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Capacity of Level Sets of Certain Stochastic Processes

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1. Introduction

Let $X(t, \omega)$, $t \in [0, T]$ be a stochastic process with continuous sample paths defined on some probability space $(\Omega, \mathscr{F}, P)$. Denote by

$$G(u, T, \omega) = \{t \colon X(t, \omega) = u, t \in [0, T]\} \tag{1.1}$$

the level set $X(t, \omega) = u$. When $X(t, \omega)$ has absolutely continuous sample paths then, under rather general conditions given in [4], the cardinality of $G(u, T, \omega) < \infty$ a.s. and one can proceed to investigate moments of $G(u, T, \omega)$. In this paper we will be concerned with processes for which the number of elements in $G(u, T, \omega)$ is uncountable and we give conditions in terms of the joint distribution of $X(t_1, \omega)$ and $X(t_2, \omega)$, $t_1, t_2 \in [0, T]$, for which the capacity of $G(u, T, \omega)$ is strictly positive, with respect to certain potential functions, on a set $\Omega' \subset \Omega$, $P(\Omega') > 0$.

The question of level sets has received a great deal of attention for many types of stochastic processes. In the case of Gaussian processes, to which the results in this paper can be readily applied, it has been studied by Orey [5] and Berman (see [2] for a listing of Berman's work on this and related topics). In Theorem 4, p. 146 [3], Kahane obtains results on the capacity of the level set of a random Fourier series which is also a Gaussian process. In Theorem 1 of this paper we extend and simplify Kahane's result although our method of proof is essentially the same as his. For an explanation of capacity and potential the reader is referred to Chapter 13, [3].

We proceed to state our results, the proofs will be given in Section 2. Assume that for all pairs $t_1, t_2 \in [0, T]$ and all (x_1, x_2) in some neighborhood of (u, u) the density function $p(x_1, x_2; t_1, t_2)$ of $X(t_1)$, $X(t_2)$ exists, is continuous in (x_1, x_2) and satisfies the following dominated condition:

$$p(x_1, x_2; t_1, t_2) \leqq g(t_1, t_2) \tag{1.2}$$

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where

$$\int_0^T \int_0^T g(t_1, t_2)\, dt_1\, dt_2 < \infty. \tag{1.3}$$

Assume also that for all $t \in [0, T]$ and x in some neighborhood of u the density $p(x; t)$ of $X(t)$ also exists and is continuous and satisfies

$$p(x; t) \leqq h(t) \tag{1.4}$$

where

$$\int_0^T h(t)\, dt < \infty \tag{1.5}$$

and

$$\int_0^T p(u; t)\, dt > 0. \tag{1.6}$$

Let $k(t_1, t_2) = k(|t_1 - t_2|)$ be a potential function such that

$$\int_0^T \int_0^T p(u, u; t_1, t_2)\, k(|t_1 - t_2|)\, dt_1\, dt_2 < \infty. \tag{1.7}$$

We denote the fact that the capacity of a set $A \in [0, T]$ is strictly positive with respect to a potential k by $\mathrm{Cap}_k(A) > 0$.

Theorem 1. *For $X(t, \omega)$, $t \in [0, T]$ as defined above, satisfying* (1.2) *through* (1.7), $\mathrm{Cap}_k(G(u, T, \omega)) > 0$ *on a set $\Omega' \subset \Omega$, $P(\Omega') > 0$.*

This result is used to find the Hausdorff dimension of the level set of a separable stationary Gaussian process. Let $Z(t, \omega)$, be a separable stationary Gaussian process, $EZ(t) = 0$, $EZ^2(t) = 1$, $r(\tau) = EZ(t + \tau) Z(t)$ and $\sigma^2(\tau) = E(Z(t + \tau) - Z(t))^2$. Following Orey [5] we say that $\sigma(\tau)$ has index α if

$$\alpha = \sup\{\beta \colon \sigma(\tau) = o(\tau^\beta), \tau \downarrow 0\} = \inf\{\beta \colon \tau^\beta = o(\sigma(\tau)), \tau \downarrow 0\}.$$

Let $\dim G(u, T, \omega)$ denote the Hausdorff dimension of $G(u, T, \omega)$. Define

$$\dim G(u, \infty, \omega) = \lim_{T \to \infty} \dim G(u, T, \omega).$$

Theorem 2. *Let $Z(t)$ be a separable stationary Gaussian process $EZ(t) = 0$, $EZ^2(t) = 1$ such that $\sigma(\tau)$ has index α, $0 < \alpha < 1$ and $\lim_{\tau \to \infty} r(\tau) = 0$. Then for each u, $-\infty < u < \infty$,* $\dim G(u, \infty, \omega) = 1 - \alpha$ a.s.

Theorem 2 is the same as Orey's Theorem 3 [5] but with weaker hypotheses. It is different from Theorem 2.1 [1] of Berman although one direction in our proof of Theorem 2 uses one part of Berman's proof. An essential difference in these two theorems is that we obtain (2.5) for a fixed level u whereas in the second part of Theorem 2.1 it is obtained for almost all u (see the line preceeding (2.6) in [1]). Under the conditions of Theorem 2 the local time for the process exists and in some sense it can be taken as a measure supported on $G(u, T, \omega)$. This is Berman's approach. However, the local time, by its nature, is only defined at almost all levels; it can be zero at a fixed level. One way around this is to establish joint continuity of the local time in both time and the level. This is the major direction of Berman's work referred to above and it leads to many interesting results, although for a restricted class of processes. (Compare Theorem 2 with Theorem 2.1 [1] in which t is contained in a finite interval.)

There is a very close relationship between the local time and the measures $Y(t, u, \omega)$ that are introduced in the proof of Theorem 1 but we will not pursue it in this paper.

2. Proofs

Proof of Theorem 1. Define

$$Y_n(t, \omega) = Y_n(t, u, \omega) = \frac{1}{2\varepsilon_n} \int_0^t \varphi_{u, \varepsilon_n}(X(s, \omega))\, ds$$

where

$$\varphi_{u, \varepsilon_n}(x) = \begin{cases} 1 & |x-u| \leqq \varepsilon_n \\ 0 & \text{otherwise} \end{cases}$$

and $\varepsilon_n = 2^{-n}$, $t \in [0, T]$. Consider

$$E\left[\frac{1}{2\varepsilon_n} \int_0^t \varphi_{u, \varepsilon_n}(X(s_1, \omega))\, ds_1 \frac{1}{2\varepsilon_m} \int_0^t \varphi_{u, \varepsilon_m}(X(s_2, \omega))\, ds_2\right].$$

Since everything is finite and positive, by Fubini's theorem this is equal to

$$\frac{1}{4\varepsilon_n \varepsilon_m} \int_0^t \int_0^t \int_{u-\varepsilon_n}^{u+\varepsilon_n} \int_{u-\varepsilon_m}^{u+\varepsilon_m} p(x_1, x_2; s_1, s_2)\, dx_1\, dx_2\, ds_1\, ds_2.$$

Therefore, it follows from the dominated convergence theorem using (1.2) and (1.3) that

$$\lim_{m, n \to \infty} E[Y_n(t, \omega)\, Y_m(t, \omega)] = \int_0^t \int_0^t p(u, u; s_1, s_2)\, ds_1\, ds_2. \tag{2.1}$$

Consequently

$$\lim_{m, n \to \infty} E[(Y_n(t, \omega) - Y_m(t, \omega))^2] = 0. \tag{2.2}$$

By the Borel-Cantelli lemma there exists a subsequence of the $Y_n(t, \omega)$ which converges a.s. to a limit which we will denote by $Y(t, \omega)$. Taking a further subsequence, if necessary, we get

$$\lim_{k \to \infty} Y_{n_k}(t, \omega) = Y(t, \omega), \qquad t \in I \tag{2.3}$$

where I is a countable dense set in $[0, T]$; we can and do include $T \in I$.

We define $Y(t, \omega)$ for all $t \in [0, T]$ by

$$Y(t, \omega) = \inf\{Y(s, \omega) \colon s \in I, s \geqq t\}$$

Clearly, $Y(t, \omega)$ is a non-decreasing function in t. It follows from (2.1) and (1.3) that

$$E Y^2(T, \omega) < \infty. \tag{2.4}$$

For each $\omega \in \bar{\Omega} \subset \Omega$, $P(\bar{\Omega})=1$ the non-decreasing functions $Y_{n_k}(t,\omega)$ are bounded measures on $[0,T]$; the same is true of $Y(t,\omega)$. By (2.3) and the fact that $Y_{n_k}(t,\omega)$ and $Y(t,\omega)$ are non-decreasing, the measures $Y_{n_k}(t,\omega)$ converge weakly to $Y(t,\omega)$ and this is true almost surely with respect to Ω.

Since $X(t,\omega)$ has continuous sample paths a.s. the measure $Y(t,\omega)$ is supported by $G(u,T,\omega)$. In order to show that $G(u,T,\omega)$ has positive capacity with respect to k on a set of positive measure with respect to $(\Omega,\mathscr{F},P)$ we show that

$$\int_0^T \int_0^T k(|t_1-t_2|)\, dY(t_1,\omega)\, dY(t_2,\omega) < \infty \quad \text{a.s.} \tag{2.5}$$

and that $\Omega' = \{\omega\colon Y(T,\omega)>0\}$ is such that $P(\Omega')>0$.

We first show (2.5). Let $k_j(|t-t'|)=k(|t-t'|)\wedge j$, $j=1,2,\ldots$. Therefore $k_j \uparrow k$. By monotone convergence

$$E\left[\int_0^T \int_0^T k(|t_1-t_2|)\, dY(t_1,\omega)\, dY(t_2,\omega)\right] \tag{2.6}$$

$$=\lim_{j\to\infty} E\left[\int_0^T \int_0^T k_j(|t_1-t_2|)\, dY(t_1,\omega)\, dY(t_2,\omega)\right]. \tag{2.7}$$

Since $Y_{n_k}(t,\omega)$ converges weakly to $Y(t,\omega)$, (2.6)

$$=\lim_{j\to\infty} E\left[\lim_{k\to\infty} \int_0^T \int_0^T k_j(|t_1-t_2|)\, dY_{n_k}(t_1,\omega)\, dY_{n_k}(t_2,\omega)\right]$$

$$=\lim_{j\to\infty} \lim_{k\to\infty} E\left[\int_0^T \int_0^T k_j(|t_1-t_2|)\, dY_{n_k}(t_1,\omega)\, dY_{n_k}(t_2,\omega)\right] \tag{2.8}$$

by dominated convergence, since $k_j \leqq j$ and $Y(T,\omega)<\infty$ a.s. (using (2.2) and (2.3)). Repeating the argument used in the beginning of the proof we see that

$$\lim_{k\to\infty} E\left[\int_0^T \int_0^T k_j(|t_1-t_2|)\, dY_{n_k}(t_1,\omega)\, dY_{n_k}(t_2,\omega)\right]$$
$$=\int_0^T \int_0^T k_j(|t_1-t_2|)\, p(u,u;t_1,t_2)\, dt_1\, dt_2.$$

Therefore (2.6) equals (2.8) equals (1.7) by the monotone convergence theorem. Thus we establish (2.5).

By the same arguments as above, using (1.4) and (1.5) we can show that

$$EY(T,\omega)=\int_0^T p(u;t)\, dt \tag{2.9}$$

To show $Y(T,\omega)>0$ in a set of positive probability we note that by the Schwarz inequality, for $0<\lambda<1$

$$P[Y(T,\omega)>\lambda EY(T,\omega)] \geqq (1-\lambda)^2 \frac{E^2 Y(T,\omega)}{EY^2(T,\omega)}>0 \tag{2.10}$$

by (1.6) and (2.4).

Proof of Theorem 2. We first show that for any $\eta>0$ we can find a T_0 so that for $T>T_0$ $\dim G(u,T,\omega) \geqq 1-\alpha$ on a set $\Omega' \subset \Omega$, $P(\Omega')>1-\eta$. It is easy to check that conditions (1.2) through (1.6) are satisfied. For $g(t_1, t_2)$ we take $p(0,0;t_1,t_2)= p(0,0;0,t_2-t_1)$. In this case (1.3) is

$$2\int_0^T \frac{T-\tau}{(1-r^2(\tau))^{1/2}}\,d\tau. \tag{2.11}$$

The function $1-r^2(\tau)=0$ for $\tau=0$ but since it is continuous and $\lim_{\tau\to\infty} r(\tau)=0$, $1-r^2(\tau)$ is bounded away from zero for $\tau\geqq\delta>0$. Therefore (2.11) is finite if

$$\int_0 \frac{d\tau}{\sigma(\tau)}<\infty. \tag{2.12}$$

Since $\sigma(\tau)$ has index $\alpha<1$, $\sigma(\tau)>\tau^\beta$ for some $\beta<1$. Therefore (2.12) is finite. For $h(t)$ in (1.4) we can take 1.

By the same change of variables that gave rise to (2.11) we see that (1.7) is equal to

$$\frac{1}{\pi}\int_0^T (T-\tau)\frac{k(|\tau|)}{(1-r^2(\tau))^{1/2}}\,e^{-\frac{u^2}{1+r(\tau)}}\,d\tau \tag{2.13}$$

and, as above, this will be finite if

$$\int_0 \frac{k(|\tau|)}{\sigma(\tau)}\,d\tau<\infty. \tag{2.14}$$

Since $\sigma(\tau)$ has index α, $\sigma(\tau)>\tau^\beta$ for all $\beta>\alpha$ and (2.14) is finite for $k(|\tau|)=\tau^{-(1-\alpha-\delta)}$ for all $\delta>0$. Therefore the capacitarian dimension of $G(u,T,\omega)$ is greater than or equal to $1-\alpha$ for all ω for which $Y(T,\omega)>0$ (Here $G(u,T,\omega)$ and $Y(T,\omega)$ are as given in the proof of Theorem 1). By the equivalence of the capacitarian and Hausdorff dimension we have $\dim G(u,T,\omega)\geqq 1-\alpha$ for all ω for which $Y(T,\omega)>0$. To complete this part of the proof we need to show that

$$\lim_{T\to\infty} P[Y(T,\omega)>0]=1 \tag{2.15}$$

(since $Y(t,\omega)$ is increasing in t). By (2.10), (2.15) will follow if we show that

$$\lim_{T\to\infty}\frac{E^2Y(T,\omega)}{EY^2(T,\omega)}=1. \tag{2.16}$$

Clearly

$$EY(T,\omega)=\frac{T}{\sqrt{2\pi}}\,e^{-u^2/2}$$

and

$$EY^2(T,\omega)=\frac{1}{\pi}\int_0^T (T-\tau)\frac{1}{(1-r^2(\tau))^{1/2}}\,e^{-\frac{u^2}{1+r(\tau)}}\,d\tau. \tag{2.17}$$

For any $\delta_1>0$ we can find a δ such that

$$\left|\frac{1}{(1-\delta^2)^{1/2}}\,e^{-\frac{u^2}{1+\delta}}-e^{-u^2}\right|\leqq\delta_1.$$

Choose τ_0 such that $r(\tau)<\delta$ for $\tau>\tau_0$. Then (2.17)

$$\leqq M(\tau_0)T+\frac{1}{\pi}\int_{\tau_0}^{T}(T-\tau)\frac{1}{(1-\delta^2)^{1/2}}e^{-\frac{u^2}{1+\delta}}d\tau \tag{2.18}$$

where $M(\tau_0)$ is a constant that depends on τ_0. (2.8) is

$$\leqq M(\tau_0)T+\frac{1}{\pi}\int_{\tau_0}^{T}(T-\tau)e^{-u^2}d\tau$$
$$+\frac{1}{\pi}\int_{\tau_0}^{T}(T-\tau)\left|\frac{1}{(1-\delta^2)^{1/2}}e^{-\frac{u^2}{1+\delta}}-e^{-u^2}\right|d\tau.$$

Consequently

$$\frac{E^2Y(T,\omega)}{EY^2(T,\omega)}\geqq\frac{\frac{T^2}{2\pi}e^{-u^2}}{M(\tau_0)T+\frac{1}{2\pi}(T-\tau_0)^2e^{-u^2}+\frac{1}{\pi}(T-\tau_0)^2\delta_1}$$
$$\geqq\frac{1}{\frac{2\pi M(\tau_0)}{T}+\left(1-\frac{\tau_0}{T}\right)^2+2\left(1-\frac{\tau_0}{T}\right)^2\delta_1}\geqq\frac{1}{1+3\delta_1}$$

for T sufficiently large. Therefore we obtain (2.16) and we have proved the first part of the theorem.

To complete the proof it suffices to show that $\dim G(u,T,\omega)\leqq 1-\alpha$ a.s. for all T. This is precisely what Berman shows in the first part of his Theorem 2.1 [1]. Even though his condition (2.1) is stronger than saying that $\sigma(t)$ has index α, the critical inequality on line 10 page 1263 still holds.

Remark. If a process $X(t)$ satisfies the hypotheses of Theorem 1 and if in addition (2.16) holds for this process, then the following result is immediate: For any η we can find a T_0 such that for $T>T_0$ $\mathrm{Cap}_k(G(u,T,\omega))>0$ on a set $\Omega'\subset\Omega$, $P(\Omega')>1-\eta$.

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