Z. Wahrscheinlichkeitstheorie verw. Gebiete
60, 453–484 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Optimal Navigation with Random Terminal Time in the Presence of Phase Constraints

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Summary. We consider the problem of controlling a system whose state at time t is given by $p(t) \in R^n$, where we assume that we can choose the velocity $r(t)$ of $p(t)$ and the terminal time of control ζ in an arbitrary manner, restricted only by the target condition $Z(p(\zeta)) \leqq 0$, the phase constraints $G_j(p(t)) \leqq 0$, $j=1, \dots, J$ for all $t \leqq \zeta$, and the requirement that the norm of r is either essentially bounded or a.s. constant. For given cost function S the loss functional to be minimized is given by $\mathfrak{E} S(p(T \wedge \zeta), T \wedge \zeta)$, where T is a nonnegative random variable with known distribution P. So we control the state effectively only up to the random terminal time $T \wedge \zeta$.

By means of the technique of Dubovitskij and Milyutin for the treatment of extremum problems in locally convex topological vector spaces, which turns out to be a powerful tool in the stochastic setting too, we derive necessary conditions on optimal controls under rather general assumptions on P, S, Z and G_j, $j=1, \dots, J$. In an important special case where we consider simple phase constraints and monotone cost function S the general theorems allow a rather complete description of locally optimal paths in simple form.

1. Formulation of the Problem and Survey

The starting point of this paper is the so-called "Fitzwilliam Street problem" posed by R. Davidson. A pedestrian walking with constant velocity wants to cross a street diagonally from one corner to another. If a car comes into sight before he has arrived at his target, he has to leave his planned path at once and walk straight to the opposite pavement, proceeding afterwards to his target on this side of the street. If the distribution of the arrival time of the first car is known, then the problem consists in choosing a planned path which minimizes the expected value of the actually travelled distance.

For exponential arrival time, E.M. Wilkinson (1974) has shown by means of geometrical considerations the existence of optimal planned paths and has

0044-3719/82/0060/0453/$06.40

given a partial differential equation for the minimal expected distance as a function of the starting point, from which one can derive necessary conditions on an optimal path. He used a technique related to dynamic programming principle, relying heavily on the lack of memory of the exponential distribution. In the following we derive a general approach to this type of navigation problem with random terminal time; in particular, we obtain necessary conditions for solutions of the "Fitzwilliam Street problem" for arbitrary distributions.

The "Fitzwilliam Street problem" is a special case of a general problem of stochastic navigation under phase constraints which can be treated by methods of the generalized calculus of variations in topological vector spaces. The solution of such problems, which will now be described, also provides a better understanding of Wilkinsons result.

We consider continuous paths $p(t)$, $t \geqq 0$, in R^n, which start at time 0 in the initial state $a \in R^n$, and whose velocities $r(t)$ have a norm essentially bounded by a constant v_s. The terminal time ζ of the paths, i.e. the time of last control, can be chosen arbitrary in $[0, \infty)$. It is required that $p(\zeta)$ be in a target region $\{x | Z(x) \leqq 0\} \subset R^n$ and not leave an admissible region of R^n, which is described by phase constraints of the type $\sup\{G(p(t)) | 0 \leqq t \leqq \zeta\} \leqq 0$. Let T be the random time of the occurence of some event which forcibly prevents further control of the system in the case $T \leqq \zeta$. Let $l(T)$ denote the length of the path of our system up to time T. We assume that the costs are given by $S(p(T), l(T), T)$ if $T \leqq \zeta$, and by $S(p(\zeta), l(\zeta), \zeta)$ if $T > \zeta$. Here $S(x, l, t)$ is a given cost function on $R^n \times [0, \infty) \times [0, \infty)$.

Now we want to determine conditions on paths which locally minimize the expected costs under the given constraints. We assume $v_s > 0$, and writing T_ζ for the minimum of T and ζ, we get the following optimization problem.

(1.1) Determine a path $p \in C^n[0, \infty)$, a function of path length $l \in C^1[0, \infty)$, a velocity $r \in L^n_\infty[0, \infty)$ and a terminal time $\zeta \geqq 0$, so that

$$L(p, l, r, \zeta) = \mathfrak{E} S(p(T_\zeta), l(T_\zeta), T_\zeta) = \min!$$

under the constraints

$$p(t) = a + \int_0^t r(s)\, ds \qquad l(t) = \int_0^t \|r(s)\|\, ds,$$

$$\|r(t)\| \leqq v_s \qquad \text{a.s.},$$

$$Z(p(\zeta)) \leqq 0,$$

$$G_j(p(t)) \leqq 0 \qquad \text{for all } t \leqq \zeta,\ j = 1, \dots, J.$$

The problem is to determine necessary conditions on a tuple (p^0, l^0, r^0, ζ^0) which locally minimizes the loss functional L under all tuples satisfying the constraints.

In the "Fitzwilliam Street problem" the speed of the controlled system is a.s. constant rather than merely essentially bounded from above, so that the time dependence and the length dependence of the cost function S coincide. In general, whenever $S(x, l, t)$ decreases as x approaches the target and increases in l and t (as in the "Fitzwilliam Street problem"), then it turns out that an optimal velocity r^0 for the problem (1.1) has to have a.s. constant norm equal to v_s.

Constraints of the type $\|r(t)\| = v_s$ a.s. in which constant speed is required a priori, appear automatically, if the terminating random event depends on path length (e.g. attrition of material), rather than time. In such situations we have paths parametrized by their own relative length, and the costs $S(x, l)$ depend on the state x, where the random event takes place, and on the length of path l up to this time. To unify terminology, we assume in the following that in this context the system really moves along the paths with constant speed, so that the length of path and time become interchangeable, and we get the following optimization problem:

(1.2) Determine a path $p \in C^n[0, \infty)$, velocity $r \in L^n_\infty[0, \infty)$ and a terminal time $\zeta \geqq 0$, so that

$$L(p, r, \zeta) = \mathfrak{E} S(p(T_\zeta), T_\zeta) = \min!$$

under the constraints

$$\begin{aligned} p(t) &= a + \int_0^t r(s)\, ds \\ \|r(t)\| &= v_s \quad \text{a.s.} \\ Z(p(\zeta)) &\leqq 0, \\ G_j(p(t)) &\leqq 0 \quad \text{for all } t \leqq \zeta,\ j = 1, \ldots, J. \end{aligned}$$

In Chap. 2 we treat both optimization problems by means of the formalism of Dubovitskij and Milyutin, which provides necessary conditions on solutions of non-convex optimization problems in general locally convex topological vector spaces [Girsanov, 1972]. The formalism applies rather directly to (1.1) whereas the "thinness" of the set $\{r \mid \|r(t)\| = v_s \text{ a.s.}\}$ of admissible controls for (1.2) requires the preliminary solution of a transformed optimization problem. (The situation here is similar to that in the proof of the Pontryagin maximum principle given by Dubovitskij and Milyutin [Girsanov, 1972, Chap. 13].) However, the transition from this transformed problem back to the original problem can be interpreted more easily in our context than in the cited literature. To save space, we give only the proof of the solution of problem (1.1) (Theorem 2.1), but we use the transformation technique which is necessary for dealing with (1.2). The proof of Theorem 2.2, which gives the analogous result for problem (1.2), can be adapted from the proof of Theorem 2.1 with some minor changes.

In Chap. 3, we apply our general results to an important special case. We consider the simplest type of phase constraints in which the parameters describing the state of the system are simply bounded from above, i.e. we choose $G_j(x) = x_j - z_j$, $j = 1, \ldots, n$, in (1.1) and (1.2). Furthermore, we assume that the cost function reflects the desire to reach the target quickly in the sense that S

decreases in the state coordinates as the state approaches the target region, and that S increases with time. This special structure, which is shared in particular by the "Fitzwilliam Street problem", makes it possible to give a complete description of all locally optimal paths in simple form.

The formalism of Dubovitskij and Milyutin, which has proved to be a powerful tool in the solution of our problem, has not been much used in probabilistic optimization problems. Its usefulness stems partly from the fact that its application is not restricted by convexity assumptions. One application that has been made is the treatment of stochastic search problem [Pursiheimo, 1977 and 1978; Lukka, 1977]. Especially the model described in the last-named paper is somewhat related to our model. But while Lukka considers deterministic paths and random cost functions, we have a fixed cost function and paths subject to random influences.

The conditions on optimal controls obtained by applying the approach of Dubovitskij and Milyutin are hard to interpret stochastically. Nevertheless, by using a heuristic dynamic programming approach it is possible to make them at least plausible. This will be carried out in the second part of this paper [to appear in this journal].

Notation

$\partial M, M^0, M^c$: boundary, interior and complement of the set $M \subset R^n$. $C^n(A)$, $L^n_1(A)$, $L^n_\infty(A)$: the spaces of continuous, integrable and essentially bounded functions from A into R^n with the usual norms $\|\cdot\|_C, \|\cdot\|_1, \|\cdot\|_\infty$.

$\nabla S(x, l, t)$: the vector of partial derivatives of S with respect to $x_1, \ldots, x_n$, where $x \in R^n$.

As usual we write $x \leqq z$ instead of $x_j \leqq z_j$, $j = 1, \ldots, n$, where $x, z \in R^n$.

2. Necessary Conditions on Optimal Controls for General Cost Function and Phase Constraints

It is possible to get necessary conditions on solutions of the optimization problems (1.1) and (1.2) directly from the Pontryagin maximum principle (e.g. Theorem 14.2 of Girsanov, 1972), if we make rather stringent assumptions, e.g. continuous differentiability of the density f of P [Franke, 1980]. But we prefer a direct approach to the results contained in Theorems 2.1 and 2.2, which not only admits weaker assumptions, but, due to the special structure of our optimization problem, is also more transparent than the related proof of the general Theorem 14.2 of Girsanov (1972).

We only treat the optimization problem (1.1) with variable norm of velocity in detail, but we choose an approach, which is more general than necessary for analyzing (1.1), from which the respective results for problem (1.2) can be derived by some minor changes of proof. Both problems could be analyzed separately by somewhat simpler techniques [Franke 1980], which we combine in the following to get Theorems 2.1 and 2.2 simultaneously.

The operator $r \mapsto \int_0^t \|r(s)\|\,ds$ from $L^n_\infty[0,\infty)$ into $C^n[0,\infty)$, which appears in the constraints of problem (1.1), is not Fréchet-differentiable in functions r, whose norm is not essentially bounded away from 0. So we would have difficulties with our functional analytic approach. We avoid this problem by treating norm and direction of velocity as separate variables of another optimization problem equivalent to (1.1).

Essentially, this idea would suffice for deriving most of the results of Theorem 2.1, but in the case of constant velocity (1.2) we would end up with a rather weak result. This effect is due to the fact that in the formulation of our optimization problems (1.1) and (1.2) we consider only paths as adjacent, which not only differ uniformly by a small amount, i.e. the C^n-norm of their difference is small, but additionally the L^n_∞-distance of their velocities has to be small too. In the terminology of the classical calculus of variations we are searching for conditions on a "weak extremum", i.e. we are searching for paths with velocities r minimizing the loss functional L locally with respect to a weak neighbourhood, i.e. in our case a L^n_∞-neighbourhood.

Now, if the paths are parametrized by their own relative length, i.e. if we consider the constraint $\|r(s)\|=1$ a.s., the set of admissible variations is "thin" in L^n_∞, i.e. it has an empty interior. The case $n=1$ is especially extreme. Here, $r(s)$ a.s. can only assume the values ± 1, so that there are no other admissible variations than r itself. The entire variability under the prescribed constraints of the optimization problem (1.2) consists of the possibility to choose different ζ.

If we consider a so-called spike variation $r(s)+\delta(s)$ of r [Girsanov 1972], where $\delta(s)$ is different from 0 only for s in some small interval, then the resultant path with velocity $r(s)+\delta(s)$ lies adjacent to the original path in C^n, and the value of the loss functional L does not change very much, even if $\|\delta\|_\infty$ is not small. So it is natural to give up the claim to uniform nearness of the velocities and to proceed to another space, e.g. L^n_1, where the norm of a spike variation is small. In the terminology of the classical calculus of variations this corresponds to searching for "strong extremum conditions", i.e. for conditions on some (p,r) locally optimal with respect to a $C^n \times L^n_1$-neighbourhood.

In the following we do not analyze the loss functional L and the respective constraints directly in $C^n \times L^n_1$, but we employ an approach due to Dubovitskij and Milyutin [Girsanov 1972, Chap. 13] and proceed to an equivalent optimization problem. Here we exploit the fact that the value of the loss functional L, which depends only on path p and relative length l, does not change, if we choose another time scale, i.e. if we consider a change of parameter $s=V(t) =\int_0^s v(\tau)\,d\tau$, where $v\in L^1_\infty[0,1]$ and $v(t)\geqq 0$ a.s.

Then $q(t)=p(V(t))$, $k(t)=l(V(t))$, $n(t)=\|r(V(t))\|$ on $\{t|v(t)>0\}$ and $u(t) =r(V(t))/n(t)$ on $\{t|v(t)>0, n(t)>0\}$ are path, relative length, norm of velocity and direction of velocity in the new parametrization; $\zeta=V(1)=\int_0^1 v(\tau)\,d\tau$ is the time up to the termination of control. On $\{t|v(t)=0\}$ we can choose $n(t)$ and $u(t)$ arbitrary as far as we have $0\leqq n(t)\leqq v_s$ and $\|u(t)\|=1$ for almost all $t\in[0,1]$, and the same is true for $u(t)$ on $\{t|n(t)=0\}$. Then (1.1) changes to the following optimization problem:

(2.1) Determine a path $q\in C^n[0,1]$, a function of relative length $k\in C^1[0,1]$, a function of direction $u\in L^n_\infty[0,1]$, a function of absolute velocity $n\in L^1_\infty[0,1]$ and a parametrization density $v\in L^1_\infty[0,1]$, so that

$$\begin{aligned}\Lambda(q,k,u,n,v)=&\int_0^1 S(q(t),k(t),V(t))f(V(t))v(t)\,dt\\ &+S(q(1),k(1),V(1))P\{T>V(1)\}=\min!\end{aligned}$$

under the constraints

$$\begin{aligned}&q(t)=a+\int_0^t u(\tau)n(\tau)v(\tau)\,d\tau\\ &k(t)=\int_0^t n(\tau)v(\tau)\,d\tau\\ &\|u(t)\|=1 \quad \text{a.s.,} \quad 0\leqq n(t)\leqq v_s \quad \text{a.s.}\\ &0\leqq v(t) \quad \text{a.s.}\\ &Z(q(1))\leqq 0\\ &G_j(q(t))\leqq 0 \quad \text{for all } 0\leqq t\leqq 1,\ j=1,\dots,J.\end{aligned}$$

Here $V(t)$ is not a proper variable of the problem, but it only serves as an abbreviation of $\int_0^t v(\tau)\,d\tau$.

This optimization problem is equivalent to the original problem (1.1) in the sense that an arbitrary change of parametrization of a fixed trajectory does not influence the value of the loss functional and the validity of the constraints.

Proposition 2.1. a) *If* (p,l,r,ζ) *satisfies the constraints of* (1.1), *then* (q,k,u,n,v) *satisfies the constraints of* (2.1) *for arbitrary* $v\in L^1_\infty[0,1]$ *with* $v(t)\geqq 0$ a.s. *and* $V(1)=\zeta$, *where* $q(t)=p(V(t))$, $k(t)=l(V(t))$, $n(t)=\|r(V(t))\|$ *on* $\{t|v(t)>0\}$ *and* $0\leqq n(t)\leqq v_s$ a.s. *on* $\{t|v(t)=0\}$, $u(t)=r(V(t))/n(t)$ *on* $\{t|v(t)>0,\ n(t)>0\}$ *and* $\|u(t)\|=1$ a.s. *else. Furthermore,* $L(p,l,r,\zeta)=\Lambda(q,k,u,n,v)$.

b) If on the other side (q,k,u,n,v) *satisfies the constraints of* (2.1), *then* (p,l,r,ζ) *satisfies the constraints of* (1.1), *where* $p(s)=q(V^{-1}(s))$, $l(s)=k(V^{-1}(s))$, $\zeta=V(1)$ *and* $r(s)=u(V^{-1}(s))\,n(V^{-1}(s))$. *Again we have* $L(p,l,r,\zeta)=\Lambda(q,k,u,n,v)$. *Here* $V^{-1}(s)=\{t|\,V(t)=s\}$ *as usual.*

Proof. As V is monotonically increasing and absolutely continuous, we have [Hewitt and Stromberg 1965, Corollary 20.5]

$$\begin{aligned}&\int_0^1 S(p(V(t)),l(V(t)),V(t))f(V(t))v(t)\,dt\\ &\quad=\int_0^{V(1)} S(p(s),l(s),s)f(s)\,ds\end{aligned}$$

and

$$a+\int_0^t r(V(s))\,v(s)\,ds=a+\int_0^{V(t)} r(s)\,ds=p(V(t))=q(t).$$

Analogously, we conclude $k(t)=\int_0^t n(s)\,v(s)\,ds$, and the rest of a) is obvious.

As $v(t)=0$ for almost all $t\geqq V^{-1}(V(1))$ and $s=V(V^{-1}(s))$, we have

$$\begin{aligned}&\int_0^\zeta S(p(t),l(t),t)\,f(t)\,dt\\&=\int_0^{V(1)} S(q(V^{-1}(s)),k(V^{-1}(s)),V(V^{-1}(s)))\,f(V(V^{-1}(s)))\,ds\\&=\int_0^1 S(q(t),k(t),V(t))\,f(V(t))\,v(t)\,dt\end{aligned}$$

and

$$\begin{aligned}a+\int_0^\tau r(t)\,dt&=a+\int_0^\tau u(V^{-1}(s))\,n(V^{-1}(s))\,ds\\&=a+\int_0^{V^{-1}(\tau)} u(t)\,n(t)\,v(t)\,dt=q(V^{-1}(\tau))=p(\tau).\end{aligned}$$

Analogously, $l(\tau)=\int_0^\tau \|r(t)\|\,dt$. Furthermore, we have $p(\zeta)=q(V^{-1}(V(1)))=q(1)$, as $q(t)=q(1)$ for all $t\geqq V^{-1}(V(1))$, and now the rest of b) is obvious. □

The following proposition provides necessary conditions on a solution of (2.1), which subsequently yield properties of solutions of (1.1) by means of Proposition 2.1.

Proposition 2.2. *Let the distribution P of the nonnegative random variable T have a density f, which is continuous in $[0,\infty)$. We assume that $S(x,l,t)$, $Z(x)$ and $G_j(x)$, $j=1,\dots,J$, are continuously differentiable in $x\in R^n$ and $l,t\in R$ and that additionally*

$$\begin{aligned}&\nabla Z(x)\neq 0 \qquad \text{on } \{x\,|\,Z(x)=0\},\\&\nabla G_j(x)\neq 0 \qquad \text{on } \{x\,|\,G_j(x)=0\} \qquad j=1,\dots,J.\end{aligned}$$

Then, if (q^0,k^0,u^0,n^0,v^0) is a solution of the optimization problem (2.1), *one of the following two sets of conditions is fulfilled:*

(i) There exist $\lambda^0\geqq 0$, $\lambda\geqq 0$ and Borel measures ν_j on $[0,1]$, whose support is contained in $N_j=\{t\,|\,G_j(q^0(t))=0\}$, $j=1,\dots,J$, *so that*

$$\tilde{\psi}(t)\,u_k^0(t)\,n^0(t)=\tilde{\psi}_k(t)\,u^0(t)\,n^0(t) \qquad k=1,\dots,n \tag{2.2}$$

for almost all t with $v^0(t)>0$,

$$(2.3)\qquad (\langle \tilde{\psi}(t), u^0(t)\rangle + \lambda^0 \tilde{\phi}(t))\, n^0(t) + \lambda^0 \tilde{\chi}(t) \begin{cases} =0 \\ \leqq 0 \end{cases}$$

for almost all t with $v^0(t) \begin{cases} >0 \\ =0, \end{cases}$

$$(2.4)\qquad (\langle \tilde{\psi}(t), u^0(t)\rangle + \lambda^0 \tilde{\phi}(t))\, v^0(t) \begin{cases} \geqq 0 & v_s = n^0(t) \\ =0 \quad \text{a.s. on } & \{t \mid v_s > n^0(t) > 0\} \\ \leqq 0 & 0 = n^0(t) \end{cases}$$

where, if we write $\xi^0(s)$ *abbreviating for the tuple* $(q^0(s), k^0(s), V(s))$, $0 \leqq s \leqq 1$,

$$\begin{aligned} \tilde{\psi}(t) = & -\lambda^0 \int_t^1 \nabla S(\xi^0(s))\, f(V^0(s))\, v^0(s)\, ds \\ & -\lambda^0 \nabla S(\xi^0(1))\, P\{T > V^0(1)\} \\ & -\lambda \nabla Z(q^0(1)) - \sum_{j=1}^{n} \int_t^1 \nabla G_j(q^0(s))\, dv_j(s), \end{aligned}$$

$$\begin{aligned} \tilde{\phi}(t) = & -\int_t^1 \frac{\partial}{\partial l} S(\xi^0(s))\, f(V^0(s))\, v^0(s)\, ds \\ & -\frac{\partial}{\partial l} S(\xi^0(1))\, P\{T > V^0(1)\}, \end{aligned}$$

$$\begin{aligned} \tilde{\chi}(t) = & \int_t^1 \langle \nabla S(\xi^0(s)), u^0(s)\rangle\, n^0(s)\, f(V^0(s))\, v^0(s)\, ds \\ & + \int_t^1 \frac{\partial}{\partial l} S(\xi^0(s))\, n^0(s)\, f(V^0(s))\, v^0(s)\, ds \\ & -\frac{\partial}{\partial t} S(\xi^0(1))\, P\{T > V^0(1)\}. \end{aligned}$$

At least one of the numbers λ^0, λ *or one of the measures* v_j, $j = 1, \ldots, J$, *does not vanish, and additionally we have* $\lambda = 0$ *or* $Z(q^0(1)) = 0$.

(ii) $\nabla S(\xi^0(t))\, f(V^0(t))\, v^0(t) = 0$ a.s.

$$\frac{\partial}{\partial l} S(\xi^0(t))\, f(V^0(t))\, v^0(t) = 0 \quad \text{a.s.}$$

$$\nabla S(\xi^0(1))\, P\{T > V^0(1)\} = 0,$$

$$\frac{\partial}{\partial l} S(\xi^0(1))\, P\{T > V^0(1)\} = 0,$$

$$\frac{\partial}{\partial t} S(\xi^0(1))\, P\{T > V^0(1)\} = 0.$$

Proof. The proof of this proposition is based on a rather straightforward application of the approach of Dubovitskij and Miljutin for analyzing extremum problems in general locally convex spaces as described e.g. by Girsanov (1972). This method allows to investigate the differential properties of the loss functional and the functionals and operators, which define the constraints, separately and then to combine the results in the so-called Euler-Lagange equation. From this equation we finally get necessary conditions for a local extremum by using the properties of our special optimization problem.

We consider the tuple (q, k, u, n, v) as element of the Banach space

$$E = C^n[0,1] \times C^1[0,1] \times L^n_\infty[0,1] \times L^1_\infty[0,1] \times L^1_\infty[0,1]$$

with the norm

$$\|(q,k,u,n,v)\|_E = \|q\|_C + \|k\|_C + \|u\|_\infty + \|n\|_\infty + \|v\|_\infty.$$

In the following we simplify our notation by writing $e, e^0, \dots$ instead of the tuples (q,k,u,n,v), $(q^0,k^0,u^0,n^0,v^0), \dots$ occasionally, where we especially assume that e^0 is a solution of the optimization problem (2.1).

a) Investigation of the loss functional Λ.

In the following we write $\xi(t)$, $\xi^0(t)$ abbreviatingly for the tuples $(q(t), k(t), V(t))$, $(q^0(t), k^0(t), V^0(t))$, where $0 \leqq t \leqq 1$. Now we show that the functional $\Lambda(e)$ is Fréchet differentiable in e^0, where the Fréchet derivative is given by

$$\begin{aligned}\Lambda_0'(e) &= \int_0^1 \langle \nabla S(\xi^0(t)), q(t) - u^0(t)\, n^0(t)\, V(t)\rangle f(V^0(t))\, v^0(t)\, dt \\ &+ \int_0^1 \frac{\partial}{\partial l} S(\xi^0(t)) \{k(t) - n^0(t)\, V(t)\} f(V^0(t))\, v^0(t)\, dt \\ &+ \left(\langle \nabla S(\xi^0(1)), q(1)\rangle + \frac{\partial}{\partial l} S(\xi^0(1))\, k(1)\right) P\{T > V^0(1)\} \\ &+ \frac{\partial}{\partial t} S(\xi^0(1))\, V(1)\, P\{T > V^0(1)\}.\end{aligned}$$

Firstly, we consider the case $V^0(1) > 0$. As S and P are both continuously differentiable, we get at once the Fréchet derivative of the second summand of Λ:

$$\begin{aligned}&S(\xi^0(1) + \xi(1))\, P\{T > V^0(1) + V(1)\} - S(\xi^0(1))\, P\{T > V^0(1)\} \\ &\quad = \left(\langle \nabla S(\xi^0(1)), q(1)\rangle + \frac{\partial}{\partial l} S(\xi^0(1))\, k(1) + \frac{\partial}{\partial t} S(\xi^0(1))\, V(1)\right) P\{T > V^0(1)\} \\ &\qquad - S(\xi^0(1))\, f(V^0(1))\, V(1) + o(\|e\|).\end{aligned}$$

Now we set $f(t) = 0$ for $t < 0$. As we have by continuity of f for almost all t with $v^0(t) > 0$:

$$f(V^0(t) + V(t)) \to f(V^0(t)) \quad \text{as } \|v\|_\infty \to 0$$

we conclude from the mean value theorem and from Lebesgue's dominated convergence theorem:

$$\int_0^1 (S(\xi^0(t)+\xi(t))-S(\xi^0(t)))\, f(V^0(t)+V(t))\,\{v^0(t)+v(t)\}\,dt$$

$$=\int_0^1 \left(\langle \nabla S(\xi^0(t)), q(t)\rangle + \frac{\partial}{\partial l} S(\xi^0(t))\, k(t)\right) f(V^0(t))\, v^0(t)\,dt$$

$$+\int_0^1 \frac{\partial}{\partial t} S(\xi^0(t))\, V(t)\, f(V^0(t))\, v^0(t)\,dt + o(\|e\|).$$

As P and S are continuously differentiable and q^0, k^0, V^0 and V are absolutely continuous, $P(V^0(t)+V(t))$, $P(V^0(t))$ and $S(\xi^0(t))=S(q^0(t), k^0(t), V^0(t))$ are absolutely continuous as functions of t too, and their densities are

$$\{v^0(t)+v(t)\}\, f(V^0(t)+V(t)),\; v^0(t)\, f(V^0(t))$$

and

$$\left\{\langle \nabla S(\xi^0(t)), u^0(t)\, n^0(t)\rangle + \frac{\partial}{\partial l} S(\xi^0(t))\, n^0(t) + \frac{\partial}{\partial t} S(\xi^0(t))\right\} v^0(t).$$

For the last term we write in abbreviated form $\sigma(t)$. Bearing in mind that $V^0(0) = V(0) = 0$, it follows by means of integration by parts [Hewitt and Stromberg, 1965, Corollary 18.20]:

$$\int_0^1 S(\xi^0(t))\,\{(v^0(t)+v(t))\, f(V^0(t)+V(t)) - v^0(t)\, f(V^0(t))\}\,dt$$

$$= -\int_0^1 \sigma(t)\,\{P(V^0(t)+V(t)) - P(V^0(t))\}\,dt$$

$$+ S(\xi^0(1))\,\{P(V^0(1)+V(1)) - P(V^0(1))\}$$

$$= -\int_0^1 \sigma(t)\, f(V^0(t))\, V(t)\,dt + S(\xi^0(1))\, f(V^0(1))\, V(1) + o(\|v\|_\infty)$$

where the last part follows by Lebesgue's dominated convergence theorem, if we take into account that for all t with $V^0(t)>0$ we have $P(V^0(t)+V(t)) - P(V^0(t)) = f(V^0(t))\, V(t) + o(\|v\|_\infty)$.

If on the other hand $V^0(1)=0$, i.e. $v^0\equiv 0$, $q^0\equiv a$ and $k^0\equiv 0$, then we have

$$\Lambda(e^0+e) - \Lambda(e) = \int_0^1 S(a+q(t), k(t), V(t))\, f(V(t))\, v(t)\,dt$$

$$+ S(a+q(1), k(1), V(1))\, P\{T > V(1)\} - S(a,0,0)$$

$$= \int_0^1 \left\{\langle \nabla S(a,0,0), q(1)\rangle + \frac{\partial}{\partial l} S(a,0,0)\, k(t)\right\} f(V(t))\, v(t)\,dt$$

$$+\int_0^1 \frac{\partial}{\partial t} S(a,0,0)\, V(t)\, f(V(t))\, v(t)\,dt$$

$$+ S(a,0,0)\int_0^1 f(V(t))\, v(t)\,dt - S(a,0,0)\, P(V(1))$$

$$+\left\{\langle \nabla S(a,0,0), q(1)\rangle + \frac{\partial}{\partial l} S(a,0,0)\, k(1)\right\} P(T > V(1))\}$$

$$+\frac{\partial}{\partial t} S(a,0,0)\, V(1)\, P\{T > V(1)\} + o(\|e\|).$$

As $\int_0^1 f(V(t))\, v(t)\, dt = P(V(1))$ [Hewitt and Stromberg, 1965, Corollary 20.5], and as $P\{T > V(1)\} \to 1$ as $\|v\|_\infty \to 0$, we finally get $\Lambda'_0(e) = \langle \nabla S(a,0,0), q(1)\rangle + \frac{\partial}{\partial l} S(a,0,0)\, k(1) + \frac{\partial}{\partial t} S(a,0,0)\, V(1)$ as Fréchet derivative of Λ in e^0 in the case $V^0(1) = 0$. So the representation of Λ'_0 given above is valid in this case too.

Now we know from Girsanov (1972, Theorem 7.5) that the loss functional is regularly decreasing in e^0, and therefore the cone of directions of decrease for the functional Λ at the point e^0, i.e. the cone of directions e, from which Λ decreases in the right sense starting from e^0 [Girsanov, 1972, Chap. 6], consists of the directions e, for which $\Lambda'_0(e) < 0$.

If Λ'_0 does not vanish identically, the cone K_0 of directions of decrease for Λ at e^0 is not empty and the corresponding dual cone K_0^+ of the continuous linear functionals, which are nonnegative on K_0, is given by [Girsanov, 1972, Theorem 10.2]:

$$K_0^+ = \{-\lambda^0 \Lambda'_0 \mid \lambda^0 \geqq 0\}.$$

b) The constraint $Q_1 = \{e \in E \mid v(t) \geqq 0 \text{ a.s.}\}$.

Q_1 is a closed, convex set with non-empty interior. So, the dual cone K_1^+ corresponding to the cone of feasible directions for Q_1 at e^0, i.e. the cone of directions in which, starting from e^0, one does not leave Q_1 too fast in the appropriate sense [Girsanov, 1972, Chap. 6], consists of the support functionals of the convex set Q_1 at e^0, and so we have

$$K_1^+ = \{\Phi_1 \in E' \mid \Phi_1(e) \geqq \Phi_1(e^0) \text{ for all } e \in Q_1\}.$$

As only the v-coordinate of the tuple $e = (q, k, u, n, v)$ is relevant for the fact that e belongs to Q_1, such a support functional Φ_1 does not explicitly depend on the other coordinates of e.

c) The constraint $Q_2 = \{e \in E \mid 0 \leqq n(t) \leqq v_s \text{ a.s.}\}$.

In exactly the same way as under b) we get as dual cone corresponding to the cone of feasible directions for Q_2 at e^0:

$$K_2^+ = \{\phi_2 \in E' \mid \Phi_2(e) \geqq \Phi_2(e^0) \text{ for all } e \in Q_2\}.$$

The functionals $\Phi_2 \in K_2^+$ depend only on the n-coordinate of e.

d) The constraint $Q_3 = \{e \in E \mid Z(q(1)) \leqq 0\}$.

As Z is continuously differentiable, we have

$$Z(q^0(1) + q(1)) = Z(q^0(1)) + \langle \nabla Z(q^0(1)), q(1)\rangle + o(\|q\|_C).$$

So the functional $e \mapsto Z(q(1))$ is Fréchet differentiable in e^0 with derivative $e \mapsto \langle \nabla Z(q^0(1)), q(1) \rangle$. If $Z(q^0(1))=0$, this functional does not vanish identically due to our assumptions on ∇Z, so Corollary 8.1 of Girsanov (1972) provides the cone of feasible directions for Q_3 in e^0: $K_3=\{e \in E | \langle \nabla Z(q^0(1)), q(1) \rangle < 0\}$. From Theorem 10.2 of Girsanov (1972) we finally get the dual cone

$$K_3^+ = \{-\lambda \langle \nabla Z(q^0(1)), q(1) \rangle | \lambda \geqq 0\}.$$

If on the other side $Z(q^0(1))<0$, then e^0 belongs to the interior of Q_3 so that all directions are feasible, and the dual cone contains only the zero functional.

e) The constraints $Q_{4,j}=\{e \in E | G_j(q(t)) \leqq 0 \text{ for all } 0 \leqq t \leqq 1\}$.

Let $N_j=\{t | G_j(q^0(t)=0\}$ be the set of parameter values t, for which the optimal path moves on the boundary of the admissible region $\{x | G_j(x) \leqq 0\} \subset R^n$, $j=1,\ldots,J$. As in the proof of Theorem 14.1 of Girsanov (1972) we get as cone of feasible directions for $Q_{4,j}$ in e:

$$K_{4,j}=\{e \in E | \sup_{t \in N_j} \langle \nabla G_j(q^0(t)), q(t) \rangle < 0\},$$

if the optimal path $q^0(t)$, $0 \leqq t \leqq 1$, comes into contact with the boundary of $\{x | G_j(x) \leqq 0\}$, and $K_{4,j}=E$, if $q^0(t)$, $0 \leqq t \leqq 1$, is contained in the interior of the admissible region. Then the dual cone is

$$K_{4,j}^+=\left\{\Psi_j \in E' | \Psi_j(e)=-\int_0^1 \langle \nabla G_j(q^0(t)), q(t) \rangle \, d\nu_j(t), \text{ where } \nu_j \text{ is a Borel measure on } [0,1], \text{ whose support is contained in } N_j\right\}.$$

f) The constraints with empty interior.

The remaining constraints can be written in the form $Q_5=\{e \in E | A(e)=0\}$ where A is the operator from E into $C^n[0,1] \times C^1[0,1] \times L^1_\infty[0,1]$ given by

$$A(e)=\left(q(t)-a-\int_0^t u(s)\,n(s)\,v(s)\,ds,\ k(t)-\int_0^t n(s)\,v(s)\,ds,\ \|u(s)\|-1\right).$$

On the basis of a theorem of Lyusternik [Girsanov, 1972, Theorem 9.1] it is sufficient to investigate the Fréchet differentiability of the operator A in a neighbourhood of e^0, if we want to determine the cone of tangent directions of Q_5 at e^0.

1. The operator A_1: $e \mapsto q(t)-a-\int_0^t u(s)\,n(s)\,v(s)\,ds$ from E into $C^n[0,1]$ is Fréchet differentiable in an open neighbourhood of e^0, and the derivatives are continuous in this neighbourhood. The derivative in e^0 is the linear operator

$$A'_{1,0}\colon e \mapsto q(t)-\int_0^t u(s)\,n^0(s)\,v^0(s)\,ds-\int_0^t u^0(s)\,n(s)\,v^0(s)\,ds - \int_0^t u^0(s)\,n^0(s)\,v(s)\,ds.$$

This result follows from an obvious modification of Example 9.2 of Girsanov (1972) for the control (u, n, v) and with $\varphi(x, (u, n, v), t) = u\,n\,v$.

2. Analogously, $A_2\colon e \mapsto k(t) - \int_0^t n(s)\,v(s)\,ds$ is Fréchet differentiable in an open neighbourhood of e^0 with derivatives, which are continuous in this neighbourhood, and the derivative in e^0 is given by

$$A'_{2,0}\colon e \mapsto k(t) - \int_0^t n(s)\,v^0(s)\,ds - \int_0^t n^0(s)\,v(s)\,ds.$$

3. As $\|u^0(t)\| = 1$ a.s., $\|\tilde{u}(t)\|$ is a.s. bounded away from 0 for all $\tilde{u}$ in an appropriate neighbourhood of u^0. By means of a Taylor expansion we get

$$\|\tilde{u}(t) + u(t)\| = \|\tilde{u}(t)\| + \langle \tilde{u}(t), u(t)\rangle / \|\tilde{u}(t)\| + o(\|u(t)\|) \quad \text{a.s.,}$$

so that the operator $e \mapsto \|u(t)\| - 1$ is Fréchet differentiable in a neighbourhood of e^0 with continuous derivative in $\tilde{e}$:

$$e \to \langle \tilde{u}(t), u(t)\rangle / \|\tilde{u}(t)\|.$$

Especially, the Fréchet derivative in e^0 is $e \mapsto \langle u^0(t), u(t)\rangle$, as $\|u^0(t)\| = 1$ a.s.

4. So finally we get that A itself has in e^0 the Fréchet derivative

$$A'_0\colon\; e \mapsto (A'_{1,0}(e), A'_{2,0}(e), \langle u^0(t), u(t)\rangle)$$

which obviously is a surjective, continuous linear functional from E onto $C^n[0,1] \times C^1[0,1] \times L^1_\infty[0,1]$, as especially $\|u^0(t)\| = 1$ a.s. So the cone of tangent directions of Q_5 at e^0 is given by $K_5 = \{e \mid A'_0(e) = 0\}$.

5. As K_5 is a subspace of E, the dual cone consists of the continuous linear functionals vanishing on K_5:

$$K_5^+ = \{\Phi_5 \in E' \mid \Phi_5(e) = 0 \text{ for all } e \in K_5\}$$

h) The Euler-Lagrange-equation in the case $\Lambda'_0 \neq 0$.

If $\Lambda'_0 \neq 0$, the theorem of Dubovitskij and Milyutin [Girsanov, 1972, Theorem 6.1] yields the existence of functionals $\Phi_i \in K_i^+$, $i = 0, 1, 2, 3, 5$, and $\Psi_j \in K^+_{4,j}$, $j = 1, \ldots, J$, where at least one of them does not vanish identically, so that the Euler-Lagrange-equation

$$\Phi_0 + \Phi_1 + \Phi_2 + \Phi_3 + \sum_{j=1}^{J} \Psi_j + \Phi_5 \equiv 0$$

is valid.

Due to the special structure of some of the dual cones this translates into the existence of $\lambda^0 \geqq 0$, $\lambda \geqq 0$ and Borel measures ν_j, whose support is contained in N_j, $j = 1, \ldots, J$, so that

$$\begin{aligned}\lambda^0 \Lambda'_0(e) + \lambda \langle \nabla Z(q^0(1)), q(1)\rangle + \sum_{j=1}^{J} \int_0^1 \langle \nabla G_j(q^0(s)), q(s)\rangle\, d\nu_j(s) \\ = \Phi_1(e) + \Phi_2(e) + \Phi_5(e) \qquad \text{for all } e \in E.\end{aligned}$$

Especially, $\lambda=0$ or $Z(q^0(1))=0$.

Now we consider especially $e=(q,k,u,n,v)\in Q_1\cap Q_2\cap K_5$, so that $\Phi_5(e)=0$, $\Phi_1(e)\geqq\Phi_1(e^0)$ and $\Phi_2(e)\geqq\Phi_2(e^0)$. Then k and q are absolutely continuous with densities $n(t)v^0(t)+n^0(t)v(t)$ and $u(t)n^0(t)v^0(t)+u^0(t)n(t)v^0(t)+u^0(t)n^0(t)v(t)$, and from Fubini's theorem follows

$$\begin{aligned}\lambda^0\int_0^1 &\langle \nabla S(\xi^0(s)), q(s)\rangle f(V^0(s))\,v^0(s)\,ds\\ &+\lambda^0\langle \nabla S(\xi^0(1)), q(1)\rangle P\{T>V^0(1)\}\\ &+\lambda\langle \nabla Z(q^0(1)), q(1)\rangle+\sum_{j=1}^{J}\int_0^1\langle \nabla G_j(q^0(s)), q(s)\rangle\, dv_j(s)\\ =&-\int_0^1\langle\tilde\psi(t), u(t)n^0(t)v^0(t)+u^0(t)n(t)v^0(t)+u^0(t)n^0(t)v(t)\rangle\,dt\end{aligned}$$

where

$$\begin{aligned}\tilde\psi(t)=&-\lambda^0\int_t^1 \nabla S(\xi^0(s))f(V^0(s))\,v^0(s)\,ds-\lambda^0\,\nabla S(\xi^0(1))\,P\{T>V^0(1)\}\\ &-\lambda\,\nabla Z(q^0(1))-\sum_{j=1}^{J}\int_t^1 \nabla G_j(q^0(s))\,dv_j(s)\end{aligned}$$

and analogously

$$\begin{aligned}\lambda^0\int_0^1\frac{\partial}{\partial l}S(\xi^0(s))\,k(s)f(V^0(s))\,v^0(s)\,ds+\lambda^0\frac{\partial}{\partial l}S(\xi^0(1))\,k(1)\,P\{T>V^0(1)\}\\ =-\lambda^0\int_0^1\tilde\phi(t)\,\{n(t)v^0(t)+n^0(t)v(t)\}\,dt\end{aligned}$$

where

$$\tilde\phi(t)=-\int_t^1\frac{\partial}{\partial l}S(\xi^0(s))f(V^0(s))\,v^0(s)\,ds-\frac{\partial}{\partial l}S(\xi^0(1))\,P\{T>V^0(1)\}.$$

Finally we define

$$\begin{aligned}\tilde\chi(t)=\int_t^1&\left\{\langle \nabla S(\xi^0(s)), u^0(s)\rangle+\frac{\partial}{\partial l}S(\xi^0(s))\right\}n^0(s)f(V^0(s))\,v^0(s)\,ds\\ &-\frac{\partial}{\partial t}S(\xi^0(1))\,P\{T>V^0(1)\},\end{aligned}$$

and, employing the fact that $e\in K_5$ and consequently $\langle u^0(t), u(t)\rangle=0$ a.s., we finally get from the Euler-Lagrange-equation the following three relations, if we take into account that Φ_1 depends only on the v-coordinate of e and Φ_2 only on the n-coordinate:

$$\int_0^1 \langle \tilde{\psi}(t), u(t) \rangle n^0(t) v^0(t) \, dt = 0$$

for all $u \in L^n_\infty[0,1]$ which are a.e. orthogonal to u^0,

$$\int_0^1 \{\langle \tilde{\psi}(t), u^0(t) n^0(t) \rangle + \lambda^0 \tilde{\phi}(t) n^0(t) + \lambda^0 \tilde{\chi}(t)\} \{v^0(t) - v(t)\} \, dt$$
$$= \Phi_1(e - e^0) \geqq 0 \quad \text{for all } v \in L^1_\infty[0,1] \quad \text{with } v(t) \geqq 0 \quad \text{a.s.,}$$

$$\int_0^1 \{\langle \tilde{\psi}(t), u^0(t) \rangle + \lambda^0 \tilde{\phi}(t)\} v^0(t) \{n^0(t) - n(t)\} \, dt$$
$$= \Phi_2(e - e^0) \geqq 0 \quad \text{for all } n \in L^1_\infty[0,1] \quad \text{with } 0 \leqq n(t) \leqq v_s \quad \text{a.s.}$$

The second and the third condition imply at once the relations (2.3) and (2.4) of the proposition. From the first condition we get (2.2) by means of the following consideration. For $\delta > 0$ we set $B^\delta_k = \{t \mid |u^0_k(t)| \geqq \delta\}$. Then, for arbitrary $u_j \in L^1_\infty[0,1]$, $j \neq k$, whose supports are contained in B^δ_k, we can define a corresponding function $u \in L^n_\infty[0,1]$, which is a.s. orthogonal to $u^0(t)$, by choosing

$$u_k(t) = -\sum_{j \neq k} u_j(t) u^0_j(t)/u^0_k(t).$$

Consequently we have

$$0 = \int_0^1 \langle \tilde{\psi}(t), u(t) \rangle n^0(t) v^0(t) \, dt$$
$$= \int_0^1 n^0(t) v^0(t) \sum_{j \neq k} \{\tilde{\psi}_j(t) - \tilde{\psi}_k(t) u^0_j(t)/u^0_k(t)\} u_j(t) \, dt,$$

and this implies

$$\tilde{\psi}(t) u^0_k(t) n^0(t) v^0(t) = \tilde{\psi}_k(t) u^0(t) n^0(t) v^0(t) \qquad \text{for almost all } t \in B^\delta_k.$$

Now, as $\|u^0(t)\| = 1$ a.s., the B^δ_k, $k = 1, \ldots, n$, cover $[0,1]$ a.s. for δ sufficiently small, so that $u^0(t)$ is parallel to $\tilde{\psi}(t)$ for almost all t with $\tilde{\psi}(t) n^0(t) v^0(t) \neq 0$.

i) The case $\Lambda'_0 \equiv 0$.

If Λ'_0 vanishes identically, then the Euler-Lagrange-equation yields no information about the optimal e^0, as $K^+_0 = E$. But from the condition $\Lambda'_0(q, k, u, n, v) = 0$ for arbitrary q, k, v, we get from the explicit expression for the Fréchet derivative of the loss functional, given under a), at once the set (ii) of relations of Proposition 2.2. □

The proof of the Pontryagin maximum principle, given by Dubovitskij and Milyutin, makes use of a special parameter transformation with density $v^0(t)$, whose support is a set of Cantor type with positive Lebesgue measure. Then, in terms of our special optimization problem, one would use the fact, that $u^0(t)$ and $n^0(t)$ can be chosen arbitrarily on the open intervals, where $v^0(t)$ is vanishing, e.g. in such a way that $u^0(t)$ takes values in a dense set of $\{x|\,\|x\|=1\}$ separately in each of these intervals. On the support of v^0 one would choose $u^0(t)\,n^0(t)=r^0(V^0(t))$. These special u^0 and n^0 determine a local minimum of the loss functional of the modified optimization problem (2.1), and by means of Propositions 2.1 and 2.2 one would get necessary conditions on the optimality of the control r^0 of the original problem (1.1).

We prefer to employ a somewhat simpler parameter transformation to carry over the results of Proposition 2.2 to our optimization problem (1.1). A direct transference of the conditions of Proposition 2.2 to the optimization problem (1.1) yields at once the condition that r^0 has to be parallel to a function derived from $\tilde{\psi}$. To determine the correct sign we take advantage of the possibility to consider different parameter transformations v^0 corresponding to the same local minimum of the original problem. Essentially, we choose v_τ^0 in such a way that the transformed path q_τ^0 stays for a whole interval of time in the point $p^0(\tau)$ of the original path. Above all, this technique is important for analyzing the related optimization problem (1.2), where the norm of velocity is fixed, and the original path p^0 must not stay in a fixed point for some time, before proceeding further to the target region. Here, we introduce the necessary additional variability by allowing the transformed path to stop for a while at some points.

The following theorem contains the result that locally optimal solutions of (1.1) in general have to correspond to "bang-bang" controls, i.e. the norm of velocity is either 0 or maximal. Then the direction of velocity is given by (2.5). As is obvious from the proof, the condition $\frac{\partial}{\partial l} S(x,l,t) \geqq 0$ is not essential, but only provides a somewhat simpler form of the conditions of the theorem. On the other hand, it is rather natural to consider only cost functions S, which increase with the relative path length l. The set (ii) of possible necessary conditions on locally optimal paths describes extreme situations, e.g. the case that T is a.s. large, so that the target set can be reached undisturbedly a.s. Then, if for instance S does not depend explicitly on l and t, every path is locally optimal, which is steering towards a local spatial minimum of the cost function.

Theorem 2.1. *Let the distribution P of the nonnegative random variable T have a density f, which is continuous in $[0,\infty)$. We assume that $S(x,l,t)$, $Z(x)$ and $G_j(x)$, $j=1,\ldots,J$, are continuously differentiable in $x\in R^n$ and $l,t\in R$ and that additionally*

$$\frac{\partial}{\partial l} S(x,l,t) \geqq 0 \quad \textit{for all } x\in R^n,\ l\in R,\ t\geqq 0,$$

$$\nabla Z(x) \neq 0 \quad \textit{on } \{x\,|\,Z(x)=0\},$$

$$\nabla G_j(x) \neq 0 \quad \textit{on } \{x\,|\,G_j(x)=0\} \quad j=1,\ldots,J.$$

Then, if (p^0, l^0, r^0, ζ^0) is a solution of the optimization problem (1.1), one of the following two sets of conditions is fulfilled:

(i) There exist $\lambda^0 \geqq 0$, $\lambda \geqq 0$ *and Borel measures* μ_j *on* $[0, \zeta^0]$, *whose support is contained in* $M_j = \{t \mid G_j(p^0(t)) = 0\}$, $j = 1, \ldots, J$, *so that*

$$(2.5)\qquad r^0(t) = \begin{cases} v_s \psi(t)/\|\psi(t)\| \\ 0 \end{cases} \quad \text{a.s. } on \left\{ t \leqq \zeta^0 \mid \|\psi(t)\| + \lambda^0 \phi(t) \begin{cases} >0 \\ <0 \end{cases} \right\}$$

where, if we write abbreviatingly $\eta^0(t)$ *for the tuple* $(p^0(t), l^0(t), t)$, $0 \leqq t \leqq \zeta^0$,

$$\psi(t) = -\lambda^0 \int_t^{\zeta^0} \nabla S(\eta^0(s))\, dP(s) - \lambda^0\, \nabla S(\eta^0(\zeta^0))\, P\{T > \zeta^0\}$$
$$- \lambda \nabla Z(p^0(\zeta^0)) - \sum_{j=1}^{J} \int_t^{\zeta^0} \nabla G_j(p^0(s))\, d\mu_j(s),$$
$$\phi(t) = -\int_t^{\zeta^0} \frac{\partial}{\partial l} S(\eta^0(s))\, dP(s) - \frac{\partial}{\partial l} S(\eta^0(\zeta^0))\, P\{T > \zeta^0\}$$

and additionally we have

$$(2.6)\qquad \{\|\psi(t)\| + \lambda^0 \phi(t)\}^+ v_s = -\lambda^0 \chi(t) \text{ a.s.}$$

where

$$\chi(t) = \int_t^{\zeta^0} \left\{ \langle \nabla S(\eta^0(s)), r^0(s) \rangle + \frac{\partial}{\partial l} S(\eta^0(s))\, \|r^0(s)\| \right\} dP(s)$$
$$- \frac{\partial}{\partial t} S(\eta^0(\zeta^0))\, P\{T > \zeta^0\}.$$

At least one of the numbers λ^0, λ *or one of the measures* μ_j, $j = 1, \ldots, J$, *does not vanish, and additionally we have* $\lambda = 0$ *or* $Z(p^0(\zeta^0)) = 0$.

(ii) $\nabla S(p^0(t), l^0(t), t) f(t) = 0$ *for almost all* $t \leqq \zeta^0$,

$\frac{\partial}{\partial l} S(p^0(t), l^0(t), t) f(t) = 0$ *for almost all* $t \leqq \zeta^0$,

$$\nabla S(p^0(\zeta^0), l^0(\zeta^0), \zeta^0)\, P\{T > \zeta^0\} = 0,$$

$$\frac{\partial}{\partial l} S(p^0(\zeta^0), l^0(\zeta^0), \zeta^0)\, P\{T > \zeta^0\} = 0,$$

$$\frac{\partial}{\partial t} S(p^0(\zeta^0), l^0(\zeta^0), \zeta^0)\, P\{T > \zeta^0\} = 0.$$

Proof. For arbitrary $v^0 \in L^1_\infty[0, 1]$ with $v^0(t) \geqq 0$ a.s. and $V^0(1) = \int_0^1 v^0(s)\, ds = \zeta^0$, the tuple $(q^0, k^0, u^0, n^0, v^0)$ is due to Proposition 2.1 a solution of the optimization problem (2.1), if we set $q^0(t) = p^0(V^0(t))$, $k^0(t) = l^0(V^0(t))$, $n^0(t) = \|r^0(V^0(t))\|$ on $\{t \mid v^0(t) \neq 0\}$ and $u^0(t) = r^0(V^0(t))/\|r^0(V^0(t))\|$ on $\{t \mid v^0(t) \neq 0, n^0(t) \neq 0\}$. Otherwise we can choose n^0 and u^0 arbitrary, as far as the constraints $0 \leqq n^0(t) \leqq v_s$ a.s. and $\|u^0(t)\| = 1$ a.s. are fulfilled. Then, $(q^0, k^0, u^0, n^0, v^0)$ has to satisfy the conditions of Proposition 2.2.

Firstly we consider the case that the set *(ii)* of conditions of Proposition 2.2 is fulfilled. By means of substitution [Hewitt and Stromberg, 1965, Corollary 20.5] we conclude that

$$\int_s^t \nabla S(q^0(\tau), k^0(\tau), V^0(\tau)) f(V^0(\tau)) v^0(\tau)\, d\tau$$

$$= \int_{V^0(s)}^{V^0(t)} \nabla S(p^0(\tau), l^0(\tau), \tau) f(\tau)\, d\tau \qquad \text{for all } 0 \leqq s < t \leqq 1,$$

and so $\nabla S(p^0(t), l^0(t), t) f(t) = 0$ a.s. follows from

$\nabla S(q^0(t), k^0(t), V^0(t)) f(V^0(t)) v^0(t) = 0$ a.s., as V^0 maps $[0,1]$ onto $[0, \zeta^0]$. The other conditions of case *(ii)* of the theorem follows analogously from the corresponding case *(ii)* of Proposition 2.2. In the following we assume that $(q^0, k^0, u^0, n^0, v^0)$ are fulfilling the set *(i)* of conditions of Proposition 2.2. We define:

$$\psi(t) = \tilde{\psi}(\{V^0\}^{-1}(t)), \quad \phi(t) = \tilde{\phi}(\{V^0\}^{-1}(t)) \quad \text{and} \quad \chi(t) = \tilde{\chi}(\{V^0\}^{-1}(t)).$$

As $V^0(\{V^0\}^{-1}(t)) = t$, we get by substitution

$$\psi(t) = -\lambda^0 \int_t^{\zeta^0} \nabla S(\eta^0(s))\, dP - \lambda^0 \nabla S(\eta^0(\zeta^0)) P\{T > \zeta^0\} - \lambda \nabla Z(p^0(\zeta^0))$$

$$- \sum_{j=1}^{J} \int_t^{\zeta^0} \nabla G_j(p^0(s))\, d\mu_j(s),$$

$$\phi(t) = -\int_t^{\zeta^0} \frac{\partial}{\partial l} S(\eta^0(s))\, dP - \frac{\partial}{\partial l} S(\eta^0(\zeta^0)) P\{T > \zeta^0\},$$

$$\chi(t) = \int_t^{\zeta^0} \left\{ \langle \nabla S(\eta^0(s)), r^0(s) \rangle + \frac{\partial}{\partial l} S(\eta^0(s)) \, \| r^0(s) \| \right\} dP$$

$$- \frac{\partial}{\partial t} S(\eta^0(\zeta^0)) P\{T > \zeta^0\}$$

where $\mu_j = v_j \circ \{V^0\}^{-1}$ is the image of the measure v_j under V^0 [Hewitt and Stromberg, 1965, 12.45]. As V^0 maps $[0,1]$ onto $[0, \zeta^0]$, $M_j = V^0(N_j)$, and from this follows that the support of μ_j is contained in M_j.

Now from (2.2) and (2.3) we get

(2.7) $\quad r^0(t)\, \psi_k(t) = \psi(t)\, r_k^0(t) \qquad k = 1, \ldots, n, \qquad \text{for almost all } t \leqq \zeta^0,$

(2.8) $\quad \langle \psi(t), r^0(t) \rangle + \lambda^0 \phi(t) \, \| r^0(t) \| + \lambda^0 \chi(t) = 0$ a.s.

as by absolute continuity and monotonicity of V^0 we have for almost all images $s = V^0(t)$: $v^0(t) > 0$.

In the same way we conclude by multiplying both sides of (2.4) with $n^0(t)$:

(2.9) $\quad \langle \psi(t), r^0(t) \rangle + \lambda^0 \phi(t) \, \| r^0(t) \| \begin{cases} \geqq 0 \\ = 0 \end{cases}$ a.s. on $\left\{ t \,\middle|\, \begin{matrix} \| r^0(t) \| = v_s \\ 0 < \| r^0(t) \| < v_s \end{matrix} \right\}.$

If $r^0(t)=0$, we have $n^0(\{V^0\}^{-1}(t))=0$ too, and by Proposition 2.1 we can choose $u^0(\{V^0\}^{-1}(t))$ arbitrary from $\{x \mid \|x\|=1\}$. If we specify $u^0(\{V^0\}^{-1}(t)) =\psi(\{V^0\}^{-1}(t))/\|\psi(\{V^0\}^{-1}(t))\|$, if the denominator does not vanish, then we get from the last line of (2.4)

$$\text{(2.10)} \quad \|\psi(t)\|+\lambda^0\phi(t)\leqq 0 \quad \text{a.s. on } \{t \mid r^0(t)=0\}.$$

In Proposition 2.2 *(i)* λ^0, λ and μ_j, $j=1,\dots,J$, depend on the special choice of v^0, so that it is possible that $\tilde{\psi}(t)$ and the function $\psi(t)$, derived from it, depend on the specification of v^0 too. On the other hand the expressions for $\phi(t)$ and $\chi(t)$ contain only the solution (p^0, l^0, r^0, ζ^0) of the original problem (1.1), and the Lagrange multipliers λ^0, λ and the measures μ_j do not appear, so that $\phi(t)$ and $\chi(t)$ are the same for all choices of v^0.

Now we consider the following special parameter transformation

$$v_\tau^0(t)=\begin{cases}\tau/t_0 & 0\leqq t\leqq t_0\\ 0 & t_0<t<t_1, \text{ where } 0<t_0<t_1<1.\\ (\zeta^0-\tau)/(1-t_1) & t_1\leqq t\leqq 1\end{cases}$$

So we go along the path p^0, which is optimal for the problem (1.1), with velocity increased by a constant factor, until we reach the state $p^0(\tau)$, where we stay for the time t_1-t_0. Then we proceed to the target region with velocity increased by a constant factor again. This v_τ^0 satisfies the conditions of Proposition 2.1, especially $V_\tau^0(1)=\int_0^1 v_\tau^0(s)\,ds=\zeta^0$, and we have $V_\tau^0(t)=\tau$ for all $t\in[t_0,t_1]$.

There is a solution to the optimization problem (2.1) associated to this specification of v^0. In the following we denote by λ_τ^0, λ_τ, $v_{\tau j}$, $j=1,\dots,J$, $\tilde{\psi}_\tau(t)$, $\psi_\tau(t)$, $\tilde{\phi}_\tau(t)$ and $\tilde{\chi}_\tau(t)$ the numbers, measures and functions corresponding to this solution on the basis of Proposition 2.2 *(i)*. We remark that specification of v^0 alone is not enough to determine the solution (2.1), as we have yet some freedom of choosing u^0 and n^0. Especially, as already mentioned, we define u^0 in such a way that for all t with $r^0(t)=0$ we have $u^0(\{V_\tau^0\}^{-1}(t))\,\|\psi_\tau(\{V_\tau^0\}^{-1}(t))\| =\psi_\tau(\{V_\tau^0\}^{-1}(t))$.

Now we direct our attention to the interval (t_0,t_1), where V_τ^0 vanishes and where we can yet choose u^0 and n^0 freely. But firstly we remark that from $V^0(t)=\tau$ for all t in this interval follows

$$q^0(t)=p^0(\tau),\quad \tilde{\psi}_\tau(t)=\psi_\tau(\tau)-\sum_{j=1}^J \nabla G_j(p^0(\tau))\int_{t_0}^t dv_{\tau j},\quad \tilde{\phi}_\tau(t)=\phi(\tau)$$

$$\text{and } \tilde{\chi}_\tau(t)=\chi(\tau) \text{ for } t_0<t<t_1.$$

So by (2.3) we have on (t_0,t_1)

$$\text{(2.11)} \quad \left\{\langle\psi_\tau(\tau),u^0(t)\rangle-\sum_{j=1}^J\langle\nabla G_j(p^0(\tau)),u^0(t)\rangle\int_{t_0}^t dv_{\tau j}\right\}n^0(t) +\lambda_\tau^0\,\phi(\tau)\,n^0(t)+\lambda_\tau^0\,\chi(\tau)\leqq 0 \text{ a.s.}$$

Now we choose u^0 and n^0 on (t_0, t_1) in such a way that

$$u^0(t)\,n^0(t)=\begin{cases}-r^0(\tau) & t_0<t<t_a\\ r^0(\tau) & t_a<t<t_b\end{cases}\qquad n^0(t)=\begin{cases}\|r^0(\tau)\| & t_0<t<t_b\\ 0 & t_b<t<t_1\end{cases}$$

where $t_0<t_a<t_b<t_1$.

Due to the specification of u^0, the second summand of (2.11) vanishes for almost all $\tau\in[0,1]$, as the density $\langle \nabla G_j(p^0(\tau)), r^0(\tau)\rangle$ of $G_j(p^0(\tau))$ vanishes for almost all $\tau\in M_j=\{t\,|\,G_j(p^0(t))=0\}$. If on the other hand $\tau\notin M_j$, it follows from $M_j=V_\tau^0 N_{\tau j}=V_\tau^0\{t\,|\,G_j(q^0(t))=0\}$ that (t_0, t_1) is disjoint to the support of $v_{\tau j}$, which is contained in $N_{\tau j}$, and consequently $\int_{t_0}^{t} dv_{\tau j}=0$ for all $t\in(t_0, t_1)$.

Putting in the definition of $u^0(t)$ and $n^0(t)$ in the relation (2.11), we get

$$\pm\langle\psi_\tau(\tau), r^0(\tau)\rangle+\lambda_\tau^0\,\phi(\tau)\,\|r^0(\tau)\|+\lambda_\tau^0\,\chi(\tau)\leqq 0 \quad \text{for almost all } \tau\leqq\zeta^0. \tag{2.12}$$

If there is some τ, for which (2.12) is true and $\lambda_\tau^0=0$, then from (2.8) $\langle\psi_\tau(t), r^0(t)\rangle=0$ a.s. follows. Now by (2.7) r^0 and ψ_τ can not be orthogonal to each other, if both of them do not vanish. So we have for almost all t: $\psi_\tau(t)=0$ or $r^0(t)=0$. But as $\lambda_\tau^0=0$, we have for almost all t with $r^0(t)=0$ by (2.10): $\psi_\tau(t)=0$ too, so that $\psi_\tau(t)=0$ for almost all $t\leqq\zeta^0$. So in this case Theorem 2.1 (i) is true, if we choose $\lambda^0=0$, $\lambda=\lambda_\tau$ and $\mu_j=\mu_{\tau j}$. But this case, where especially the theorem does not provide conditions on r^0 itself, is degenerate, as is shown by the corollary following this proof.

Now we let take τ all values in $[0, \zeta^0]$, and we assume $\lambda_\tau^0>0$ for almost all $\tau\leqq\zeta^0$. Then we conclude from (2.12) that $\phi(\tau)\|r^0(\tau)\|+\chi(\tau)\leqq 0$ a.s. To establish this relation we had in mind, as we introduced the special parameter transformations v_τ^0. As ϕ and χ do not depend on the special choice of v^0, we now consider again a fixed, but arbitrary v^0 and the function ψ, parameters λ^0, λ and measures μ_j corresponding to it.

Given the above relation, now we are able to conclude from (2.8) that $\langle\psi(t), r^0(t)\rangle=-\lambda^0\phi(t)\|r^0(t)\|-\lambda^0\chi(t)\geqq 0$ a.s., and together with (2.7) we get that ψ and r^0 are not only parallel, but their orientation is the same too, i.e. we have

$$r^0(t)/\|r^0(t)\|=\psi(t)/\|\psi(t)\| \quad \text{for almost all } t \text{ with } r^0(t)\neq 0 \text{ and } \psi(t)\neq 0. \tag{2.13}$$

Taking into account this relation, it follows from (2.9) and (2.10)

$$\|r^0(t)\|=\begin{cases}v_s\\ 0\end{cases}\quad \text{for almost all } t \text{ with } \|\psi(t)\|+\lambda^0\phi(t)\begin{cases}>0\\ <0\end{cases}. \tag{2.14}$$

Additionally, if we take into consideration (2.13) and the fact that $\|\psi(t)\|+\lambda^0\phi(t)=0$ a.s. on the set $\{t\,|\,0<\|r^0(t)\|<v_s\}$, (2.8) assumes the form: $\{\|\psi(t)\|+\lambda^0\phi(t)\}^+ v_s=-\lambda^0\chi(t)$ a.s.

Finally, if we assume $\frac{\partial}{\partial l} S(x, l, t) \geqq 0$ for all $t \geqq 0$, x and l, we have $\lambda^0 \phi(t) \leqq 0$ a.s. and consequently $\psi(t) \neq 0$ for all t with $\|\psi(t)\| + \lambda^0 \phi(t) > 0$. So by (2.13) and (2.14) we get $r^0(t) = v_s \psi(t)/\|\psi(t)\|$ for almost all t with $\|\psi(t)\| + \lambda^0 \phi(t) > 0$. □

The set of conditions (i) of Theorem 2.1 allow some conclusions about r^0 only if $\lambda^0 > 0$. But this is true for reasonable phase constraints and target regions. We encounter difficulties, if, e.g., the closures of the regions $\{x \mid G_j(x) \leqq 0\}$ and $\{x \mid G_k(x) \leqq 0\}$ have exactly one point y in common, if $\nabla G_j(y) = -\nabla G_k(y)$, and if the path $p^0(t)$ touches the boundaries of the two regions exactly in this point y and nowhere else. In this situation the phase constraints do not satisfy condition a) of the following corollary.

Corollary 2.1. *If a locally optimal* (p^0, l^0, r^0, ζ^0) *satisfies conditions* (i) *of Theorem* 2.1, *then* $\lambda^0 > 0$, *if additionally.*

a) *The closed cone generated by* $\nabla G_k(p^0(t))$, $k \in K$, *in* R^n *is a proper cone,* i.e. *it does not contain a linear subspace. Here, K is an arbitrary subset of* $\{1, \ldots, J\}$, *and* $t \in \bigcap \{M_k; k \in K\}$.

b) *If* $Z(p^0(\zeta^0)) = 0$ *and* $G_k(p^0(\zeta^0)) = 0$ *for all* $k \in K \subset \{1, \ldots, J\}$, *then the closed cone in* R^n, *generated by* $\nabla Z(p^0(\zeta^0))$ *and* $\nabla G_k(p^0(\zeta^0))$, $k \in K$, *is proper.*

Proof. Let $v = \mu_1 + \ldots + \mu_J$, and denote by ρ_j the density of μ_j with respect to v, $j = 1, \ldots, J$. If $\lambda^0 = 0$, we get from Theorem 2.1 (i) and especially from (2.6) for almost all $t \leqq \zeta^0$:

$$\psi(t) = -\lambda \nabla Z(p^0(\zeta^0)) - \sum_{j=1}^{J} \int_t^{\zeta^0} \nabla G_j(p^0(s))\, d\mu_j = 0, \tag{2.15}$$

where λ or one of the measures μ_j, i.e. equivalently v, does not vanish.

If $Z(p^0(\zeta^0)) < 0$, we have $\lambda = 0$. Otherwise,

$$0 = \psi(\zeta^0) = -\lambda \nabla Z(p^0(\zeta^0)) - \sum_{j=1}^{J} \mu_j(\{\zeta^0\}) \nabla G_j(p^0(\zeta^0)).$$

If $\zeta^0 \notin M_j$, we have $\mu_j(\{\zeta^0\}) = 0$, and then we conclude from the assumption b) that $\lambda = 0$ too, and that additionally $\mu_j(\{\zeta^0\}) = 0$, $j = 1, \ldots, J$.

Therefore we get from (2.15) $\sum_{j=1}^{J} \nabla G_j(p^0(t)) \rho_j(t) = 0$ v-a.s. But then we know from assumption a) that $\rho_j(t) = 0$ v-a.s., $j = 1, \ldots, J$, so that finally we see that $\lambda^0 = 0$ implies $\lambda = 0$ and $v = 0$ too. But this is a contradiction to Theorem 2.1 (i) □

In those parts of the time axis, where we have $\|\psi(t)\| + \lambda^0 \phi(t) = 0$, (2.5) yields no direct conclusion about the optimal control r^0. If we denote by

$$H(x, l, u, \psi, \phi, t) = \langle \psi, u \rangle + \phi \|u\| - \lambda^0 S(x, l, t) f(t)$$

the generalized Hamiltonian of the optimization problem (1.1), then the special Pontryagin maximum principle following from Theorem 2.1 (i) signifies that for almost all t $H(p^0(t), l^0(t), \psi(t), \phi(t), t)$ assumes its maximum in $u = r^0(t)$ under all $u \in R^n$ with $\|u\| \leqq v_s$. If $\psi(t)$ and $\phi(t)$ vanish simultaneously, H assumes the same value for other controls too. In this case the optimization problem (1.1) possesses a singular extremum or, if $r^0(t)$ is not determined by the Euler-Lagrange equation for t contained in some subset of $[0, \zeta^0]$ only, an extremum with singular arcs [Bell and Jacobson, 1975].

There are methods for analyzing singular optimization problems by considering second order variations, but they are not applicable to problems with phase constraints of the type $\sup\{G(p(t)) \mid t \geqq 0\} \leqq 0$. But the singularity of control arcs will not cause difficulties in the application of our general theorems to more restricted optimization problems in Chap. 3, as $\psi(t) = \phi(t) = 0$ implies at once $r^0(t) = 0$ for the cost functions, which we consider there. The following corollary is an immediate consequence of (2.6) and the continuity of ϕ and χ.

Corollary 2.2. *Under the assumptions of Theorem 2.1 we have*

$$\begin{aligned}
&\{\|\psi(\zeta^0-)\| + \lambda^0 \phi(\zeta^0)\}^+ v_s \\
&= \left\{ \left\| \lambda^0 \nabla S(\eta^0(\zeta^0)) P\{T > \zeta^0\} + \sum_{j=1}^{J} \nabla G_j(p^0(\zeta^0)) \mu_j(\{\zeta^0\}) \right.\right. \\
&\qquad \left.\left. + \lambda \nabla Z(p^0(\zeta^0)) \right\| - \lambda^0 \frac{\partial}{\partial l} S(\eta^0(\zeta^0)) P\{T > \zeta^0\} \right\}^+ v_s \\
&= \lambda^0 \frac{\partial}{\partial t} S(\eta^0(\zeta^0)) P\{T > \zeta^0\},
\end{aligned}$$

where $\eta^0(\zeta^0) = (p^0(\zeta^0), l^0(\zeta^0), \zeta^0)$ *as usual.*

Essentially, this corollary provides a condition on the choice of the optimal terminal time ζ^0. If especially $\lambda^0 > 0$, $P\{T > \zeta^0\} > 0$, $\|\psi(t)\| + \lambda^0 \phi(t) > 0$ for almost all t in some open interval (t_0, ζ^0), so that $r^0(t) \neq 0$ for t near ζ^0, and if additionally the terminal state $p^0(\zeta^0)$ is contained neither in the boundary of the target region $\{x \mid Z(x) \leqq 0\}$ nor in the boundary of the admissible set $\bigcap_j \{x \mid G_j(x) \leqq 0\}$ given by the phase constraints, then the corollary is simplified to

$$\|\nabla S(p^0(\zeta^0), l^0(\zeta^0), \zeta^0)\| - \frac{\partial}{\partial l} S(p^0(\zeta^0), l^0(\zeta^0), \zeta^0) = \frac{\partial}{\partial t} S(p^0(\zeta^0), l^0(\zeta^0), \zeta^0)/v_s. \tag{2.16}$$

But if we consider the loss functional as function of ζ only and fix the path p^0 and its relative length l^0:

$$g(\zeta) = \int_0^\zeta S(p^0(t), l^0(t), t)\, dP(t) + S(p^0(\zeta), l^0(\zeta), \zeta) P\{T > \zeta\},$$

then (2.16) implies nothing else than the vanishing of the derivative g' of g with respect to ζ in the optimal point ζ^0 as

$$g'(\zeta^0)=\left\{\langle \nabla S(\eta^0(\zeta^0)), r^0(\zeta^0)\rangle+\frac{\partial}{\partial l}S(\eta^0(\zeta^0))\,\|r^0(\zeta^0)\|+\frac{\partial}{\partial t}S(\eta^0(\zeta^0)\right\}P\{T>\zeta^0\}$$

$$=\left\{-v_s\|\nabla S(\eta^0(\zeta^0))\|+v_s\frac{\partial}{\partial l}S(\eta^0(\zeta^0))+\frac{\partial}{\partial t}S(\eta^0(\zeta^0))\right\}P\{T>\zeta^0\}$$

by (2.5), for in the case $\lambda^0>0$ and $\psi(\zeta^0)\neq 0$ we have $r^0(\zeta^0)=v_s\psi(\zeta^0)/\|\psi(\zeta^0)\|$. Here we have used the fact that $\psi(t)$ and consequently $r^0(t)$ are continuous, if t is close to ζ^0, due to our assumptions. So, if neither phase constraints nor terminal constraints induce the termination of control, an optimal control is terminated only in a situation, where the possible decrease of cost, which we can achieve by varying the state $p^0(t)$ further, is counterbalanced by the infinitesimal increase of cost, which is due to elongation of the path and the passing of more time, i.e.

$$v_s\|\nabla S(\eta^0(\zeta^0))\|=v_s\frac{\partial}{\partial l}S(\eta^0(\zeta^0))+\frac{\partial}{\partial t}S(\eta^0(\zeta^0)).$$

Finally, we formulate the general theorem giving necessary conditions for a solution of the optimization problem (1.2), where the norm of velocity is assumed to be constant. As already mentioned, this theorem can be proved in a completely analogous way to Theorem 2.1. The proof is even simplified by the fact that in the modified optimization problem, which corresponds to (2.1), $n(t)=v_s$ and $k(t)=v_s V(t)$, so that these two variables do not appear in the proof of the following theorem, which we do not give here.

Theorem 2.2. *Let the distribution P of the nonnegative random variable T have a density f, which is continuous in $[0,\infty)$. We assume that $S(x,t)$, $Z(x)$ and $G_j(x)$, $j=1,\ldots,J$, are continuously differentiable in $x\in R^n$ and $t\in R$, and that additionally*

$$\nabla Z(x)\neq 0 \quad \text{on } \{x\mid Z(x)=0\},$$
$$\nabla G_j(x)\neq 0 \quad \text{on } \{x\mid G_j(x)=0\}\ j=1,\ldots,J.$$

Then, if (p^0,r^0,ζ^0) is a solution of the optimization problem (1.2), one of the following two sets of conditions is fulfilled:

(i) *There exist $\lambda^0\geqq 0$, $\lambda\geqq 0$ and Borel measures μ_j on $[0,\zeta^0]$, whose support is contained in $M_j=\{t\mid G_j(p^0(t))=0\}$, $j=1,\ldots,J$, so that*

(2.17) $$r^0(t)=v_s\psi(t)/\|\psi(t)\| \quad \text{a.s. on } \{t\leqq\zeta^0\mid\psi(t)\neq 0\}$$

where

$$\psi(t)=-\lambda^0\int_t^{\zeta^0}\nabla S(p^0(s),s)\,dP-\lambda^0\nabla S(p^0(\zeta^0),\zeta^0)\,P\{T>\zeta^0\}$$
$$-\lambda\nabla Z(p^0(\zeta))-\sum_{j=1}^{J}\int_t^{\zeta^0}\nabla G_j(p^0(s))\,d\mu_j(s)$$

and additionally we have

$$\|\psi(t)\| = -\lambda^0 \chi(t) \text{ a.s.} \tag{2.18}$$

where

$$\chi(t) = \int_t^{\zeta^0} \langle \nabla S(p^0(s), s), r^0(s) \rangle \, dP - \frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0) P\{T > \zeta^0\}.$$

At least one of the numbers λ^0, λ *or one of the measures* μ_j, $j = 1, \ldots, J$, *does not vanish, and additionally we have* $\lambda = 0$ *or* $Z(p^0(\zeta^0)) = 0$.

(ii) $\nabla S(p^0(t), t) f(t) = 0$ *for almost all* $t \leqq \zeta^0$

$$\nabla S(p^0(\zeta^0), \zeta^0) P\{T > \zeta^0\} = 0$$

$$\frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0) P\{T > \zeta^0\} = 0.$$

Analogously to Corollary 2.1 we have

Corollary 2.3. *If a locally optimal* (p^0, r^0, ζ^0) *satisfies conditions* (i) *of Theorem* 2.2, *and if the assumptions* a) *and* b) *of Corollary* 2.1 *are fulfilled, then* $\lambda^0 > 0$.

3. Constant Cost Functions and Simple Phase Constraints

In the following chapter we consider the case that we have phase constraints in the form of upper bounds of the state coordinates, i.e. $G_j(x) = x_j - z_j$, $j = 1, \ldots, n$, where $z \in R^n$ is known and $z \geqq a$. If we are considering the optimization problem (1.1), it is favourable to select a velocity, which is constant and maximal, if we make some monotonicity assumptions on the cost function described in the following. Therefore, the dependence of the cost function on path length and time are essentially equivalent. So, for simplicity of notation, in this chapter we consider only cost functions $S(x, t)$, which do not depend on path length l explicitly. The "Fitzwilliam Street problem", for instance, is a special case of an optimization problem with phase constraints and cost function of the type, which we shall consider now.

For the present we assume only that the costs are increasing with time, i.e. $\frac{\partial}{\partial t} S(x, t) > 0$ for all $t \geqq 0$ and $x \leqq z$. Furthermore, we suppose $P\{T > t\} > 0$ for all $t \geqq 0$ to exclude degenerate cases, e.g. the situation, where T is a.s. less than the time of arrival at the target region for all possible paths. These two assumptions already suffice to prevent the occurence of case (ii) of Theorem 2.1, which corresponds to the vanishing of the loss derivative.

We think of z as a target state we are striving for. Therefore, the target region $\{x \mid Z(x) \leqq 0\}$ will be a neighbourhood of z. As z is contained in the boundary of the admissible state set $\{x \mid x \leqq z\}$ given by our special constraints, we have to assume $Z(z) < 0$ due to technical reasons. Additionally, we assume that $\frac{\partial}{\partial x_j} Z(x) < 0$ for all $x \in \{x \mid x_j = z_j, Z(x) = 0\}$, $j = 1, \ldots, n$. This and the special type of phase constraints guarantees that $\lambda^0>0$ by Corollary 2.1, and so we can choose $\lambda^0=1$. Simple target regions, for which all assumptions are fulfilled, are defined e.g. by $Z(x)=\|z-x\|^2-\delta$ or by $Z(x)=\sum_{j=1}^{n}(z_j-x_j)-\delta$ for some $\delta>0$.

Firstly, we formulate a special version of Theorem 2.1:

Proposition 3.1. *Let the distribution P of the nonnegative random variable T have a density f, which is continuous in $[0,\infty)$, and $P\{T>t\}>0$ for all $t\geqq 0$. We assume that $S(x,t)$ and $Z(x)$ are continuously differentiable in $x\in R^n$ and $t\in R$, and that additionally*

$$\frac{\partial}{\partial t}S(x,t)>0 \quad \textit{for all } x\leqq z \textit{ and } t\geqq 0,$$

$$\frac{\partial}{\partial x_j}Z(x)<0 \quad \textit{for all } x\leqq z \textit{ with } x_j=z_j \textit{ and } Z(x)=0,\ j=1,\dots,n.$$

Then, if (p^0,l^0,r^0,ζ^0) is a solution of the optimization problem (1.1) with $G_j(x)=x_j-z_j$, $j=1,\dots,n$, and $z\geqq a$, there exist $\lambda\geqq 0$ and Borel measures μ_j on $[0,\zeta^0]$, whose support is contained in $M_j=\{t\,|\,p_j^0(t)=z_j\}$, $j=1,\dots,n$, so that

$$r^0(t)=v_s\,\psi(t)/\|\psi(t)\| \qquad \text{a.s. } \textit{on } \{t\leqq\zeta^0\,|\,\psi(t)\neq 0\}$$

where

$$\psi(t)=-\int_t^{\zeta^0}\nabla S(p^0(s),s)\,dP-\nabla S(p^0(\zeta^0),\zeta^0)\,P\{T>\zeta^0\}$$
$$-\lambda\nabla Z(p^0(\zeta^0))-\sum_{j=1}^{n}\int_t^{\zeta^0}d\mu_j(s)$$

and additionally we have

$$\|\psi(t)\|\,v_s=-\chi(t) \text{ a.s.}$$

where

$$\chi(t)=\int_t^{\zeta^0}\langle\nabla S(p^0(s),s),r^0(s)\rangle\,dP$$
$$-\frac{\partial}{\partial t}S(p^0(\zeta^0),\zeta^0)\,P\{T>\zeta^0\}.$$

Especially, we have $\lambda=0$ or $Z(p^0(\zeta^0))=0$, and μ_j is absolutely continuous with the possible exception of the points ζ^0 and $t\in\partial M_j\cap\partial M_k$ for some $k\neq j$, and its density is given by

$$\frac{d}{dt}\mu_j(t)=\begin{cases}-\dfrac{\partial}{\partial x_j}S(p^0(t),t)\,f(t) & t\in M_j^0\\ 0 & t\in M_j^c\end{cases}.$$

Proof. With the exception of the additional properties of the measures μ_j, $j=1,\dots,n$, the proposition follows from Theorem 2.1 and Corollary 2.1, if we bear in mind that $\phi(t)=0$ for all t.

On M_j the j-th coordinate of p^0 is identical to z_j, and so we have $r_j^0(t)=0$ for almost all t in the interior M_j^0 of M_j, and

$$\frac{\partial}{\partial x_j} S(p^0(\zeta^0), \zeta^0) P\{T>\zeta^0\} + \lambda \frac{\partial}{\partial x_j} Z(p^0(\zeta^0)) + \int_t^{\zeta^0} \frac{\partial}{\partial x_j} S(p^0(t), t) f(t)\, dt + \int_t^{\zeta^0} d\mu_j = -\psi_j(t) = -r_j^0(t)\, \|\psi(t)\|/v_s = 0 \qquad \text{for almost all } t \in M_j^0.$$

As additionally the support of μ_j is contained in M_j, we get from this result the absolute continuity of μ_j on $M_j^c + M_j^0$ and the representation of the density given in the proposition.

Finally, if $t \in \partial M_j$, $t<\zeta^0$ and $t \notin \partial M_k$ for all $k \neq j$, then the measures μ_k, $k \neq j$, and consequently ψ_k, $k \neq j$, are absolutely continuous in an open neighbourhood of t. On $(0, \zeta^0)$ $\|\psi(s)\|$ coincides with the absolutely continuous function $\chi(s)$, so that we conclude the absolute continuity of $\psi_j^2(s) = \|\psi(s)\|^2 - \sum_{k \neq j} \psi_k^2(s)$ and then of $\psi(s)$ in a neighbourhood of t. □

We can not expect a better statement about the smoothness of ψ, as difficulties possibly appear, if the locally optimal path p^0 passes through states, which are contained in the intersection of two or more of the boundaries $\{x \mid x_j = z_j\}$, $j=1, \ldots, n$. [Franke, 1980].

If an optimal path $p^0(t)$ coincides for t in some open interval with one of the boundaries $\{x \mid x_j = z_j\}$, then we have $-\frac{\partial}{\partial x_j} S(p^0(t), t)\, f(t) = \frac{d}{dt} \mu_j(t) \geqq 0$. So, an optimal path stays on one of the boundaries, only if at this moment we do not have to reckon with the occurrence of the random event, i.e. $f(t)=0$, or if a return to the interior of the admissible region $\{x \mid x \leqq z\}$ is not advantageous. Otherwise, if $f(t)>0$ and $\frac{\partial}{\partial x_j} S(p^0(t), t) > 0$, the path leaves the j-th boundary immediately. So, here the optimal control takes into account the local behaviour of the cost function. The influence of the μ_j-term in the function ψ keeps the path on the boundary and prevents the violation of the constraints in the case that it would be advantageous to leave the admissible region and cross the boundary $\{x \mid x_j = z_j\}$, i.e. if $\frac{\partial}{\partial x_j} S(p^0(t), t) < 0$.

Under the assumptions, which we have made up to now, a situation can occur, where paths minimizing the expected costs do not approach the target region. If e.g. $S(x, t)$ decreases much more faster for $x \to -\infty$ than it increases with respect to t, possibly there are locally optimal paths, which start in a and then move away from z, until they are stopped by the random event. Such situations are in contradiction to the spirit of a navigation problem, where the pre-eminent object is to get to the target region, and they hint at an inadequate choice of the cost function. To exclude those situations, we assume in the following that the cost function decreases in all phase coordinates, i.e.

$\nabla S(x,t)<0$ for all $x \leqq z$ and $t \geqq 0$. Especially, we then can conclude that in general locally optimal paths are approaching the target region monotonously, if additionally $f(t)>0$ a.s.:

Lemma 3.1. *Additionally to the assumptions of Proposition* 3.1 *we assume* $\frac{\partial}{\partial x_j} S(x,t)<0$ *for all* $x \leqq z$ *and* $t \in (0, \zeta^0)$, *and* $f(t)>0$ *for almost all* $t \in (0,\zeta^0)$, *where* $j \in \{1, \ldots, n\}$. *Then, if* (p^0, l^0, r^0, ζ^0) *is locally optimal, we have* $r_j^0(t) \geqq 0$ a.s. *in* $[0, \zeta^0]$.

Proof. a) Firstly, we assume that there exists $t_j \in [0, \zeta^0)$, so that $r_j^0(t) \leqq 0$ a.s. in (t_j, ζ^0) and the Lebesgue measure of the set $\{t \in (t_j, \zeta^0) \mid r_j^0(t)<0\}$ does not vanish, i.e. after time t_j $p_j^0(t)$ is non-increasing and not constant. If we define $r_k = r_k^0$ for all $k \neq j$, $r_j(t) = r_j^0(t)$ for $t<t_j$ and $r_j(t)=(1-\varepsilon) r_j^0(t)$ for $t>t_j$, $p(t)=a + \int_0^t r(s)\,ds$, $l(t)=\int_0^t \|r(s)\|\,ds$, then the tuple (p, l, r, ζ^0) satisfies the constraints of the optimization problem (1.1) with $G_k(x)=x_k - z_k$, $k=1, \ldots, n$, $p(t) \geqq p^0(t)$ for all $t \leqq \zeta^0$ and $p_j(t) > p_j^0(t)$ for all t in a set of non-vanishing Lebesgue measure. As $\varepsilon>0$ is arbitrary, we get in an arbitrary neighbourhood of (p^0, l^0, r^0, ζ^0) another tuple (p, l, r, ζ^0), satisfying the constraints so that, due to our assumptions of the lemma, $L(p, l, r, \zeta^0) < L(p^0, l^0, r^0, \zeta^0)$ in contradiction to the local optimality of the latter.

b) Otherwise, if we define $\zeta_j = \sup\{t \in (0, \zeta^0) \mid p_j^0(t) < z_j\}$, then for every $s \in [0, \zeta_j)$ the Lebesgue measure of the set $\{t \in (s, \zeta_j) \mid r_j^0(t)>0\}$ does not vanish. If $\zeta_j = 0$, then $r_j^0(t)=0$ a.s. in $(0, \zeta^0)$, so that we assume $\zeta_j>0$ in the following. Furthermore, if $p_j^0(t)$ is not non-decreasing, then there exist t', s', so that $0 \leqq t' < s' < \zeta_j$ and $p_j^0(t') > p_j^0(s')$. Together with the consideration above we conclude that for suitable $\eta>0$ there exist $\delta_1, \delta_2 > 0$, so that $t' < s'-\delta_1 < s'+\delta_2 < \zeta_j$, $p_j^0(t) \leqq p_j^0(t') - \eta$ for all t in the interval $(s'-\delta_1, s'+\delta_2)$ and the Lebesgue measures of the sets $A=(s'-\delta_1, s') \cap \{t \mid r_j^0(t)<0\}$ and $B=(s', s'+\delta_2) \cap \{t \mid r_j^0(t)>0\}$ do not vanish.

Now we define $r_k = r_k^0$ for all $k \neq j$, $r_j(t)=(1-\varepsilon_1) r_j^0(t)$ for $t \in A$, $r_j(t)=(1-\varepsilon_2) r_j^0(t)$ for $t \in B$ and $r_j(t)=r_j^0(t)$ else, $p(t)=a+\int_0^t r(s)\,ds$, $l(t)=\int_0^t \|r(s)\|\,ds$, where $0<\varepsilon_1 \leqq \eta / \int_A \|r_j^0(s)\|\,ds$ and ε_1 small enough so that $\varepsilon_2 = \varepsilon_1 \int_A \|r_j^0(s)\|\,ds / \int_B r_j^0(s)\,ds < 1$. Then the tuple (p, l, r, ζ^0) satisfies the constraints of the optimization problem, which we are considering, for

$$p_j(t) \leqq p_j^0(t) + \varepsilon_1 \int_A \|r_j^0(s)\|\,ds \leqq p_j^0(t') \leqq z_i \qquad \text{for all } t$$

in the interval $(s'-\delta_1, s'+\delta_2)$, and $p_j(t)=p_j^0(t)$ else. Furthermore, we have $p(t) \geqq p^0(t)$ for all t and $p_j(t) > p_j^0(t)$ in an open interval around s'. By choosing $\varepsilon_1 > 0$ small enough, as in part a) of the proof we get again a contradiction to the local optimality of (p^0, l^0, r^0, ζ^0). So, necessarily $r_j^0(t) \geqq 0$ a.s. in $(0, \zeta^0)$. □

If we consider a globally optimal (p^0, l^0, r^0, ζ^0), we can conclude $r^0(t) \geqq 0$ a.s. under weaker assumptions, e.g. $\nabla S(x,t) \leqq 0$ for all $x \leqq z$ and $t \geqq 0$ and $\nabla S(x,t) < 0$ for x contained in the target region. But degenerated cases are possible, where paths, which are not non-decreasing, are locally optimal, because their neighbourhoods do not contain enough tuples compatible with the constraints [Franke, 1980].

If $r^0(t) \geqq 0$ a.s., then we know especially that the corresponding optimal path stays on the boundary $\{x \mid x_j = z_j\}$ as soon as it has arrived there, i.e. if we define

$$\zeta_j = \begin{cases} \min\{t \mid p_j^0(t) = z_j\} & \text{if } p_j^0(\zeta^0) = z_j \\ \zeta^0 & \text{if } p_j^0(t) < z_j \quad \text{for all } t \leqq \zeta^0 \end{cases}$$

then we get the following form of the set of points of time, at which $p^0(t)$ is contained in the j-th boundary: $M_j = [\zeta_j, \zeta^0]$, $p_j^0(\zeta^0) = z_j$ or $M_j = \emptyset$, $p_j^0(\zeta^0) < z_j$. To avoid considering some special cases, we now assume additionally $\nabla S(x,t) < 0$ for all $x \leqq z$ and $t \geqq 0$ and $f(t) > 0$ a.s. in $[0, \zeta^0]$, so that we have $r^0(t) \geqq 0$ a.s. for all possible locally optimal paths.

If $\zeta_j < \zeta^0$ and if we take into account the form of the density of μ_j in M_j^0 given in Proposition 3.1, then we get for almost all $t > \zeta_j$:

$$\begin{aligned} 0 &= r_j^0(t)\,\|\psi(t)\|/v_s = \psi_j(t) \\ &= -\frac{\partial}{\partial x_j} S(p^0(\zeta^0), \zeta^0)\, P\{T > \zeta^0\} - \lambda \frac{\partial}{\partial x_j} Z(p^0(\zeta)) - \mu_j\{\zeta^0\}. \end{aligned}$$

From the right-continuity of ψ we conclude additionally $\psi_j(\zeta_j) = 0$ and $\psi_j(\zeta_j -) = -\mu_j\{\zeta_j\}$. As $\|\psi(t)\|$ is absolutely continuous by Proposition 3.1, we have

$$\begin{aligned} 0 = \|\psi(\zeta_j)\|^2 - \|\psi(\zeta_j -)\|^2 &= \sum_{\{k \mid \zeta_k = \zeta_j\}} (\psi_k^2(\zeta_j) - \psi_k^2(\zeta_j -)) \\ &= \sum_{\{k \mid \zeta_k = \zeta_j\}} (\mu_k\{\zeta_j\})^2, \end{aligned}$$

and so especially we have $\mu_j\{\zeta_j\} = 0$.

Putting things together, we get the following form of $\psi_k(t)$, $k = 1, \ldots, n$:

$$\psi_k(t) = \begin{cases} -\int_t^\zeta \frac{\partial}{\partial x_k} S(p^0(s), s)\, dP + \rho_k & t < \zeta_k \\ 0 & t > \zeta_k \end{cases} \tag{3.1}$$

where $\rho_k = -\frac{\partial}{\partial x_k} S(p^0(\zeta^0), \zeta^0)\, P\{T > \zeta^0\} - \lambda \nabla Z(p^0(\zeta^0)) - \mu_k\{\zeta^0\}$ and especially $\rho_k = 0$ for all k with $\zeta_k < \zeta^0$.

Theorem 3.1. *Let the distribution P of the nonnegative random variable T have a density f, which is continuous in $[0, \infty)$, and $P\{T > t\} > 0$ for all $t \geqq 0$, $f(t) > 0$ a.s. in $[0, \zeta^0]$. We assume that $S(x,t)$ and $Z(x)$ are continuously differentiable in $x \in R^n$ and $t \in R$, and that additionally:*

$$\frac{\partial}{\partial t} S(x,t) > 0 \qquad \nabla S(x,t) < 0 \qquad \textit{for all } x \leqq z \textit{ and } t \geqq 0,$$

$$\frac{\partial}{\partial x_j} Z(x) < 0 \qquad \textit{for all } x \leqq z \textit{ with } x_j = z_j \textit{ and } Z(x) = 0, \quad j = 1, \ldots, n.$$

Then, if (p^0, l^0, r^0, ζ^0) *is a solution of the optimization problem* (1.1) *with* $G_j(x) = x_j - z_j$, $j = 1, \ldots, n$, *and* $z \geqq a$, *there exist* $\zeta_1, \ldots, \zeta_n \in [0, \zeta^0]$, $\eta_1, \ldots, \eta_n \geqq 0$ *with* $\|\eta\| = 1$, *so that* $\eta_j = 0$ *or* $\zeta_j = \zeta^0$ *for all* $j = 1, \ldots, n$, *and*

$$r^0(t) = v_s \psi(t) / \|\psi(t)\| \qquad \textit{for almost all } t \leqq \zeta^0 \tag{3.2}$$

where

$$\psi_j(t) = -\int_t^{\zeta_j} \frac{\partial}{\partial x_j} S(p^0(s), s)\, dP + \eta_j v_s^{-1} \frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0) P\{T > \zeta^0\}$$

if $t < \zeta_j$, *and* $\psi_j(t) = 0$, *if* $t > \zeta_j$.

Especially, $p_j^0(t) = z_j$ *for all* $t \in [\zeta_j, \zeta^0)$, $j = 1, \ldots, n$. *Furthermore, one of the two following sets of conditions is satisfied:*

(i) $Z(p^0(\zeta^0)) < 0$, $\eta \frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0) \leqq -v_s \nabla S(p^0(\zeta^0), \zeta^0)$,

$$\eta_j \frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0) = -v_s \frac{\partial}{\partial x_j} S(p^0(\zeta^0), \zeta^0) \qquad \textit{for all } j \textit{ with } p_j^0(\zeta^0) < z_j.$$

(ii) $Z(p^0(\zeta^0)) = 0$; *there exists* $\Lambda \geqq 0$, *so that*

$$\eta \frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0) \leqq -v_s \nabla S(p^0(\zeta^0), \zeta^0) - \Lambda \nabla Z(p^0(\zeta^0)),$$

$$\eta_j \frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0) = -v_s \frac{\partial}{\partial x_j} S(p^0(\zeta^0), \zeta^0) - \Lambda \nabla Z(p^0(\zeta^0))$$

for all j *with* $p_j^0(\zeta^0) < z_j$.

Proof. The theorem essentially follows from Proposition 3.1 and the special form (3.1) of ψ already.

If we take into account that $\frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0)\, P\{T > \zeta^0\} > 0$, and if we define η by $\rho v_s = \eta \frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0) P\{T > \zeta^0\}$, then we get $\eta \geqq 0$ by (3.1), as by Lemma 3.1 $\psi(t) = r^0(t) \|\psi(t)\| / v_s \geqq 0$ for almost all $t \leqq \zeta^0$.

$\|\eta\| = 1$ follows from $\|\psi(t)\|\, v_s = -\chi(t)$ a.s., as

$$\|\eta\|\, v_s^{-1} \frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0) P\{T > \zeta^0\} = \|\psi(\zeta^0 -)\| = -\chi(\zeta^0)\, v_s^{-1}$$

$$= v_s^{-1} \frac{\partial}{\partial t} S(p^0(\zeta^0), \zeta^0) P\{T > \zeta^0\}.$$

Especially, we have $\psi(\zeta^0-)\neq 0$, so that we conclude $\psi(t)\neq 0$ for all $t<\zeta^0$, for ψ is nonnegative and non-increasing, as $r^0(t)\geqq 0$ and $-\nabla S(p^0(t),t)f(t)>0$ a.s. So, our assumptions exclude the existence of locally optimal paths with singular arcs, and we have in the whole interval $[0,\zeta^0]$: $r^0(t)=v_s\psi(t)/\|\psi(t)\|$ a.s.

If $Z(p^0(\zeta^0))<0$, then $\lambda=0$ by Proposition 3.1. From the definitions of η and ρ we conclude immediately the set (i) of conditions, as we have $P\{T>\zeta^0\}>0$ and $M_j=\emptyset$, especially $\mu_j\{\zeta^0\}=0$, for all j with $p_j^0(\zeta^0)<z_j$.

If $Z(p^0(\zeta^0))=0$, then we define $\Lambda=v_s\lambda/P\{T>\zeta^0\}$, and (ii) follows analogously from the definitions of η and ρ. □

In Theorem 3.1 (i) we have $3n+1$ parameters characterizing a possible locally optimal (p^0,l^0,r^0,ζ^0): the terminal state $p^0(\zeta^0)\in R^n$, the terminal time ζ^0, the terminal direction of velocity $\eta\in R^n$ and the time points ζ_j, $j=1,\dots,n$, where we stop controlling the j-th state coordinate, as it has assumed its maximal admissible value. But only n of these parameters can be chosen arbitrarily, which corresponds to the n initial conditions $p^0(0)=a\in R^n$. If, for instance, we have $p_j^0(\zeta^0)=z_j$, then we conclude from Theorem 3.1 $\zeta_j=\zeta^0$ and

$$\eta_j\frac{\partial}{\partial t}S(p^0(\zeta^0),\zeta^0)=-v_s\frac{\partial}{\partial x_j}S(p^0(\zeta^0),\zeta^0).$$

So, for all $j=1,\dots,n$ only one of the three parameters

$$\eta_j\in\left[0,-v_s\frac{\partial}{\partial x_j}S(p^0(\zeta^0),\zeta^0)\left\{\frac{\partial}{\partial t}S(p^0(\zeta^0),\zeta^0)\right\}^{-1}\right],$$

$p_j^0(\zeta^0)\in(-\infty,z_j]$ and $\zeta_j\in[0,\zeta^0]$ is not contained in the boundary of its interval of possible values, and additionally we have the condition $\|\eta\|=1$, which further reduces the number of free parameters by one.

Analogously, the additional parameter Λ in case (ii) of the theorem is generally compensated by the additional constraint $Z(p^0(\zeta^0))=0$.

The set of conditions (i) of Theorem 3.1 implies especially that $\frac{\partial}{\partial t}S(p^0(\zeta^0),\zeta^0)=v_s\|\nabla S(p^0(\zeta^0),\zeta^0)\|$, if $Z(p^0(\zeta^0))<0$ and $p_j^0(\zeta^0)<z_j$, $j=1,\dots,n$. This fact corresponds to the remarks following Corollary 2.2. In general, the time derivative and the components of the spatial gradient of the cost function S determine also the choice of the terminal time ζ^0, if the path is not stopped immediately at arrival at the target region. The first part a) of the following corollary implies that the control is terminated only if locally a further increasing of the coordinates of p^0, which have not yet assumed the maximal admissible value, does not compensate the increasing of the costs caused by the lengthening of the time interval of control. Conversely, the decreasing of costs, caused by changing the state, must have been at least as large as the increasing of costs, caused by lengthening of the control interval, locally before the terminal time; otherwise, one could decrease the loss by stopping earlier.

Corollary 3.1. *Let*

$$g_j=\begin{cases}\dfrac{\partial}{\partial x_j}S(p^0(\zeta^0),\zeta^0) & \text{if } p_j^0(\zeta^0)<z_j\\ 0 & \text{if } p_j^0(\zeta^0)=z_j,\end{cases}$$

$$h_j=\begin{cases}\dfrac{\partial}{\partial x_j}S(p^0(\zeta^0),\zeta^0) & \text{if } \zeta_j=\zeta^0\\ 0 & \text{if } \zeta_j<\zeta^0.\end{cases}$$

Under the assumptions of Theorem 3.1 we have

a) $\|g\|\,v_s\leqq\dfrac{\partial}{\partial t}S(p^0(\zeta^0),\zeta^0)$

b) $\dfrac{\partial}{\partial t}S(p^0(\zeta^0),\zeta^0)\leqq\|h\|\,v_s,\quad$ *if* $Z(p^0(\zeta^0))<0$.

Proof. a) If $g=0$, e.g. if $p^0(\zeta^0)=z$, then the assertion is true, as $0<\dfrac{\partial}{\partial t}S(p^0(\zeta^0),\zeta^0)$.

Otherwise, we assume

$$\langle \nabla S(p^0(\zeta^0),\zeta^0),g/\|g\|\rangle=\|g\|>\frac{\partial}{\partial t}S(p^0(\zeta^0),\zeta^0),$$

and we consider the path p, which is identical to p^0 continued with a small straight line, i.e. we choose $p(t)=a+\int\limits_0^t r(s)\,ds$ with $r(s)=r^0(s)$ for $s\leqq\zeta^0$ and $r(s)=-g/\|g\|$ for $\zeta^0<s\leqq\zeta^0+\tau$. Here, we let $\tau>0$ be so small that $p_j(\zeta^0+\tau)<z_j$ for all j with $p_j^0(\zeta^0)<z_j$ and that

$$\langle \nabla S(p(t),t),g/\|g\|\rangle>\frac{\partial}{\partial t}S(p(t),t)\qquad\text{for all } t\in[\zeta^0,\zeta^0+\tau].$$

Then, $S(p(t),t)$ decreases in the interval $[\zeta^0,\zeta^0+\tau]$, as for $t\in[\zeta^0,\zeta^0+\tau]$ we have

$$\frac{d}{dt}S(p(t),t)=\frac{\partial}{\partial t}S(p(t),t)-\langle \nabla S(p(t),t),g/\|g\|\rangle<0.$$

From this we finally conclude

$$L(p,l,r,\zeta^0+\tau)-L(p^0,l^0,r^0,\zeta^0)=\int\limits_{\zeta^0}^{\zeta^0+\tau}S(p(t),t)\,dP$$
$$+S(p(\zeta^0+\tau),\zeta^0+\tau)\,P\{T>\zeta^0+\tau\}-S(p^0(\zeta^0),\zeta^0)\,P\{T>\zeta^0\}<0,$$

contradicting the optimality of p^0.

b) The inequality follows at once from Theorem 3.1 (i) together with $\|\eta\| = 1$ and $\eta_j = 0$ for all j with $\zeta_j < \zeta^0$. □

Finally, we return to the "Fitzwilliam Street problem". Here, the loss functional has the simple form $\mathfrak{E}\left\{v_s T_\zeta - \sum_{j=1}^{2} (z_j - p_j(T_\zeta))\right\}$, which corresponds to the cost function $S(x,t) = -\sum_{j=1}^{2} x_j + v_s t$, as constant summands do not influence optimality considerations. If we assume $0 < v_s \leqq 1$, then the function Z, defining the target region, is not relevant, as in any case we can choose optimal paths ending in the target state z itself, i.e. $p^0(\zeta^0) = z$ [Franke, 1980].

As the gradient of the cost function is constant, Theorem 3.1 supplies an explicit representation of the optimal control $r^0(t)$. If we take into account that we have assumed $p^0(\zeta^0) = z$, especially $Z(p^0(\zeta^0)) < 0$, we get as possible locally optimal velocities $r^0(t) = \psi(t)/\|\psi(t)\|$ for almost all $t \leqq \zeta^0$, where

$$\psi_j(t) = \begin{cases} P(\zeta^0) - P(t) + \eta_j(1 - P(\zeta^0)) & t < \zeta_j \\ 0 & t > \zeta_j \end{cases} \qquad j = 1, 2$$

with $\zeta^0 \geqq 0$; $\zeta_1, \zeta_2 \in [0, \zeta^0]$; $\eta_1, \eta_2 \geqq 0$, $\|\eta\| = 1$; $\eta_j = 0$ or $\zeta_j = \zeta^0$ for $j = 1, 2$. If P is the exponential distribution, this representation coincides with the solution given by Wilkinson (1974).

Now it is possible to choose a set of consistent terminal parameters ζ^0, ζ_1, ζ_2, η_1, η_2 and, starting in the terminal state $p^0(\zeta^0) = z$, to integrate the paths backwards numerically, to get numerical approximations for locally optimal paths, which start in different initial states a [Franke, 1980].

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Received September 5, 1980