|h\| \leqq 1$) where C_3 depends only on q. Furthermore, by the second relation of (2.1) and $\alpha\beta \geqq 12$ we have

$$\|h\| = \|f - s_{[N^{1/6}]^\beta}\| \leqq A[N^{1/6}]^{-\beta\alpha} \leqq CN^{-2}. \tag{5.27}$$

(5.26) and (5.27) evidently imply (3.3) for sufficiently large N.

b) Let us now drop condition (5.12) (but keep the assumption that n_k are integers). Then the above proof breaks down (in step 1. we used (5.12) in an essential way, see (5.17)) but a simple modification makes the argument applicable in the present case, too. Namely, instead of defining $g(t)$ (in step 1.) as the partial sum or order H^β of the Fourier series of f, let us define rather

$$g(t) = \sigma_{H^\beta}(t)$$

where $\sigma_n(t)$ denotes the n-th $(C, 1)$ (Fejér) mean of the partial sums of the Fourier series of f. The first relation of (2.1) implies (see [18] p. 89)

$$|\sigma_n(t)| \leqq M \qquad (n \geqq 1,\ 0 \leqq t \leqq 1) \tag{5.28}$$

furthermore we have

$$g(t) = \sum_{k=1}^{H^\beta} d_k \cos 2\pi k t \qquad \text{with } |d_k| \leqq |c_k| \tag{5.29}$$

(actually, $d_k = (1 - k/(H^\beta + 1))\, c_k$). Going back once more to Takahashi's proof in [16] §2 and using (5.28), (5.29) we see that with this choice of g (and defining $U_m(t)$ again by the second relation of (5.14)) relations (5.15) and (5.16) will be valid and thus our proof above will hold in an unchanged form except a little modification in (5.26) and (5.27). In the present case the Fourier series of h is $h \sim \sum_{k=1}^{\infty} c_k^* \cos 2\pi k t$ where $c_k^* = c_k - d_k = k c_k/([N^{1/6}]^\beta + 1)$ for $k \leqq [N^{1/6}]^\beta$ and $c_k^* = c_k$ for $k > [N^{1/6}]^\beta$. Hence we have $|c_k^*| \leqq |c_k|$ for $k \geqq 1$ and thus ((1.1) and) the second relation of (2.1) hold for h instead of f (with the same A, α) also in the present case. Thus (5.26) is valid also in the present case, furthermore instead of (5.27) we now have (using Lemma (5.2))

$$\|h\| = \|f - \sigma_{[N^{1/6}]^\beta}\| \leqq C[N^{1/6}]^{-\alpha\beta} \leqq CN^{-2}.$$

(Note that Lemma (5.2) was not proved for $\alpha = 1$ but throughout in our proof we can assume, without loss of generality, that $0 < \alpha < 1$.) The proof of Lemma (3.4) (for integer n_k) is hence completed.

c) Let us now drop also the assumption that n_k are integers. In this case the proof of relations (5.15) and (5.16) (as we already remarked there) breaks down. It can be saved, however, by using an observation due to Hartman (see [5]). Indeed, instead of (5.15), (5.16) let us prove first that

$$\int_{-\infty}^{+\infty} \left(\frac{\sin t}{t}\right)^2 \exp\left\{\lambda \sum_{m=0}^{k-1} U_{2m}(t)\right\} dt \leqq e^{C\lambda^2 Hk + C|\lambda|^3 H^3 k} \tag{5.15'}$$

and

$$\int_{-\infty}^{+\infty} \left(\frac{\sin t}{t}\right)^2 \exp\left\{\lambda \sum_{m=1}^{k} U_{2m-1}(t)\right\} dt \leqq e^{C\lambda^2 Hk + C|\lambda|^3 H^3 k}. \tag{5.16'}$$

The proofs of (5.15′), (5.16′) are the same as those of (5.15), (5.16) but here no problem arises for non-integral n_k since we have

$$\int_{-\infty}^{+\infty} \left(\frac{\sin t}{t}\right)^2 \cos u t \, dt = 0$$

for any real $u > 2$. (Here we need the fact that $n_1 \geqq 4$ but this can be assumed without loss of generality.) It remains now to observe that $(\sin t/t)^2 \geqq 1/4$ for $0 < t \leqq 1$ and thus (5.15′), (5.16′) imply (5.15), (5.16) with an extra coefficient 4 on the right hand side. The remaining parts of the proof of Lemma (3.4) require only trivial changes.

Proof of Lemma (3.5). For any distribution function $H(x)$ with $\int_{-\infty}^{+\infty} x^2 \, dH(x) < \infty$ we have

$$\int_b^\infty x^2 \, dH(x) = 2 \int_b^\infty (1 - H(x)) x \, dx + b^2 (1 - H(b)) \qquad (b \geqq 0).$$

Using this formula and $1 - G(x) \leqq (1 - F_1(x/2)) + (1 - F_2(x/2))$ $(x \geqq 0)$ we get

$$\int_{\sqrt{a}}^\infty x^2 \, dG(x) \leqq 4 \int_{\sqrt{a}/2}^\infty x^2 \, dF_1(x) + 4 \int_{\sqrt{a}/2}^\infty x^2 \, dF_2(x). \tag{5.30}$$

A similar argument yields

$$\int_{-\infty}^{-\sqrt{a}} x^2 \, dG(x) \leqq 4 \int_{-\infty}^{-\sqrt{a}/2} x^2 \, dF_1(x) + 4 \int_{-\infty}^{-\sqrt{a}/2} x^2 \, dF_2(x) \tag{5.31}$$

which, together with (5.30), proves the statement of the lemma.

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