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A Note on Specifications

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Let S be a countably infinite set. For each $\alpha \in S$, suppose we have a finite state space E_α and a random variable X_α – on underlying probability space (Ω, B, μ) – with values in E_α. The collection $\{X_\alpha\}$ is said to define a random field over $(S, \{E_\alpha\})$.

For any $\Lambda \subset S$ let $E_\Lambda = \underset{\alpha\in\Lambda}{\times} E_\alpha$ be the state space over Λ and let $\mathscr{F}_\Lambda = \sigma\{X_\alpha \mid \alpha \in \Lambda^c\}$[1] be the σ-algebra of events observable *outside* of Λ. We will denote E_S by E, and elements of E (possible configurations of the random field) will be denoted by $x = \{x_\alpha \mid \alpha \in S, x_\alpha \in E_\alpha\}$. For any $x \in E$ and any $\Lambda \subset S$, let $x_\Lambda = \{x_\alpha \mid \alpha \in \Lambda\} \in E_\Lambda$ be the configuration over Λ. Let L be the collection of finite subsets of S. For $\Lambda \in L$, elements of E_Λ will be denoted by y or y'.

The Gibbs states are perhaps the best known examples of random fields. They are defined in terms of their conditional probability distributions given $\mathscr{F}_\Lambda$ for all $\Lambda \in L$: For μ a Gibbs state, $\Lambda \in L$, $y \in E_\Lambda$

$$\operatorname{Prob}\{X_\Lambda = y \mid \mathscr{F}_\Lambda\}(x) = \frac{e^{-h_\Lambda(y,x)}}{\sum_{y' \in E_\Lambda} e^{-h_\Lambda(y',x)}} \quad \text{for } \mu \quad \text{a.e. } x \in E. \tag{1}$$

Here $h_\Lambda(y,x)$ is $\mathscr{F}_\Lambda$-measurable in x and should be thought of as the energy possessed by the configuration y over Λ when in the presence of the configuration x_{Λ^c} outside of Λ. For example, for the Ising model, $S = \mathbb{Z}^d$, $E_\alpha = \{-1, 1\}$, and

$$h_\Lambda(y,x) = -J\Big(\tfrac{1}{2}\sum_{\substack{\alpha,\beta\in\Lambda \\ \|\alpha-\beta\|=1}} y_\alpha y_\beta + \sum_{\substack{\alpha\in\Lambda \\ \beta\in\Lambda^c \\ \|\alpha-\beta\|=1}} y_\alpha x_\beta\Big). \tag{2}$$

Given a collection $h = \{h_\Lambda\}$, random fields satisfying (1) are called Gibbs states with energy h and this collection will be denoted by $G(h)$.

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[1] As defined, $\mathscr{F}_\Lambda \subset \mathscr{B}$. We also regard $\mathscr{F}_\Lambda$ as a σ-algebra on E in the obvious way. We will also regard μ as a measure on E.

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Let us denote the right hand side of (1) by $\pi_\Lambda(y,x)$. Then, typically, (e.g., when h_Λ is defined by (2)), π_Λ will satisfy the conditions

$$\pi_\Lambda(y,x) \geqq 0 \quad \text{for any } y \in E_\Lambda, \quad x \in E. \tag{3}$$

$$\sum_{y' \in E_\Lambda} \pi_\Lambda(y',x) = 1 \quad \text{for any } x \in E. \tag{4}$$

$$\pi_\Lambda(y,\cdot) \quad \text{is } \mathscr{F}_\Lambda\text{-measurable, for any } y \in E_\Lambda. \tag{5}$$

For $\Lambda \subset \bar{\Lambda} \in L$, there exists a function $\phi \in \mathscr{F}_\Lambda$[2] such that for all $x \in E$ and $y \in E_\Lambda$ (6)

$$\pi_{\bar{\Lambda}}(y\, x_{\bar{\Lambda}-\Lambda}, x) = \phi(x)\, \pi_\Lambda(y,x)^3. \tag{C}$$

(Here we are using the following notation:
For $y_1 \in E_{\Lambda_1}$, $y_2 \in E_{\Lambda_2}$, $\Lambda_1 \cap \Lambda_2 = \phi$, $y_1\, y_2 \in E_{\Lambda_1 \cup \Lambda_2}$ is given by

$$(y_1\, y_2)_\alpha = \begin{cases} (y_1)_\alpha & \alpha \in \Lambda_1 \\ (y_2)_\alpha & \alpha \in \Lambda_2 \end{cases}.$$

Remark 1. (3) and (4) say that $\pi_\Lambda(y,x)$ defines a probability distribution in y for fixed x.

(6) says that $\{\pi_\Lambda\}$, $\Lambda \in L$ defines a consistent family of conditional probabilities.

Definition 2. A collection $\pi = \{\pi_\Lambda \mid \Lambda \in L\}$ satisfying (3)–(6) is called a *specification* [1, 2].

Thus Gibbs states have conditional probabilities given by a specification. For any random field we may define the conditional probabilities P_x^Λ given $\mathscr{F}_\Lambda$. "A priori", P_x^Λ is defined for μ a.e. x. For a random field with conditional probabilities given by a specification π, $P_x^\Lambda = \pi_\Lambda(\cdot, x)$ is defined for all x. "A priori", for $\Lambda \subset \bar{\Lambda}$, $P_x^{\bar{\Lambda}}$ need be consistent with $P_\cdot^\Lambda$ only for μ a.e. x. If the random field is given by a specification, we have everywhere consistency (6).

For the study of Gibbs states, it is very important that they be defined by *everywhere* defined conditional probabilities. This can perhaps best be appreciated if we note that the extreme members of $G(h)$ are mutually singular.

What we show in this note is that *every* random field (over $(S, \{E_\alpha\})$) has conditional probabilities given by a specification.

Theorem. *Suppose $\{X_\alpha\}$ is a random field over $(S, \{E_\alpha\})$. Then there exists a specification π such that for $\Lambda \in L$, $y \in E_\Lambda$*

$$\operatorname{Prob}\{X_\Lambda = y \mid \mathscr{F}_\Lambda\}(x) = \pi_\Lambda(y,x) \quad \textit{for } \mu \textit{ a.e. } x. \tag{7}$$

Proof. Let $\{\Lambda_n\}$ be an increasing sequence with $\Lambda_n \in L$, and $\bigcup_n \Lambda_n = S$.

By the Martingale convergence theorem

$$\operatorname{Prob}\{X_\Lambda = y \mid \mathscr{F}_\Lambda\}(x) = \lim_{n\to\infty} \operatorname{Prob}\{X_\Lambda = y \mid X_{\Lambda_n - \Lambda} = x_{\Lambda_n - \Lambda}\} \quad \text{for } \mu \text{ a.e. } x. \tag{8}$$

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[2] "$\phi \in \mathscr{F}_\Lambda$" means "$\phi$ is bounded and $\mathscr{F}_\Lambda$-measurable"

[3] ϕ depends upon Λ and $\bar{\Lambda}$

The sequence on the right hand side (RHS) of (8) is defined by elementary conditional probabilities.

Definition 3. Let $G_\Lambda = \{x \in E \mid$ the limit on RHS of (8) exists for all $y \in E_\Lambda\}$. Denote this limit by $p_\Lambda(y, x)$.

Lemma 4. Suppose $\Lambda \subset \bar{\Lambda}$ and $x \in G_\Lambda \cap G_{\bar{\Lambda}}$. Then, there exists $\phi \in \mathscr{F}_\Lambda$ such that for every $y \in E_\Lambda$,

$$p_{\bar{\Lambda}}(y\, x_{\bar{\Lambda}-\Lambda}, x) = \phi(x)\, p_\Lambda(y, x). \tag{9}$$

Proof. By elementary probability

$$\begin{aligned}&\mathrm{Prob}\{X_{\bar{\Lambda}} = y\, x_{\bar{\Lambda}-\Lambda} \mid X_{\Lambda_n-\bar{\Lambda}} = x_{\Lambda_n-\bar{\Lambda}}\}\\ &\quad = \mathrm{Prob}\{X_{\bar{\Lambda}-\Lambda} = x_{\bar{\Lambda}-\Lambda} \mid X_{\Lambda_n-\bar{\Lambda}} = x_{\Lambda_n-\bar{\Lambda}}\}\ \mathrm{Prob}\{X_\Lambda = y \mid X_{\Lambda_n-\Lambda} = x_{\Lambda_n-\Lambda}\}.\end{aligned} \tag{10}$$

Passing to the limit $n \to \infty$, we obtain

$$p_{\bar{\Lambda}}(y\, x_{\bar{\Lambda}-\Lambda}, x) = \Big(\sum_{y'} p_{\bar{\Lambda}}(y'\, x_{\bar{\Lambda}-\Lambda}, x)\Big)\, p_\Lambda(y, x) \tag{11}$$

which is of the form (9).

Definition 5. For $\Lambda \subset \Lambda_n$, let

$$P_\Lambda^{\Lambda_n} = \Big\{x \in G_{\Lambda_n} \,\Big|\, \sum_{y' \in E_\Lambda} p_{\Lambda_n}(y'\, x_{\Lambda_n-\Lambda}, x) > 0\Big\}.$$

Definition 6. $\mathscr{H}_\Lambda = \bigcup_n \bigcap_{j=n}^{\infty} P_\Lambda^{\Lambda_j}$.

Definition 7. For each $\alpha \in S$ let ν_α be a probability measure on E_α. Then we define a specification π by

$$\pi_\Lambda(y, x) = \begin{cases} p_\Lambda(y, x) & \text{if } x \in \mathscr{H}_\Lambda \\ \underset{\alpha \in \Lambda}{\times} \nu_\alpha(y_\alpha) & \text{if } x \in \mathscr{H}_\Lambda^c \end{cases}.$$

To complete the proof we must show (a) that π is a specification and (b) that π satisfies (7). (b) will follow immediately from (8) once we have established the next lemma.

Lemma 8. $\mu(\mathscr{H}_\Lambda) = 1$.

Proof. It suffices to show that

$$\mu(P_\Lambda^{\Lambda_n}) = 1. \tag{12}$$

But,

$$\begin{aligned}\mu((P_\Lambda^{\Lambda_n})^c) &= \int d\mu(x)\, \mu((P_\Lambda^{\Lambda_n})^c \mid \mathscr{F}_{\Lambda_n})(x)\\ &\leqq \int d\mu(x) \sum_{\{z \in E_{\Lambda_n} \mid p_{\Lambda_n}(z, x) = 0\}} p_{\Lambda_n}(z, x) = 0.\end{aligned}$$

It remains to check (a). Since $p_\Lambda(y, x)$ is defined only for $x \in G_\Lambda$, it may not be immediately clear that π_Λ is well defined for all $x \in \mathscr{H}_\Lambda$. This follows from the next lemma.

Lemma 9. $P_\Lambda^{\Lambda_m} \subset G_\Lambda$.

Proof. We must show that if $x \in P_\Lambda^{\Lambda_m}$, then $\lim_{n \to \infty} \text{Prob}\{X_\Lambda = y \mid X_{\Lambda_n - \Lambda} = x_{\Lambda_n - \Lambda}\}$ exists for all $y \in E_\Lambda$. But $x \in P_\Lambda^{\Lambda_m} \Rightarrow$

$$\lim_{n \to \infty} \text{Prob}\{X_{\Lambda_m - \Lambda} = x_{\Lambda_m - \Lambda} \mid X_{\Lambda_n - \Lambda_m} = x_{\Lambda_n - \Lambda_m}\} > 0.$$

We may therefore pass to the limit in

$$\text{Prob}\{X_\Lambda = y \mid X_{\Lambda_n - \Lambda} = x_{\Lambda_n - \Lambda}\}$$
$$= \frac{\text{Prob}\{X_{\Lambda_m} = y\, x_{\Lambda_m - \Lambda} \mid X_{\Lambda_n - \Lambda_m} = x_{\Lambda_n - \Lambda_m}\}}{\text{Prob}\{X_{\Lambda_m - \Lambda} = x_{\Lambda_m - \Lambda} \mid X_{\Lambda_n - \Lambda_m} = x_{\Lambda_n - \Lambda_m}\}}, \tag{13}$$

(since the limits on the RHS exist because $x \in G_{\Lambda_m}$). This proves the lemma.

We now show that π is a specification. (3)–(5) are obvious. ((5) is true because $\mathscr{H}_\Lambda \in \mathscr{F}_\Lambda$.) To check (6) fix $\Lambda \subset \bar{\Lambda} \in L$ and fix $y \in E_\Lambda$. We must show that (C) is satisfied for all $x \in E$. Four cases have to be considered:

(i) $x \in \mathscr{H}_\Lambda \cap \mathscr{H}_{\bar{\Lambda}}$
(ii) $x \in \mathscr{H}_\Lambda \cap \mathscr{H}_{\bar{\Lambda}}^c$
(iii) $x \in \mathscr{H}_\Lambda^c \cap \mathscr{H}_{\bar{\Lambda}}$

and

(iv) $x \in \mathscr{H}_\Lambda^c \cap \mathscr{H}_{\bar{\Lambda}}^c$

Case (i): That (C) holds in case (i) is the content of Lemma 4.
Case (ii):

Lemma 10. *Suppose* $\Lambda \subset \bar{\Lambda} \subset \Lambda_m$. *Then*

$$P_\Lambda^{\Lambda_m} \subset P_{\bar{\Lambda}}^{\Lambda_m}. \tag{14}$$

Proof.

$$\sum_{y' \in E_{\bar{\Lambda}}} p_{\Lambda_m}(y'\, x_{\Lambda_m - \bar{\Lambda}}, x) \geqq \sum_{\substack{y' \in E_\Lambda \\ V}} p_{\Lambda_m}(y'\, x_{\Lambda_m - \Lambda}, x).$$

Therefore $x \in P_\Lambda^{\Lambda_m} \Rightarrow x \in P_{\bar{\Lambda}}^{\Lambda_m}$.

Lemma 11. $\mathscr{H}_\Lambda \subset \mathscr{H}_{\bar{\Lambda}}$.

Proof. Follows immediately from Lemma 10.

That case (ii) presents no problem follows from Lemma 11, which implies that $\mathscr{H}_\Lambda \cap \mathscr{H}_{\bar{\Lambda}}^c = \phi$.

Case (iii):

Definition 12. For $x \in P_{\Lambda}^{\Lambda_m}$ and $y \in E_{\Lambda}$, we define

$$p_{\Lambda}^{\Lambda_m}(y, x) = \frac{p_{\Lambda_m}(y\, x_{\Lambda_m - \Lambda}, x)}{\sum_{y' \in E_{\Lambda}} p_{\Lambda_m}(y'\, x_{\Lambda_m - \Lambda}, x)}$$

Lemma 13. If $x \in P_{\Lambda}^{\Lambda_m}$, then

$$p_{\Lambda}(y, x) = p_{\Lambda}^{\Lambda_m}(y, x). \tag{15}$$

Proof. Passing to the limit in Equation (13) leads directly to (15).

Lemma 14. Suppose $\Lambda \subset \bar{\Lambda} \in L$ and $x \in \mathscr{H}_{\Lambda}^c \cap \mathscr{H}_{\bar{\Lambda}}$. Then

$$\sum_{y' \in E_{\bar{\Lambda}}} p_{\bar{\Lambda}}(y'\, x_{\bar{\Lambda} - \Lambda}, x) = 0. \tag{16}$$

Proof. $x \in \mathscr{H}_{\Lambda}^c \cap \mathscr{H}_{\bar{\Lambda}} \Rightarrow$ there exists an m such that $\bar{\Lambda} \subset \Lambda_m$, $x \in P_{\bar{\Lambda}}^{\Lambda_m}$, and

$$\sum_{y' \in E_{\Lambda}} p_{\Lambda_m}(y'\, x_{\Lambda_m - \Lambda}, x) = 0. \tag{17}$$

For this m,

$$p_{\bar{\Lambda}}(\cdot, x) = p_{\bar{\Lambda}}^{\Lambda_m}(\cdot, x) \qquad \text{(Lemma 13)} \tag{18}$$

and

$$\sum_{y' \in E_{\Lambda}} p_{\bar{\Lambda}}^{\Lambda_m}(y'\, x_{\bar{\Lambda} - \Lambda}, x) = 0. \tag{19}$$

(from Eq. (17) and Definition 12). Combining (18) and (19) gives (16).

Case (iii) thus presents no problem: Because of Lemma 14, (C) is satisfied by letting $\phi(x) = 0$ whenever $x \in (\mathscr{H}_{\Lambda}^c \cap \mathscr{H}_{\bar{\Lambda}}) \in \mathscr{F}_{\Lambda}$.

Case (iv): In this case

$$\begin{aligned} \pi_{\bar{\Lambda}}(y\, x_{\bar{\Lambda} - \Lambda}, x) &= \Big(\mathop{\times}_{\alpha \in \bar{\Lambda} - \Lambda} v(x_{\alpha})\Big)\Big(\mathop{\times}_{\alpha \in \Lambda} v(y_{\alpha})\Big) \\ &= \phi(x)\, \pi_{\Lambda}(y, x). \end{aligned}$$

This completes the proof.

Very briefly, the idea of the construction of π is as follows. G_{Λ} is a set on which it would seem that $\pi_{\Lambda}(\cdot, y)$ may be unambiguously defined by the limit in (8). However G_{Λ} is too large. Configurations x in G_{Λ} which are not in $\mathscr{H}_{\Lambda}$ are such that the "boundary" x_{Λ^c} is "atypical", so that the definition of $\pi_{\Lambda}(\cdot, x)$ is not really determined by the random field. For such configurations x we define $\{\pi_{\Lambda}(\cdot, x)\}$ in such a way that it corresponds to an independent random field, so that consistency is easily satisfied.

Several open problems remain:

(a) Does the theorem remain valid if the E_{α} are allowed to be countably infinite?

(b) Does the theorem remain valid if $E_\alpha = \mathbb{R}$, for all $\alpha \in S$? This requires some explanation, since for this case the definition of specification must be reformulated. Suppose $\{\pi_\Lambda\}$ is a specification over $(S, \{E_\alpha\})$ with E_α finite (or countable). Let

$$\pi_\Lambda(A, x) = \sum_{y \in A} \pi_\Lambda(y, x)$$

for any $A \subset E_\Lambda$, $x \in E$.

Then (3)–(6) may be replaced by

(i) For fixed $x \in E$, $\pi_\Lambda(\cdot, x)$ is a probability measure on E_Λ.
(ii) For fixed $A \subset E_\Lambda$, $\pi_\Lambda(A, \cdot) \in \mathscr{F}_\Lambda$.
(iii) For any $f \in \mathscr{B}_S$[4], $\Lambda \subset \bar{\Lambda}$ and $x \in E$

$$\int \pi_{\bar{\Lambda}}(d\bar{y}, x) f(\bar{y}\, x_{\Lambda^c}) = \int \pi_{\bar{\Lambda}}(d\bar{z}, x)\left(\int \pi_\Lambda(dy, \bar{z}\, x_{\bar{\Lambda}^c}) f(y\, \bar{z}_{\bar{\Lambda}-\Lambda}\, x_{\bar{\Lambda}^c})\right).$$

(i)–(iii) may be used to define the concept of a specification for $E_\alpha = \mathbb{R}$, if we regard E_Λ as equipped with the Borel sets $\mathscr{B}_\Lambda$ (so that in (ii) we replace $A \subset E_\Lambda$ by $A \in \mathscr{B}_\Lambda$. The conclusion of the desired theorem would be of the form: Then there exists a specification π such that for $\Lambda \in L$ and $A \in \mathscr{B}_\Lambda$,

$$\operatorname{Prob}\{X_\Lambda \in A \mid \mathscr{F}_\Lambda\}(x) = \pi_\Lambda(A, x) \qquad \text{for } \mu \text{ a.e. } x.$$

(c) The most general setting is perhaps the following:

L is any directed set, and $\{\mathscr{F}_\Lambda \mid \Lambda \in L\}$ is a decreasing collection of sub σ-algebras of $\mathscr{B}$ ($(\Omega, \mathscr{B}, \mu)$ is the underlying probability space) indexed by L – for $\Lambda \leqq \bar{\Lambda}$, $\mathscr{F}_{\bar{\Lambda}} \subset \mathscr{F}_\Lambda$. A specification will be an indexed collection $\pi = \{\pi_\Lambda(A, \omega) \mid \Lambda \in L, A \in \mathscr{B}, \omega \in \Omega\}$ such that

(i) For fixed $\omega \in \Omega$, $\pi_\Lambda(\cdot, \omega)$ is a probability measure on $(\Omega, \mathscr{B})$.
(ii) For $f \in \mathscr{F}_\Lambda$, $\int \pi_\Lambda(d\omega', \omega) f(\omega') = f(\omega)$, and for $A \in \mathscr{B}$, $\pi_\Lambda(A, \cdot) \in \mathscr{F}_\Lambda$.
(iii) For any $f \in \mathscr{B}$, $\Lambda \leqq \bar{\Lambda}$, and $\omega \in \Omega$,

$$\int \pi_{\bar{\Lambda}}(d\omega', \omega) f(\omega') = \int \pi_\Lambda(d\omega', \omega)\left(\int \pi_\Lambda(d\omega'', \omega') f(\omega'')\right).$$

And here, the desired conclusion is of the form: Then there exists a specification π such that for $\Lambda \in L$ and $f \in \mathscr{B}$

$$E_\mu(f \mid \mathscr{F}_\Lambda)(\omega) = \int \pi_\Lambda(d\omega', \omega) f(\omega').$$

(π thus defines a system of regular conditional probabilities for the $\mathscr{F}_\Lambda$'s.)

These problems, at least (b) and (c), require the use of more sophisticated methods than have been used here. In particular, the fact that for decent measure spaces conditional expectations are given by regular conditional probabilities would get things started. It should also be clear that the theorem is certainly false if complete generality is allowed in (c), since at the very least the theorem would require that the conditional expectation given $\mathscr{F}_\Lambda$ be given by regular conditional probabilities, which is not true with complete generality. For (c), the following additional assumptions seem reasonable:

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[4] $\mathscr{B}_S$ is the natural product σ-algebra on E.

(1) $(\Omega, \mathscr{B})$ is a standard Borel space[5]

(2) L is countably final.

(2) means that there exists a sequence $\Lambda_n \in L$ such that for any $\Lambda \in L$, $\Lambda \leqq \Lambda_n$ for some n.

A problem related to the one investigated here is that of determining which random fields are Gibbs states for some energy h which is defined in terms of a potential [3]. This problem has been investigated by Kozlov [4], Sullivan [5], and Averintsev [6]. Such a random field it is clear, should have (i) strictly positive conditional probabilities which (ii) depend in some sense continuously on the boundary x_{Λ^c}. (i) and (ii) greatly simplify the problem of finding a specification.

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[5] Measurable spaces which arise naturally are almost always standard Borel spaces.