Z. Wahrscheinlichkeitstheorie verw. Gebiete 32, 323–331 (1975)

The Markov Processes of Schrödinger

Benton Jamison*

Introduction

Let $\{X_t, t\in I\}$ be a stochastic process on a finite or semi-infinite interval I. For each $J\subset I$, let $\mathscr{A}_J$ be the σ-field of events generated by X_t, $t\in J$, and $\mathscr{B}_J$ be the σ-field generated by X_t, $t\in I\setminus J$. We say that $\{X_t, t\in I\}$ is a *reciprocal process* if, for any subinterval $J=(s,t)$ of I,

$$P(A\cap B|X_s, X_t)=P(A|X_s)\,P(B|X_t), \qquad A\in\mathscr{B}_I,\ B\in\mathscr{B}_J. \tag{1}$$

The concept was formulated in 1932 by Bernstein [2] in connection with the processes introduced in 1931 by Schrödinger [10]. Consider the transition $q(s,x;t,y)$ for Brownian motion $\{Y_t, a\leqq t\leqq b\}$ on an interval $[a,b]$:

$$q(s,x;t,y)=\frac{1}{\sqrt{2\Pi(t-s)}}\,e^{-\frac{(y-x)^2}{2(t-s)}}, \qquad a\leqq s<t\leqq b. \tag{2}$$

If we prescribe an initial distribution μ_a for Y_a, the finite dimensional distributions of $\{Y_t, a\leqq t\leqq b\}$ are determined by (2) and μ_a; in particular the distribution of Y_b is so determined. Schrödinger asks the following question. Suppose we prescribe in advance not only a distribution μ_a of Y_a but an arbitrary distribution μ_b of Y_b as well, what is the most likely way for Y_t to evolve as t goes from a to b? His answer amounts to the construction of a new process $\{X_t, a\leqq t\leqq b\}$. He obtains from the original process the "intermediate probabilities"

$$P(s,x;t,y;u,z)=\frac{q(s,x;t,y)\,q(t,y;u,z)}{q(s,x;u,z)}, \qquad a\leqq s<t<u\leqq b. \tag{3}$$

We see that $p(s,x;t,y;u,z)$ is the value at y of the conditional density of Y_t given $Y_s=x$ and $Y_u=z$. Let μ be any two dimensional probability measure with marginals μ_a and μ_b. (There are in general many such measures.) The distribution μ is used as the joint distribution of X_a and X_b in the new process $\{X_t, a\leqq t\leqq b\}$, whose finite-dimensional distributions are as follows. Let $a<t_1<\cdots<t_n<b$. Let $A, B, E_1, \ldots, E_n$ be measurable sets in the state space. Let $E=\prod_{i=1}^n E_i$, $\Lambda=A\times E\times B$. Define

$$P_\mu(\Lambda)=\int_{A\times B} d\mu(x,y)\int_E p(a,x;t_1,x_1;b,y)\,p(t_1,x_1;t_2\,x_2;b,y)\ldots \\ p(t_{n-1},x_{n-1};t_n,x_n;b,y)\,dx_1\ldots dx_n. \tag{4}$$

It follows from the results of [8] that (3) defines a consistent set of finite-dimensional distributions such that the stochastic process $\{X_t, a\leqq t\leqq b\}$ defined by them is a

* Partially supported by National Science Foundation Grants GD-177-2, GU 3171.

reciprocal process. Furthermore $p(s, x; t, y; u, z)$ is the value at y of the conditional density of X_t given $X_s = x$ and $X_u = z$. We say that the process $\{X_t, a \leqq t \leqq b\}$ is *derived* from $\{Y_t, a \leqq t \leqq b\}$. All this goes through if the Brownian transition function (2) is replaced by any strictly positive Markov transition density $q(s, x; t, y)$. Schrödinger's processes are not so general. Given marginals μ_a and μ_b, he uses as endpoint measure a particular μ with these marginals, one which can be written in the form

$$\mu(E) = \int_E q(a, x; b, y)\, \nu_a(dx)\, \nu_b(dy), \tag{5}$$

where E is an arbitrary two-dimensional Borel set and where ν_a and ν_b are σ-finite one-dimensional measures. Schrödinger's physical intuition convinced him of the existence of a unique such μ with the given marginals μ_a and μ_b. Indeed, a slight extension of a result of Beurling [3] yields existence and uniqueness (see [8]).

In [8], it is shown that it is precisely for those measures which have the representation (5) that the derived process $\{X_t, a \leqq t \leqq b\}$ has the Markov property. In this paper we show that these Markov processes of Schrödinger are h-path processes in the sense of Doob [5], where h is a space-time harmonic function for the original process $\{Y_t, a \leqq t \leqq b\}$. This fact is exploited to show that under certain conditions the sample paths of $\{X_t, a \leqq t \leqq b\}$ are almost surely continuous on $[a, b]$; in particular the $\{Y_t\}$ process "tied down" at $t = a$ and $t = b$ is well-defined. We then treat the case where $\{Y_t, a \leqq t \leqq b\}$ is a diffusion in Euclidean space. We show that if the coefficients of the diffusion equation satisfy some regularity conditions, the derived process $\{X_t, a \leqq t \leqq b\}$ turns out also to be a diffusion, with the same diffusion coefficients, plus an additional drift term.

§1

Let (S, d) be a σ-compact metric space, with Σ the σ-field generated by the open sets of S. Let $[a, b]$ be a closed interval of real numbers. Let Ω, Ω_0, and Ω_u be, respectively, the set of all functions from $[a, b]$, $[a, b)$, and $[a, \mu]$ into S (here $a < u \leqq b$). We denote by X_t the coordinate function on Ω; that is, $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$, $t \in [a, b]$. We also use X_t to denote the coordinate functions on Ω_0 and Ω_u. The smallest σ-field $\mathscr{G}$ on Ω relative to which X_t is $\mathscr{G} - \Sigma$ measurable for each $t \in [a, b]$ is denoted by $\mathscr{I}$. The σ-fields $\mathscr{I}_0$ and $\mathscr{I}_u$ are defined in the same way on Ω_0 and Ω_u respectively. Let $Q(s, x; t, E)$, $a \leqq s < t \leqq b$, $x \in S$, $E \in \Sigma$, be a Markov transition probability function. We assume that Q is given by a strictly positive density relative to some σ-finite measure λ on Σ; that is, there is a strictly positive function $q(s, x; t, y)$ defined for $a \leqq s < t \leqq b$ and $(x, y) \in S \times S$, Σ-measurable in (x, y) for each s and t, and for which

$$Q(s, x, t, E) = \int_E q(s, x; t, y)\, \lambda(dy), \qquad a \leqq t \leqq b, \ x \in S, \ E \in \Sigma. \tag{6}$$

For each $a \leqq s < t < u \leqq b$ and $(x, y, z) \in S \times S \times S$ we define $p(s, x; t, y; u, z)$ by (3). Now set

$$P(s, x; t, E; u, z) = \int_E p(s, x; t, y; u, z)\, \lambda(dy), \qquad a \leqq s < t < u \leqq b, \ (x, y) \in S \times S, \ E \in \Sigma. \tag{7}$$

It is observed in Section 3 of [8] that $P(s, x; t, E; u, y)$ is a reciprocal transition function. Let μ be a probability measure on $\Sigma \times \Sigma$. Then μ and $P(s, x; t, E; u, y)$ define a measure P_μ on $\mathscr{I}$ relative to which $\{X_t, a \leqq t \leqq b\}$ is a reciprocal process. (See Theorem 2.1 of [8]. It is clear that 2.2 of [8] can be written as (4) of this paper.) We assume that there are σ-finite measures v_a and v_b on Σ for which (5) holds for $E \in \Sigma \times \Sigma$. Then, by virtue of Theorem 3.1 of [8], $\{X_t, a \leqq t \leqq b\}$ is a Markov process relative to the probability space $(\Omega, \mathscr{I}, P_\mu)$. We denote by μ_a and μ_b the marginals of μ, that is, $\mu_a(E) = \mu(E \times S)$ and $\mu_b(E) = \mu(S \times E)$ for each $E \in \Sigma$. Let P be the probability measure on $\mathscr{I}$ (constructed in the usual way) for which $\{X_t, a \leqq t \leqq b\}$ is a Markov process with initial distribution μ_a and transition function $Q(s, x; t, E)$. We will show that P_μ can be obtained from P by means of a multiplicative functional; in fact we can use Doob's construction of an "h-path process" to go from P to P_μ. First, define h on $[a, b) \times S$ by

$$h(t, x) = \int q(t, x; b, y)\, v_b(dy) \qquad x \in S,\ t \in [a, b). \tag{8}$$

It is easily verified that h is space-time harmonic relative to Q; that is,

$$h(s, x) = \int Q(s, x; t, dy)\, h(t, y) \qquad x \in S,\ a \leqq s < t < b. \tag{9}$$

Following Doob [5] we define a transition probability operator Q^h by

$$Q^h(s, x; t, E) = \frac{1}{h(s, x)} \int_E Q(s, x; t, dy)\, h(t, y), \qquad a < s < t < b,\ x \in S,\ E \in \Sigma. \tag{10}$$

Let P^h be the measure on $(\Omega^0, \mathscr{I}^0)$ for which $\{X_t, a \leqq t < b\}$ is a Markov process with initial distribution μ_a and transition probability operator $Q^h(s, x; t, E)$. Note that for $a \leqq s < t < b$, $Q^h(s, x; t, E) = \int_E q^h(s, x; t, y)\, \lambda(dy)$, where

$$q^h(s, x; t, y) = \frac{q(s, x; t, y)\, h(t, y)}{h(s, x)}. \tag{11}$$

Let $a < s < t < b$. By (3), (4) and (5)

$$\begin{aligned} &P_\mu(X_s \in E, X_t \in F) \\ &\quad = \iint v_a(dx)\, v_b(dy) \int_F \int_E q(a, x; s, z_1)\, q(s, z_1; t, z_2)\, q(t, z_2; b, y)\, \lambda(dz_1)\, \lambda(dz_2) \\ &\quad = \int v_a(dx) \iint q(a, x; s, z_1)\, q(s, z_1; t, z_2)\, h(t, z_2)\, \lambda(dz_1)\, \lambda(dz_2). \end{aligned}$$

But by (5),

$$\mu_a(A) = \int_A v_a(dx) \int q(a, x; b, y)\, v_b(dy) = \int_A v_a(dx)\, h(a, x),$$

so the last expression for $P_\mu(X_s \in E, X_t \in F)$ is equal to

$$\int \frac{\mu_a(dx)}{h(a, x)} \int_F \int_E q(a, x; s, z_1)\, q(s, z_1; t, z_2)\, h(t, z_2)\, \lambda(dz_1)\, \lambda(dz_2),$$

which is equal to $P^h(X_s \in E, X_t \in F)$. A similar argument shows that

$$P_\mu(X_a \in E, X_t \in F) = P^h(X_a \in E, X_t \in F) \qquad \text{for } a < t < b.$$

Since $P_\mu(X_a \in E) = P^h(X_a \in E) = \mu_a(E)$, and since $\{X_t, a<t<b\}$ is a Markov process relative to both P_μ and P^h, this shows that, if $a<t<b$, the restriction of P_μ to $\{X_s, a \leqq s<t\}$ (that is, to $\mathscr{I}_t$) is given by P^h. If we define

$$M_s^t = \frac{h(t, X_t)}{h(s, X_t)} \qquad a \leqq s \leqq t < b, \tag{12}$$

M_s^t is a multiplicative functional for the Markov process $\{X_t, a \leqq t<b\}$ relative to P and $P_\mu(=P_h)$ on each $\mathscr{I}_t$, $a<t<b$, is obtained from P via transformation by M_s^t [6]. This amounts to the same thing as observing, as does Doob [4], that

$$P_h(A) = \int_A k(t, X_t)\, dP, \qquad A \in \mathscr{I}_t, \tag{13}$$

where $k(t,x) = h(t,x)/\int h(a,y)\, \mu_a(dy)$. The measure P_μ on $\mathscr{I}$, as distinct from the sub-σ-fields $\mathscr{I}_t$, may not be obtained from P by a multiplicative functional transformation, because we need not have $P_\mu \ll P$ on $\mathscr{I}$. For instance, the marginal distributions μ_b and $Q_b \mu_a$ of P_μ and P respectively, where

$$Q_b \mu_a(E) = \int_E Q(a, x; b, E)\, \mu_a(dx),$$

may be mutually singular, as they indeed are if μ_b and λ are mutually singular.

For each $x \in S$, we denote by P_x the measure on $\mathscr{I}$ for which $\{X_t, a \leqq t \leqq b\}$ is a Markov process with initial measure δ_x (where $\delta_x(\{x\})=1$) and transition function $Q(s, x; t, E)$.

Theorem 1. *Suppose that for each $x \in S$, the set C of all continuous paths on $[a,b]$ has outer measure one relative to P_x. Suppose in addition that for each $y_0 \in S$ the following holds: given $\delta>0$ and $\varepsilon>0$ there is a t_0 such that if $d(y, y_0) \geqq \delta$ and $t_0<t<b$, then $q(t, y; b, y_0) \leqq \varepsilon$. Then, for each probability measure μ on Σ, the measure P_μ defined by* (4) *also assigns outer measure* 1 *to C.*

Proof. It is clear that it suffices to prove the theorem for measures μ concentrating all their mass on an arbitrary pair $(x_0, y_0) \in S \times S$. Let μ be such a measure. Then $h(x,t) = q(t, x; b, y_0)$. Since $P_\mu = P_h \ll P_{x_0}$ on $\mathscr{I}_t$ for each $a<t<b$ by virtue of (11), it is clear that P_μ assigns outer measure 1 to the set of all paths on $[a,b]$ which are continuous on $[a,b)$. Observe with Doob [5] that $\{1/h(t, X_t): a \leqq t<b\}$ is a martingale with respect to P^h-measure on $\mathscr{I}^0$. It follows from the non-negativity of h that as $t \uparrow b$ $1/h(t, X_t)$ converges P^h-almost surely to a finite limit, hence $h(t, X_t) = q(t, X_t; b, y_0)$ converges to a non-zero limit. Now the assumption on q contained in the hypothesis of the theorem shows that, given f: $[a,b) \to S$, $q(t, f(t); b, y_0)$ cannot converge to a non-zero limit as $t \uparrow b$ unless $f(t)$ converges to y_0 as $t \uparrow b$. It easily follows that P^h, hence P_μ, also assigns outer measure 1 to the set of all paths on $[a,b)$ which have limit y_0 as $t \uparrow b$. Thus the P_μ-outer measure of C is 1, and the proof is complete.

§2

We now consider the case where S is equal to d-dimensional Euclidean space E^d and Σ is the σ-field of Borel subsets of S. We assume that the underlying Markov process is a diffusion in the sense of Stroock and Varadhan [11]. Let a_{ij}, $i, j=1, \ldots, d$ and b_i, $i=1, \ldots, d$ be real-valued functions on $E^d \times [a, b)$ such that the following hold

(D 1) a_{ij} is continuous and bounded on $E^d \times [a, b-\varepsilon)$ for each $\varepsilon>0$ and $i, j=1, \ldots, d$.

(D 2) The matrix $((a_{ij}(x, t)))$ is positive definite for each $(x, t) \in E^d \times [a, b)$.

(D 3) b_i is measurable and bounded on $E^d \times [a, b-\varepsilon]$ for each $\varepsilon>0$ and $i=1, \ldots, d$.

We denote the matrix $((a_{ij}))$ by α and the vector (b_i) by β. Let $\sigma=\alpha^{1/2}$ (see [11], p. 347). For each $s \in [a, b)$ let $C[a, s]$ be the subclass of Ω^s consisting of all continuous real valued functions on $[a, s]$, and let $\mathscr{M}^s=\{E \cap C[a, s]: E \in \mathscr{I}^s\}$. Then $\mathscr{M}^s$ is a σ-field over $C[a, s]$: in fact if we consider $C[a, s]$ as a metric space with metric given by the uniform norm, $\mathscr{M}^s$ is the σ-field generated by the open spheres. Under conditions D1–D3, Stroock and Varadhan show that for each $(s, x) \in [a, b) \times E^d$ there is a probability measure $P_{s,x}$ on $\mathscr{M}^s$ which solves what they call "the martingale problem." That $P_{s,x}$ is a solution to the martingale problem is equivalent to the existence of a Wiener process $\{W_t, a \leqq t \leqq s\}$ on $(C[a, s], \mathscr{M}^s, P_{s,x})$ for which

$$X_t = x + \int_a^t \beta(u, X_u)\, du + \int_a^t \sigma(u, X_u)\, dW_u \qquad a<t \leqq s, \tag{14}$$

where the second integral on the right is a stochastic integral in the sense of Ito. Relative to $P_{s,x}$, $\{X_t: a \leqq t \leqq s\}$ is a strong Markov process with a transition function denoted in [11] by $P(t, x; u, E)$ for $a \leqq t<u \leqq s$. From the results of [11] and assumptions D1–D3 it is easy to see that the processes $\{X_t, a \leqq t \leqq s\}$, $s<b$ can be extended to a process $\{X_t, a \leqq t<b\}$ with transition function $Q(s, x; t, E)$. We refer to $\{X_t, a \leqq t<b\}$ as the diffusion corresponding to the *diffusion matrix* α and *drift vector* β. Eq. (14) holds throughout $[a, b)$; that is, there is a Wiener process $\{W_t, a \leqq t<b\}$ constructed from $\{X_t, a \leqq t<b\}$ for which

$$X_t - X_s = \int_s^t \beta(u, X_u)\, du + \int_s^t \sigma(u, X_u)\, dW_u, \qquad a \leqq s<t<b. \tag{15}$$

We say that a function f on $E^d \times [a, b)$ is *smooth* if the partial derivatives $\frac{\partial f}{\partial t}$ and $\frac{\partial^2 f}{\partial x_i \partial y_j}$ exist and are continuous throughout $E^d \times [a, b)$, $i, j=1, \ldots, d$.

Theorem 2. *Let h be a smooth and everywhere positive space-time harmonic function for the process with diffusion matrix α and drift vector β. Then the process with diffusion matrix α and drift vector $\beta+\alpha\,\mathrm{grad}(\log h)$ has the transition density $q^h(s, x; t, E)$ given by* (11).

Proof. The proof consists mainly of calculations based on Ito's lemma. From (15) and Ito's lemma (as stated in [11], p. 352) we have

$$h(t, X_t) - h(s, X_s) = \int_s^t G(u, X_u)\, du + \int_s^t H(u, X_u)\, dW_u \qquad a \leqq s<t<b, \tag{16}$$

where G and H are defined on $[a, b) \times E^d$ by

$$G = \frac{\partial h}{\partial t} + \sum_i \frac{\partial h}{\partial x_i} b_i + \frac{1}{2} \sum_{i,j,k} \frac{\partial^2 h}{\partial x_i \partial x_j} \sigma_{ik} \sigma_{jk},$$
$$H = (H_1, \ldots, H_d), \qquad H_k = \sum_i \frac{\partial h}{\partial x_i} \sigma_{ik}, \qquad k = 1, \ldots, d. \tag{17}$$

The space-time process $\{(X_t, t), a \leqq t \leqq b\}$ can be considered a Markov process with state space $E^d \times [a, b)$. The process can be started out at any point (x, s) in its state space, and we use $P^{(s,x)}$ and $E^{(s,x)}$ to denote the corresponding probability functions expectation operators. Because h is space-time harmonic, we have

$$E_{(s,x)} h(t, X_t) = h(s, x) \qquad (s, x) \in E^d \times [\dot{a}, b); \tag{18}$$

in fact, from the optional stopping theorem for martingales ([4], p. 376) we have

$$E_{(s,x)} h(\tau, X_\tau) = h(s, x) \tag{19}$$

for any stopping time τ with $s \leqq \tau < b' < b$. Let $U = \{(s, x) : a < s < b, G(s, x) > 0\}$. Suppose $U \neq \emptyset$, and $(s, x) \in U$. Let $b' \in (s, b)$ and define τ to be the first exit time after s of $\{X_t\}$ from U, or b', whichever is smaller. Because $\{X_t\}$ has continuous sample paths and U is open, $P_{(s,x)}[\tau > s] = 1$. Taking $t = \tau$ in (16), we have

$$h(\tau, X_\tau) - h(s, X_s) = \int_s^\tau G(u, X_u)\, du + \int_s^\tau H(u, X_u)\, dW_u. \tag{20}$$

Since $\{\int_s^t H(u, X_u)\, dW_u, t \in (s, b)\}$ is a martingale, $E_{(s,x)}[\int_s^\tau H(u, X_u)\, dW_u] = 0$, again by the optional stopping theorem for martingales. Applying $E_{(s,x)}$ to both sides of (20) and using (19) we obtain $E_{(s,x)}[\int_s^\tau G(u, X_u)\, du] = 0$. Since $G(u, X_u) > 0$ for $u \in (s, \tau)$, and since $P_{(s,t)}(\tau > s) = 1$, we have a contradiction. Thus $U = \emptyset$, so $G \leqq 0$. A similar argument shows that $G \geqq 0$, so $G = 0$. Since $a_{ij} = \sum_k \sigma_{ik} \sigma_{kj} = \sum_k \sigma_{ik} \sigma_{jk}$, (17) yields

$$\frac{\partial h}{\partial t} + \frac{1}{2} \sum_{i,j} a_{ij} \frac{\partial h}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial h}{\partial x_i} = 0 \tag{21}$$

for any smooth space-time harmonic function h. Applying (15) and Ito's lemma to $\log h$ instead of h, we have

$$\log h(t, X_t) - \log h(s, X_s) = \int_s^t \bar{G}(u, X_u)\, du + \int_s^t \bar{H}(u, X_u)\, dW_u \qquad a \leqq s < t < b, \tag{22}$$

where $\bar{G}$ and $\bar{H}$ are defined on $E^d \times [a, b)$ by

$$\bar{G} = \frac{\partial(\log h)}{\partial t} + \sum_i \frac{\partial(\log h)}{\partial x_i} \beta_i + \frac{1}{2} \sum_{i,j,k} \frac{\partial^2(\log h)}{\partial x_i \partial x_j} \sigma_{ik} \sigma_{jk},$$

$$\bar{H} = (\bar{H}_1, \ldots, \bar{H}_d), \qquad \bar{H}_k = \sum_i \frac{\partial(\log h)}{\partial x_i} \sigma_{ik}.$$

Using matrix multiplication and interpreting vectors as column vectors, we can write $\bar{H}=\operatorname{grad}(\log h)\sigma$. Simple computations together with an application of (21) yield $\bar{G}=-(1/2)\langle\bar{H},\bar{H}\rangle$. Setting $\gamma=\operatorname{grad}(\log h)$ we have $\bar{G}=-(1,2)\langle\sigma\gamma,\sigma\gamma\rangle$, and (22) becomes

$$\log h(t,X_t)-\log h(s,X_s)=\int\langle\sigma\gamma,dW_u\rangle-\tfrac{1}{2}\int\langle\sigma\gamma,\sigma\gamma\rangle\,du. \tag{23}$$

Since $dX_t=\sigma\,dW_t+\beta\,dt$, $\langle\gamma,dX_t\rangle=\langle\gamma,\sigma\,dW_t\rangle+\langle\gamma,\beta\rangle\,dt$, and since σ is symmetric, we have

$$\log h(t,X_t)-\log h(s,X_s)=\int_s^t\langle\gamma,dX_u\rangle-\int_s^t(\langle\gamma,\beta\rangle+\tfrac{1}{2}\langle\sigma\gamma,\sigma\gamma\rangle)\,du. \tag{24}$$

Let $Q_{s,x}$, $a\leqq s<t<b$, $x\in E^d$ be the solution to the martingale problem with the same diffusion matrix α but with β replaced by $\delta=\beta+\alpha\operatorname{grad}(\log h)=\beta+\alpha\gamma$. To prove the theorem we must show that, for each $a\leqq s<t<b$ and $x\in E^d$, the restriction $Q_{s,x,t}$ of $Q_{s,x}$ to $\mathscr{M}_s^t$ is absolutely continuous with respect to the restriction $P_{s,x,t}$ of $P_{s,x}$ to $\mathscr{M}_s^t$ and that

$$\frac{dQ_{s,x,t}}{dP_{s,x,t}}=M_s^t=\frac{h(t,X_t)}{h(s,X_s)}. \tag{25}$$

(Here $\mathscr{M}_s^t$ is the sub-field of $\mathscr{M}^t$ generated by the family $\{X_u;\,s\leqq u\leqq t\}$.) For this purpose, we introduce $R_{s,x}$, $a\leqq s<t<b$, $s\in E^d$, the solution to the martingale problem with diffusion coefficient α and zero drift vector. On the one hand

$$\frac{dQ_{s,x,t}}{dP_{s,x,t}}=\frac{dQ_{s,x,t}}{dR_{s,xt}}\cdot\frac{dR_{s,x,t}}{dP_{s,x,t}}. \tag{26}$$

On the other hand, by the Cameron-Martin formula (Lemma 6.1 of [11])

$$\begin{aligned}\frac{dQ_{s,x,t}}{dR_{s,x,t}}&=\exp\left[\int_s^t\langle\alpha^{-1}\delta,dX_u\rangle-\tfrac{1}{2}\int_s^t\langle\delta,\alpha^{-1}\delta\rangle\,du\right]\\ \frac{dP_{s,x,t}}{dR_{s,x,t}}&=\exp\left[\int_s^t\langle\alpha^{-1}\beta,dX_u\rangle-\tfrac{1}{2}\int_s^t\langle\beta,\alpha^{-1}\delta\rangle\,du\right].\end{aligned} \tag{27}$$

From (26) and (27) it follows that

$$\frac{dQ_{s,x,t}}{dP_{z,x,t}}=\exp\left[\int_s^t\alpha^{-1}(\delta-\beta),dX_u-\tfrac{1}{2}\int_s^t(\langle\delta,\alpha^{-1}\delta\rangle-\langle\beta,\alpha^{-1}\beta\rangle)\,du\right]. \tag{28}$$

But $\delta-\beta=\alpha\gamma$, and elementary algebra yields

$$\langle\delta,\alpha^{-1}\delta\rangle-\langle\beta,\alpha^{-1}\beta\rangle=2\langle\gamma,\beta\rangle+\langle\alpha\gamma,\gamma\rangle.$$

Since $\langle\alpha\gamma,\gamma\rangle=\langle\sigma\gamma,\sigma\gamma\rangle$, comparison of (28) and (24) yields

$$\begin{aligned}\frac{dQ_{s,x,t}}{dP_{s,x,t}}&=\exp\left[\int_s^t\langle\gamma,dX_u\rangle-\int_s^t(\langle\gamma,\beta\rangle+\tfrac{1}{2}\langle\sigma\gamma,\sigma\gamma\rangle)\,du\right]\\ &=\exp[\log h(t,X_t)-\log h(s,X_s)]\\ &=\frac{h(t,X_t)}{h(s,X_s)}=M_s^t\end{aligned} \tag{29}$$

which establishes (25) and completes the proof of the theorem.

Since h is defined in terms of ν_b, and since ν_b can rarely be obtained explicitly, there is little possibility of applying this last theorem unless we can establish under rather general conditions that space-time harmonic functions h of the sort defined by (8) are indeed smooth. To this end, we now assume that the coefficients a_{ij} and b_i satisfy appropriate Hölder conditions, specifically, that there is a $\kappa > 0$ and an $\alpha \in [0, 1]$ for which

$$|a_{ij}(x,t) - a_{ij}(y,t)| \leqq \kappa\, |x-y|^\alpha,$$

$$|b_i(x,t) - b_i(y,t)| \leqq \kappa\, |x-y|^\alpha$$

for each x and y in E^d and $t \in [a, b]$ and for which

$$|a_{ij}(x,s) - a_{ij}(x,t)| \leqq \kappa\, |s-t|^\alpha$$

for each $x \in E^d$ and s, t in $[a, b]$, these conditions holding for each $1 \leqq i, j \leqq d$. We also assume there is a $\gamma > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(s,t)\, \lambda_i \lambda_j \geqq \gamma \sum_{i,j=1}^{d} \lambda_i^2,$$

where $(x,t) \in E^d \times [a, b]$. These assumptions imply that conditions 0.23 B_1, B_2 and B_3 on p. 227 of volume II of [6] hold in the strip $E^d \times [a, b]$. Eq. (8) shows that $h(t,x) = \int q(t,x;b,y)\, \nu_b(dy)$. The transition density q is the fundamental solution to Eq. (21) on the strip $E^d \times [a, b)$, $\varepsilon > 0$ (see Section 0.23 of [6] vol. II for the terminology here). For each fixed $t \in [a, b)$ and $y \in E^d$, $q(s,x;t,y)$ is a smooth function of $(s,x) \in (a,t) \times E^d$, and the inequalities (0.33)–(0.36) on p. 227 of [6], volume II imply that $h(s,x)$ is smooth if ν_b is a finite measure. This is a serious restriction, for in general ν_b is merely σ-finite. However, we know that there are compact sets $C_m \uparrow E^d$ with $\nu_b(C_m) < \infty$, $m = 1, 2, \ldots$. Let ν_m be the restriction of ν_b to $C_{m'}$ and let $h_m(t,x) = \int q(t,x;b,y)\, \nu_m(dy)$. Then h_m is smooth, and a solution to (21), and $h_m \uparrow h$. By virtue of Theorem 15 on p. 80 of [7], the smoothness of h follows once we show that h is bounded on sets of the form $D \times [a+\varepsilon, b-\varepsilon]$ where $\varepsilon > 0$ and D is a bounded open subset of E^d. If the coefficients a_{ij} are once continuously differentiable, with derivatives satisfying a Hölder condition in the space coordinate, (21) can be written as

$$\frac{\partial h}{\partial t} + \frac{1}{2} \sum_{i,j=1} \frac{\partial^2}{\partial x_i \partial x_j} (\tilde{a}_{ij} h) + \sum_i \tilde{b}_i \frac{\partial h}{\partial x_i} = 0,$$

and the Harnack inequalities of Moser [9] as extended by Aronson and Serrin [11] then imply that the convergence of h_m to h is uniform on sets $R \times [a+\varepsilon, b-\varepsilon]$, where R is a bounded rectangle in E^d. Thus h is bounded on sets $D \times [a+\varepsilon, b-\varepsilon]$, and is therefore smooth. (In applying results from the theory of parabolic equations to the analysis of Eq. (4.14), one usually reverses the direction of time: this is discussed in the footnote on p. 227 of volume II of [6].)

References

1. Aronson, D.G., Serrin, J.: Local behavior of solutions of quasilinear parabolic equations. Arc. Rational Mech. Anal. **25**, 81–122 (1967)
2. Bernstein, S.: Sur les liasons entre les grandeurs aléatoires. Verh. des intern. Mathematikerkongr., Zurich, 1932, Band I

3. Beurling, A.: An automorphism of product measures, Ann. of Math. **72**, 189–200 (1960)
4. Doob, J.L.: Stochastic processes. New York: Wiley 1953
5. Doob, J.L.: A Markov chain theorem. Probability and Statistics (The Harald Cramér Volume). New York: Wiley 1959
6. Dynkin, E.B.: Markov processes, Vols. 1 and 2. New York: Academic Press 1965
7. Friedman, A.: Partial differential equations of parabolic type, Englewood Cliffs, N.J.: Prentice Hall 1964
8. Jamison, B.: Reciprocal processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **30**, 65–86 (1974)
9. Moser, J.: A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. **80**, 931–954 (1958)
10. Schrödinger, E.: Über die Umkehrung der Naturgesetze. Sitz, Ber. der Preuss. Akad. Wissen., Berlin Phys. Math. **144** (1931)
11. Stroock, D.W., Varadhan, S.R.S.: Diffusion processes with continuous coefficients. Comm. Pure Appl. Math. **22**, I, 345–400; II, 479–530 (1969)

Benton Jamison
State University of New York at Albany
1400 Washington Ave.
Albany, New York 12222, USA

(Received September 27, 1974)