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# A Convexity Theorem for Positive Operators

By

M. A. Akcoglu* and R. V. Chacon**

The purpose of this note is to prove the following

**Theorem 1.** *Let* $(X, \mathfrak{F}, \mu)$ *be a* $\sigma$*-finite measure space,* $1 < p_0$, *and let* $T$ *be a positive linear operator on* $L_p = L_p(X, \mathfrak{F}, \mu)$, $1 \leqq p \leqq p_0$, *such that* $\| T \|_p \leqq 1$ *for* $1 \leqq p \leqq p_0$, *and such that* $T$ *is regular as an operator on* $L_1$. *Then* $T$ *may be extended in such a way that* $\| T \|_p \leqq 1$ *for* $1 \leqq p \leqq \infty$.

**Definition.** *A positive contraction on* $L_1$ *is regular if for some* $f > 0$ *in* $L_1$, $\sum_{k=0}^{\infty} T^k f = +\infty$ *almost everywhere.*

We remark that it follows from [*1*] and it was first proved by E. Hopf, that an operator which is regular with respect to one strictly positive $f$, is regular with respect to all strictly positive functions.

We obtain as a corollary of Theorem 1 the following:

**Corollary 1.** *Let* $(X, \mathfrak{F}, \mu)$ *be a* $\sigma$*-finite measure space,* $1 < p_0$, *and let* $T$ *be a positive linear operator on* $L_p = L_p(X, \mathfrak{F}, \mu)$, $1 \leqq p \leqq p_0$, *such that* $\| T \|_p \leqq 1$, *for* $1 \leqq p \leqq p_0$. *Then for any* $f \in L_1$,

$$\lim_{n \to \infty} \left( \frac{1}{n} \sum_{\nu=0}^{n-1} T^\nu f(x) \right)$$

*exists and is finite almost everywhere.*

If $p_0 = \infty$, this result is a special case of the Dunford-Schwartz Theorem [*4*]. On the other hand, if $p_0 = 1$, then the assertion of Corollary 1 is not necessarily true [*3*]. A weaker form of Corollary 1, assuming that the measure space is finite, is proved by Y. Ito [*5*], using a different argument.

The proof of the corollary follows from Theorem 1 and from the following result obtained in [*2*]:

**Theorem 2.** *Let* $\{p_n\}$ *be a sequence of non-negative measurable functions and let* $T$ *be a linear operator of* $L_1$ *to* $L_1$ *with* $\| T \|_1 \leqq 1$ *and with* $|T g(x)| \leqq p_{n+1}(x)$ *almost everywhere whenever* $|g(x)| \leqq p_n(x)$ *almost everywhere and* $g \in L_1$. *Then for any* $f \in L_1$,

$$\lim_{n \to \infty} \frac{\sum_{\nu=0}^{n} T^\nu f(x)}{\sum_{\nu=0}^{n} p_\nu(x)}$$

*exists and is finite a. e. on* $\{x \,|\, 0 < \sum_{\nu=0}^{\infty} p_\nu(x) \leqq +\infty\}$.

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* Corinna Borden Keen Research Fellow at Brown University.

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The proof of Theorem 1 will depend on the following two lemmas.

**Lemma 1.** *Let* $1 < p_0$, *and* $T$ *be a positive linear operator on* $L_p$ *such that* $\| T \|_p \leqq 1$ *for* $1 \leqq p \leqq p_0$. *Let, for any measurable set* $A$, $L_p(A)$ *denote the subset of* $L_p$ *consisting of functions whose supports are contained in* $A$. *Then there exists a positive linear operator* $T_A$ *such that*

a) $T_A : L_p(A) \to L_p(A)$ *and* $\| T_A \|_p \leqq 1$ *for* $1 \leqq p \leqq p_0$.

b) *For any* $f \in L_p(A)$, $f \geqq 0$ *and for any* $n \geqq 0$

$$\sum_{\nu=0}^{n} T^{\nu} f(x) \leqq \sum_{\nu=0}^{n} T_A^{\nu} f(x) \quad \text{for} \quad \text{a. a. } x \in A .$$

*Proof.* Let $f \in L_p(A)$, $f \geqq 0$ and define two sequences of positive functions

$$f_1 = \chi T f \qquad g_1 = \chi' T f$$
$$f_{n+1} = \chi T g_n \qquad g_{n+1} = \chi' T g_n , \quad n \geqq 1 ,$$

where $\chi$ and $\chi'$ are the characteristic functions of $A$ and of the complement of $A$ respectively.

Let

$$T_A f(x) = \sum_{i=1}^{\infty} f_i(x)$$

which is defined for a. a. $x \in A$. Now we prove that $T_A f \in L_p(A)$, hence the linear extension of $T_A$ to all $L_p(A)$-functions defines a transformation on $L_p(A)$. To this end consider the relation

$$T(f + \sum_{i=1}^{n-1} g_i) = \sum_{i=1}^{n} f_i + \sum_{i=1}^{n} g_i , \quad n \geqq 2 ,$$

which follows from the definition of $f_i$ and $g_i$. Since the $f_i$'s and the $g_i$'s have disjoint supports and since $\| T \|_p \leqq 1$, we have

$$\int (\sum_{i=1}^{n} f_i)^p + \int (\sum_{i=1}^{n} g_i)^p = \int (\sum_{i=1}^{n} f_i + \sum_{i=1}^{n} g_i)^p = \int [T(f + \sum_{i=1}^{n-1} g_i)]^p$$
$$\leqq \int (f + \sum_{i=1}^{n-1} g_i)^p = \int f^p + \int (\sum_{i=1}^{n-1} g_i)^p ,$$

which implies that

$$\int (\sum_{i=1}^{n} f_i)^p \leqq \int f^p , \quad \text{for all } n = 1, 2, \ldots ,$$

or, by the Lebesgue monotone convergence theorem, that

$$\int (\sum_{i=1}^{\infty} f_i)^p = \int (T_A f)^p \leqq \int f^p .$$

Hence $T_A f \in L_p(A)$ and $\| T_A \|_p \leqq 1$. This proves a).

The linearity and positiveness of $T_A$ are obvious. To prove b) note that this inequality is trivial for $n = 1$ and for all $f \in L_p(A)$, $f \geqq 0$. Now assume that the assertion is proved for $n \leqq N - 1$. Since

$$T f = f_1 + g_1 \qquad T g_n = f_{n+1} + g_{n+1},$$

one obtains

$$\chi T^n f = \chi T^{n-1} f_1 + \chi T^{n-2} f_2 + \cdots + \chi f_n,$$

or,

$$\chi \sum_{n-1}^{N} T^n f = \chi \sum_{n=0}^{N-1} T^n f_1 + \chi \sum_{n=0}^{N-2} T^n f_2 + \cdots + \chi f_N,$$

which gives, by the induction hypothesis and by the positiveness of individual terms

$$\chi \sum_{n=1}^{N} T^n f \leqq \chi \sum_{n=0}^{N-1} T_A^n f_1 + \chi \sum_{n=0}^{N-2} T_A^n f_2 + \cdots + \chi f_N$$
$$\leqq \chi \sum_{n=0}^{N-1} T_A^n (f_1 + \cdots + f_N) \leqq \chi \sum_{n=1}^{N} T_A^n f.$$

This, essentially, is the required inequality for $n = N$.

**Lemma 2.** *Let $T$ be an operator satisfying the hypotheses of Theorem 1. Then for any $f \in L_1 \cap L_\infty$,*

$$\| T f \|_\infty \leqq \| f \|_\infty.$$

*Proof.* 1. First assume that the measure space is finite. Then it suffices to show that $T 1 \leqq 1$ almost everywhere, where 1 denotes the function which is identically 1 on $X$. Let

$$T 1 (x) = 1 + \alpha(x), \quad \text{where} \quad -1 \leqq \alpha.$$

Since the dissipative part [1] of $T$ has measure zero

$$\int T 1 = \int 1,$$

which implies that

$$\int \alpha = 0.$$

Now assume that $\alpha(x) > 0$ on a set $Q$ of positive measure and let $1 < p \leqq p_0$. Then $\| T 1 \|_p^p = \int (T 1)^p = \int (1 + \alpha)^p > \int (1 + p\alpha) = \int 1 + p \int \alpha = \int 1 = \| 1 \|_p^p$ where the inequality follows from the fact that, for any $p > 1$ and $-1 < \alpha$, $0 \neq \alpha$

$$(1 + \alpha)^p > 1 + p\alpha.$$

But $\| T 1 \|_p^p > \| 1 \|_p^p$ contradicts the condition $\| T \|_p \leqq 1$. Therefore the measure of $Q$ must be zero.

2. Now consider the general case. Assume that there exists a function $f \in L_1 \cap L_\infty$, $f \geqq 0$, such that

$$a < T f(x), \quad \text{with } a = \| f \|_\infty,$$

on a set $Q$ of positive, and necessarily finite, measure. Let $A$ be any set of finite measure which includes $Q$, and such that

$$\int_{A'} f < \int_{Q} (T f - a)$$

where $A'$ is the complement of $A$. Then it is easy to see that

$$\|\chi f\|_\infty < T\chi f(x)$$

on a subset of $Q$, with positive measure. Here $\chi$ denotes the characteristic function of $A$. Therefore, by Lemma 1,

$$1 < T_A\chi(x)$$

on a subset of $A$ with positive measure. But this is a contradiction to the first part of the proof. In fact, again by Lemma 1, $T_A$ is conservative [1] on almost all $A$, which has a finite measure.

*Proof of Theorem 1:* Let $E_1 \subset E_2 \subset \ldots$ be an increasing sequence of sets of finite measure such that $X = \bigcup_{i=1}^{\infty} E_i$. Let $\chi_i$ be the characteristic function of $E_i$. Then, for any $f \in L_p$, with $1 \leqq p \leqq \infty$, $f \geqq 0$ we have, for a.a. $x \in X$,

$$f(x) = \lim_{i\to\infty} f_i(x)\,,$$

where $f_i = f\chi_i \in L_1$. Let

$$\widetilde{T} f(x) = \lim_{i\to\infty} T f_i(x)$$

which is defined for a.a. $x \in X$. If $f$ is not positive, let

$$\widetilde{T} f = \widetilde{T} f^+ - \widetilde{T} f^-\,.$$

Also, if $1 \leqq p \leqq p_0$, then $T$ is continuous as an operator on $L_p$ and it is easy to see that $\widetilde{T} f = T f$ for all $f \in L_p$, $1 \leqq p \leqq p_0$. Then we can conclude the proof noticing that $\|\widetilde{T}\|_\infty \leqq 1$ by virtue of Lemma 2 and applying the Riesz convexity theorem to $\widetilde{T}$.

*Proof of Corollary 1:* Let the conservative and dissipative parts of $T$ be $C$ and $D$ and let $\chi$ denote the characteristic function of $C$. Theorem 1, together with the fact that $C$ is an invariant set [1], shows that $p_n = \chi$; $n = 0, 1, 2, \ldots$ is an admissible sequence to apply Theorem 2. Therefore, by this theorem, for any $f \in L_1$ and for almost all $x \in C$.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{\nu=0}^{n-1} T^\nu f(x) = \lim_{n\to\infty} \frac{\sum_{\nu=0}^{n-1} T^\nu f(x)}{\sum_{\nu=0}^{n-1} p_\nu(x)}$$

exists and is finite. On the other hand, for almost all $x \in D$,

$$\left|\sum_{\nu=0}^{\infty} T^\nu f(x)\right| < \infty\,,$$

i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{n-1} T^\nu f(x) = 0$$

almost everywhere on $D$.

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University of Toronto
Dept. of Mathematics
Toronto 5, Ontario (Canada)

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