Z. Wahrscheinlichkeitstheorie verw. Gebiete
50, 97 – 118 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete


# Weakly Coupled Gibbs Measures*

Garrett S. Sylvester**†
Department of Mathematics, Rockefeller University, N.Y., N.Y. 10021, USA

Dedicated to Professor Leopold Schmetterer

**Summary.** We formulate an abstract functional-analytic framework for the study of Gibbs measures on infinite product spaces. Working in this framework, we present a detailed analysis of the weak-coupling regime. Specifically, we derive general theorems on existence of the Gibbs measure, analyticity in its component Gibbs factors, and exponential decay of correlations and truncated expectations in the spread of distant families of random variables. In translation-invariant situations we obtain a central limit theorem. Our main tool is a series expansion in truncated expectations, which we analyze with $L^p$ methods.

## Section 1: Introduction

On infinite product spaces $\prod_{i \in \mathscr{L}} X_i$, we study a class of non-product probability measures. These measures are the Gibbs measures, which arise in the classical statistical mechanics of crystals. A Gibbs measure $\mu$ differs from a product measure $d\nu = \prod_{i \in \mathscr{L}} d\nu_i$ by an infinite product of coupling factors $g_E(x)$, the Gibbs factors, according to the heuristic formula

$$d\mu = \frac{1}{Z} \cdot \prod_{E \subset \mathscr{L}} g_E(x) \cdot d\nu. \tag{1.1}$$

Here the function $g_E(x)$ depends only on the variables labelled by the finite subset $E \subset \mathscr{L}$, and $Z$ is a normalization factor. We impose the geometric restriction

$$\sup_E |\{E' : E' \cap E \neq \emptyset \,\&\, g_E, g_{E'} \not\equiv 1\}| < \infty \tag{1.2}$$

* Original title: Analyticity and Decay of Correlations in Weakly Coupled Lattice Models.

** Supported by N.S.F. Grant PHY76-17191

† Present address: Dept. of Mathematics, Oklahoma St. Univ., Stillwater, Ok. 74074, USA

0044-3719/79/0050/0097/$04.40

in order to make the product in (1.1) locally well-defined in $\mathscr{L}$. (The absolute value $|\;|$ in (1.2) denotes set cardinality.)

The weak coupling hypothesis is that the Gibbs factors are all close to one in a suitable norm:

$$\|g_E-1\|<\delta \qquad \forall E. \tag{1.3}$$

However, in typical applications we also find that

$$|g_E(x)-1|\geqq\varepsilon(x)>0 \tag{1.4}$$

for an infinite number of $E$'s, causing the full infinite product $\prod_{E\subset\mathscr{L}} g_E(x)$ in (1.1) to diverge. To obviate this problem and give a rigorous meaning to the heuristic formula (1.1), we construct $\mu$ as a limit of measures defined on increasingly large finite product spaces, where (1.1) has a direct meaning. The weak coupling hypothesis (1.3) in tandem with the geometric condition (1.2) will enable us to perform this limiting procedure and analyze the result.

In the weak coupling region, one anticipates that the Gibbs measure will be well-behaved and amenable to detailed analysis. This has been established in many examples [13]. However, the results obtained for these examples generally depend on further properties of the model in question, such as compactness of the $X_i$ [5], or restrictions on the form of the coupling factors $g_E$. In this paper we set up and investigate a unifying functional-analytic framework for the study of Gibbs measures which reflects the basic mathematical structure of the problem while suppressing details irrelevant in the weak-coupling regime. With $L^p$ analysis as our basic technique, we derive a number of theorems in this abstract framework which confirm the expected behaviour of the measure.

We describe our results in greater detail. In Sect. 2 we collect some relevant terminology, and formulate the framework in which we study Gibbs measures. This framework admits physical models having arbitrary many-body interactions, not necessarily translation-invariant but essentially finite-range, with spin variables in an arbitrary probability space $X_i$. It also includes models on unphysical lattices such as Cayley trees. Section 3 is the technical heart of the paper. In it we obtain uniform estimates for the approximate measures on finite product spaces $\prod_{\alpha=1}^{N} X_{i_\alpha}$ which are used in the limiting procedure to construct the Gibbs measure. The estimates concern truncated expectations (i.e. cumulants, or semi-invariants), and express the exponential decay of correlations between factor spaces $X_i, X_j$ with widely separated indices $i, j$. The method we use to derive these bounds is a series expansion. While similar in spirit to other expansions in statistical mechanics, our method makes use of novel estimates on truncated expectations with respect to product measures in order to demonstrate convergence (Lemma 3.1; this lemma also makes unexpected contact with graph-coloring and finite-geometry problems). In Sect. 4 we use the uniform weak-coupling bounds of Sect. 3 to prove that the approximate measures on finite product spaces converge to a limiting Gibbs measure on the infinite-product space $\prod_{i\in\mathscr{L}} X_i$ having strong regularity properties. Specifically, we show that the limit is approached uniformly over the small-coupling region, that it is analytic in the Gibbs factors when they are regarded as elements of suitable Banach spaces, that correlations and truncated expectations decay exponentially in the separation of distant variables, and that the central limit theorem holds.

Some of the problems we study in this paper have been simultaneously attacked by other workers [6, 10, 17], who obtain results of somewhat the same nature as ours.

## Section 2: Terminology

In this section we review some useful terminology from graph theory and analysis. We then define lattice models, the functional-analytic structures in which we study Gibbs measures.

In graph theory we largely follow the definitions of [1], some of which we recall now. Let $\mathscr{L}$ be a set, not necessarily finite. A hypergraph $\mathscr{G}$ on $\mathscr{L}$ is a family of finite nonempty subsets of $\mathscr{L}$. Although $\mathscr{G}$ may have repeated elements, by an abuse of notation we shall use set-theoretic terminology in connection with $\mathscr{G}$. The members $i \in \mathscr{L}$ are *vertices*; the members $E \in \mathscr{G}$, *edges*. An alternate notation for the set of vertices $\mathscr{L}$ of $\mathscr{G}$ is $V\mathscr{G}$. A *subhypergraph* $\Lambda$ of $\mathscr{G}$ is a subset $\Lambda \subset \mathscr{G}$. If $\mathscr{L}_1 \subset \mathscr{L}$, the restriction $\mathscr{G} \upharpoonright \mathscr{L}_1$ is

$$\mathscr{G} \upharpoonright \mathscr{L}_1 = \{E \in \mathscr{G} : E \subset \mathscr{L}_1\}. \tag{2.1}$$

The *degree* $d'_{\mathscr{G}}(A)$ of any subset $A \subset \mathscr{L}$ is

$$d'_{\mathscr{G}}(A) = |\{E \in \mathscr{G} : E \cap A \neq \emptyset\}|. \tag{2.2}$$

(We use the notation $|\ |$ for set cardinality.) However, if $E$ is an edge $E \in \mathscr{G}$ it is convenient to modify (2.2) slightly:

$$d_{\mathscr{G}}(E) = d'_{\mathscr{G}}(E) - 1 = |\{F \in \mathscr{G} : F \cap E \neq \emptyset \,\&\, F \neq E\}|. \tag{2.3}$$

We drop both the prime and the subscript when the intended degree is clear from context, and we write $d'_{\mathscr{G}}(i)$ for $d'_{\mathscr{G}}(\{i\})$, $i \in \mathscr{L}$. The overall *degree* $d_{\mathscr{G}}$ of $\mathscr{G}$ is

$$d_{\mathscr{G}} = \sup_{E \in \mathscr{G}} d_{\mathscr{G}}(E). \tag{2.4}$$

A *path* $\gamma$ in $\mathscr{G}$ is a finite sequence of edges $E_1, E_2, \ldots$ such that $\forall j$,

$$E_j \cap E_{j+1} \neq \emptyset.$$

$\mathscr{G}$ decomposes into *path components* in the usual way: two edges lie in the same component if and only if there is a path beginning at one and ending at the other. A *connected hypergraph* is one with a single path component. The *spread* $\rho_{\mathscr{G}}(\{A_j\})$ of a family of finite subsets $A_j \subset \mathscr{L}$, $j \in J$ is the smallest number of edges $E_k \in \mathscr{G}$, $k \in K$, such that the hypergraph with edges $\{A_j : j \in J\} \cup \{E_k : k \in K\}$ is connected:

$$\rho_{\mathscr{G}}(A_1, A_2, \ldots) = \rho_{\mathscr{G}}(\{A_j\}) = \inf\{|\Lambda| : \Lambda \cup \{A_j : j \in J\} \text{ is connected}\}. \tag{2.5}$$

The spread is a *metric* on pairs of vertices, though not on pairs of edges. The *line graph* (or representative graph) $L(\mathscr{G})$ is the true graph which has as vertices the edges of $\mathscr{G}$ and which joins two vertices $e, f \in VL(\mathscr{G})$ by an edge if and only if when $e, f$ are regarded as edges $E, F \in \mathscr{G}$ they overlap: $E \cap F \neq \emptyset$.

We shall occasionally omit the prefix "hyper" when it is clear that the graph in question is a hypergraph.

We next summarize three notions from analysis: truncated expectations, analytic complex-valued functions on a Banach space, and $L^p$ spaces with vector weights $p$.

Let $\{X_i\}_{i\in\{1,\dots,n\}}$ be a family of $n$ random variables on some probability space, and denote the expectation integral of this space by $\mathfrak{E}$. One may define the truncated expectation $u(x_1, \dots, x_n)$ of the family $\{x_i\}_{1\leqq i \leqq n}$ by means of a formal generating function as

$$u(x_1, \dots, x_n) = u(\{x_i\}) = \frac{\partial^n}{\partial\lambda_1 \partial\lambda_2 \dots \partial\lambda_n} \log \mathfrak{E}\left(\exp\left[\sum_{i=1}^{n} \lambda_i x_i\right]\right)\Bigg|_{\lambda=0}. \tag{2.6}$$

Truncated expectations are also called cumulants, semi-invariants, connected expectations, and Ursell functions. Notice that if $\{x_i\}_{1\leqq i\leqq n}$ and $\{y_j\}_{1\leqq j \leqq m}$ are two families of random variables which are independent of each other, then

$$u(x_1, \dots x_n, y_1, \dots, y_m) = 0 \tag{2.7}$$

because the expectation in (2.6) factors. One may also define the truncated expectation recursively:

$$\mathfrak{E}(x_1 x_2 \dots x_n) = \sum_{\mathfrak{P}} \prod_{P\in\mathfrak{P}} u(\{x_i\colon i \in P\}). \tag{2.8}$$

Here $\mathfrak{P}$ is an arbitrary partition of $\{1, \dots, n\}$, and $P \in \mathfrak{P}$ is a typical block in $\mathfrak{P}$. Observe that if we isolate the blocks $P_1 \in \mathfrak{P}$ with $1 \in P_1$ and resum over the remaining blocks, we find

$$\mathfrak{E}(x_1 x_2 \dots x_n) = \sum_{i\in P_1 \subset \{1,\dots,n\}} u(\{x_i\colon i \in P_1\})\, \mathfrak{E}(\prod_{j\notin P_1} x_j). \tag{2.9}$$

This expansion plays a key role in the analysis of Sect. 3. Finally, one may give an explicit formula for $u(x_1, \dots, x_n)$:

$$u(x_1, \dots, x_n) = \sum_{\mathfrak{P}} (-1)^{|\mathfrak{P}|-1}(|\mathfrak{P}|-1)! \prod_{P\in\mathfrak{P}} \mathfrak{E}(\prod_{i\in P} x_i), \tag{2.10}$$

where again $\mathfrak{P}$ is a partition of $\{1, \dots, n\}$. As we shall discuss in Sect. 3, (2.10) actually follows from (2.8) by a Möbius inversion.

We next turn to consideration of analytic functions on Banach spaces. Let $Y$ be a Banach space over $C$, and let $U \subset Y$ be open. Essentially, a map $f\colon U \to C$ is analytic if it has a convergent Taylor expansion about every point in $U$. Formally, $f$ is analytic at $x_0 \in U$ if and only if there are continuous multilinear forms $\phi_n\colon \prod_1^n Y \to C$ such that the series

$$\sum_{n=0}^{\infty} \phi_n(x-x_0, x-x_0, \ldots)$$

converges uniformly to $f$ in some ball $B_r(x_0)=\{x\in Y\colon \|x-x_0\|\leqq r\}$ of positive radius $r$. A function is analytic over $U$ if it is analytic at every point $x_0\in U$. Reference [9] and further works cited therein set forth the elementary properties of analytic functions on Banach spaces.

We conclude our discussion of terminology with some comments on $L^p$ spaces, over product measures, that have vector weights $p$ (briefly, vector $L^p$ spaces). Let $(X_i, \mathfrak{B}_i, \nu_i)_{i\in E}$ be a family of probability spaces indexed by the finite set $E$, $|E|=n$, and let the vector weight $p$ be a member of $\prod_{i\in E}[1,\infty]$. If $f\colon \prod_E X_i \to \mathbb{C}$ is measurable with respect to the product $\sigma$-algebra $\prod_E \mathfrak{B}_i$, set

$$\|f\|_p=\Big[\int_{X_{i_1}} \ldots \Big[\int_{X_{i_{n-1}}} \Big[\int_{X_{i_n}} (f)^{p_{i_n}} d\nu_{i_n}\Big]^{p_{i_{n-1}}/p_{i_n}} d\nu_{i_{n-1}}\Big]^{p_{i_{n-2}}/p_{i_{n-1}}} \ldots d\nu_{n_i}\Big]^{1/p_{i_1}} \tag{2.11}$$

where the ordering $p_{i_1}, p_{i_2}, \ldots, p_{i_n}$ of the $n$ components of $p$ is chosen such that

$$p_{i_1} \leqq p_{i_2} \leqq \ldots \leqq p_{i_n}. \tag{2.12}$$

It is easy to see that $\|\ \|_p$ is a norm, and we take $L^p_C(\prod_E X_i, \prod_E \nu_i)$ to be the (equivalence classes of measurable) functions $f\colon \prod_E X_i \to C$ with $\|f\|_p<\infty$. An ordering of the components of $p$ which differs from (2.12) also gives rise to a norm, which in general is (possibly strictly) less than $\|\ \|_p$. The following proposition, which we state without proof, summarizes some elementary properties of vector $L^p$ spaces.

**Proposition 2.1.** *Let $(X_i, \mathfrak{B}_i, \nu_i)_{i\in E}$ be a family of probability spaces indexed by the finite set $E$. Let $p, p', q, r\in\prod_{i\in E}[1,\infty]$. Then:*

(a) *If all components $p_i$ are equal to a common value $p_0\in[1,\infty]$,*

$$L^p(\prod_E X_i) = L^{p_0}(\prod_E X_i). \tag{2.13}$$

*where $L^{p_0}$ is defined in the usual manner.*

(b) *If $f\in L^p(\prod_E X_i)$ is a product*

$$f=f_A f_{A'}, \qquad A\subset E \,\&\, A'=E-A$$

*where $f_A$ (resp. $f_{A'}$) depends only on those variables indexed by $A$ (resp. $A'$), then*

$$\|f\|_p = \|f_A\|_{p_A} \|f_{A'}\|_{p_{A'}}. \tag{2.14}$$

*Here $p_A$ (resp. $p_{A'}$) is the restriction of $p$ to $A$ (resp. $A'$).*

(c) *If $f\in L^p(\prod_E X_i)$, $g\in L^q(\prod_E X_i)$, then $f\cdot g\in L^r(\prod_{i\in E} X_i)$ where $1/p+1/q=1/r$ (componentwise). Further,*

$$\|f\cdot g\|_r \leqq \|f\|_p \cdot \|g\|_q \qquad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}. \tag{2.15}$$

(d) *Define* $p^*$ *by* $1/p+1/p^*=1$ *(componentwise). Then*

$$L^{p^*}(\prod_E X_i) \subset L^p(\prod_E X_i)^*, \quad \frac{1}{p}+\frac{1}{p^*}=1. \tag{2.16}$$

(e) *If* $p \leqq p'$ *(componentwise), then*

$$L^p(\prod_E X_i) \supset L^{p'}(\prod_E X_i). \tag{2.17}$$

We remark that (e) is the only part of Proposition 2.1 dependent on the fact that the measures $\nu_i$ are normalized, and that the containment in (2.16) may in general be strict even if all components of $p$ are finite.

We close this section by defining a lattice model and its Gibbs measure. A lattice model $\mathfrak{M}$ consists of:

(1) a denumerable set $\mathscr{L}$;
(2) a family of probability spaces $(X_i, \mathfrak{B}_i, \nu_i)_{i\in\mathscr{L}}$ indexed by $\mathscr{L}$;
(3) a hypergraph $\mathscr{G}$ on $\mathscr{L}$ of finite degree $d_{\mathscr{G}} < \infty$; and,
(4) a family of weights $\{p_E\}_{E\in\mathscr{G}}$, $p_E \in \prod_{i\in E} [1,\infty]$, indexed by $\mathscr{G}$ which is conformable for integration.

(Conformability for integration means that

$$\sum_{\{E\in\mathscr{G}: i\in E\}} (p_{E,i})^{-1} \leqq 1 \qquad \forall i \tag{2.18}$$

where $p_{E,i}$ is the $i^{\text{th}}$ component of $p_E$.) We make the additional technical assumption that every vertex $i\in\mathscr{L}$ is covered by at least two edges of $\mathscr{G}$. This assumption simplifies the statement of several estimates that would otherwise require a bound on the edge size $|E|$, $E\in\mathscr{G}$. Since the Gibbs measure of a lattice model factors over the path components of its hypergraph we may also suppose that $\mathscr{G}$ is connected with no loss of generality.

The Gibbs factors in a lattice model $\mathfrak{M}$ are a family of complex functions $g_E \in L_C^{p_E}(\prod_E X_i, \prod_E \nu_i)$, $E\in\mathscr{G}$. If $\Lambda \subset \mathscr{G}$ is a finite subhypergraph then since the weights $\{p_E\}_{E\in\mathscr{G}}$ are conformable for integration we have $\prod_{E\in\Lambda} g_E \in L^1(\prod_{\mathscr{L}} X_i, \prod_{\mathscr{L}} \nu_i)$, with

$$\|\prod_{E\in\Lambda} g_E\|_1 \leqq \prod_{E\in\Lambda} \|g_E\|_{p_E} \tag{2.19}$$

by Proposition 2.1c.

In the lattice models of primary physical interest, the set $\mathscr{L}$ is $Z^N$ for some $N$ and the remaining structure of the model – probability spaces, hypergraph, integration weights, and Gibbs factors – is invariant under translation in $Z^N$. Moreover, the Gibbs factors are nonnegative. Translation invariance means explicitly that: all probability spaces are the same

$((X_i, \mathfrak{B}_i, \nu_i) = (X, \mathfrak{B}, \nu)\ \forall i \in Z^N)$; $E \in \mathscr{G}$ if and only if $E + i \in \mathscr{G}\ \forall i \in Z^N$, $E \in \mathscr{G}$, where $E + i$ is the translate of $E$ by $i$; $p_E = p_{E+i}\ \forall i \in Z^N$, $E \in \mathscr{G}$; and

$$g_{E+i}(x_{e_1+i}, x_{e_2+i}, \dots) = g_E(x_{e_1}, x_{e_2}, \dots) \qquad \forall i \in Z^N,\ E \in \mathscr{G} \tag{2.20}$$

where $e_1, e_2, \dots$ are the elements of $E$. In a translation-invariant model we may select a (minimal) representative set of edges which generates all other edges by translation. We call such a set fundamental. Notice that a translation-invariant hypergraph $\mathscr{G}$ of finite degree is necessarily finite-range:

$$\sup_{E \in \mathscr{G}} \operatorname{diam}(E) < \infty \tag{2.21}$$

where $\operatorname{diam}(E) \equiv \sup_{i,j \in E} |i - j|_{\mathbb{R}^N}$. However, (2.21) need not hold without the invariance.

The Gibbs measure $\mu$ of an arbitrary lattice model $\mathfrak{M}$ is the measure on $(\prod_{i \in \mathscr{L}} X_i, \prod_{i \in \mathscr{L}} \mathfrak{B}_i)$ given heuristically by the formula

$$d\mu = Z^{-1} \prod_{E \in \mathscr{G}} g_E\, d\nu \tag{2.22a}$$

where

$$Z^{-1} = \int_{\prod_{\mathscr{L}} X_i} \prod_{E \in \mathscr{G}} g_E\, d\nu \tag{2.22b}$$

and

$$d\nu = \prod_{i \in \mathscr{L}} d\nu_i . \tag{2.22c}$$

As discussed in Sect. 1, (2.22) normally is not rigorously meaningful, commonly as a result of translation invariance. We use a limiting process to circumvent this problem. Partially order the finite subhypergraphs $\Lambda \subset \mathscr{G}$ by containment, and adjoin a largest element $\infty$ to this partially ordered set. By (2.19), the partition function

$$Z(\Lambda) \equiv \int_{\prod X_i} \prod_{E \in \Lambda} g_E\, d\nu \tag{2.23}$$

of the subgraph $\Lambda$ is finite. Furthermore, if $Z(\Lambda) \neq 0$ the Gibbs measure

$$d\mu_\Lambda = Z(\Lambda)^{-1} \prod_{E \in \Lambda} g_E \cdot d\nu \tag{2.24}$$

for the subgraph $\Lambda$ is well-defined. We attempt to give rigorous meaning to the heuristic formula (2.22) by defining the full Gibbs measure $d\mu$ as the limit

$$d\mu = \lim_{\Lambda \to \infty} d\mu_\Lambda \tag{2.25}$$

provided it exists.

Convergence in (2.25) is a modified weak convergence that we now describe. Let $A \subset \mathscr{L}$ be a finite subset $|A| < \infty$, and let $r_A \in \prod_{i \in A} [1, \infty]$ be a weight for $A$ conformable for integration with the weights $\{p_E\}$ already assigned to $\mathscr{G}$. (Of course, the weight $r_A = \infty$ is always conformable.) When $Z(\Lambda) \neq 0$ and $f_A \in L^{r_A}(\prod_{i \in A} X_i, \prod_{i \in A} v_i)$, denote by $\mathfrak{E}(f_A)_\Lambda$ the expectation of $f_A$ with respect to $\mu_\Lambda$, and by $\mathfrak{E}(f_A)_0$ the expectation with respect to $\mu_\emptyset = v$. Thus we have

$$\mathfrak{E}(f_A)_\Lambda = \int f_A \, d\mu_\Lambda = \frac{\mathfrak{E}(f_A \cdot \prod_{E \in \Lambda} g_E)_0}{\mathfrak{E}(\prod_{E \in \Lambda} g_E)_0} = Z(\Lambda)^{-1} \int f_A \cdot \prod_\Lambda g_E \cdot dv. \tag{2.26}$$

Our primary objectives in this paper are to show that the limiting expectations

$$\mathfrak{E}(f_A)_\infty = \lim_{\Lambda \to \infty} \mathfrak{E}(f_A)_\Lambda \tag{2.27}$$

exist for all possible $f_A$, and, this established, to study the properties of the limit. One may recover a measure from the expectations by $C^*$-algebraic techniques, or in some cases, by more direct representation theorem arguments.

Physically, the Gibbs measures derive from consideration of a crystal in a heat bath. The set $\mathscr{L}$ labels the atoms of the crystal. The probability space $(X_i, \mathfrak{B}_i, v_i)$ represents some physical quantity associated with the atom at $i$ whose statistical behavior is under analysis, most commonly the (classical) spin. The Gibbs factor $g_E$ is $\exp(-\beta H_E)$, where $\beta$ is an inverse temperature parameter and $H_E$ is the pure $|E|$-body energy of the atoms in $E \subset \mathscr{L}$. Loosely speaking, if $B \subset \prod_{i \in \mathscr{L}} X_i$ then $\mu(B)$ is the probability that the configuration $x \in \prod_{\mathscr{L}} X_i$ of the crystal will lie in $B$ when the crystal is in thermal equilibrium with a heat bath at inverse temperature $\beta$.

## Section 3: Uniform Decay Estimates

In this section we use a series expansion to derive uniform bounds for decay of correlations in weakly coupled lattice models. We first describe the expansion, and indicate the ideas employed to control it.

Let $\mathfrak{M}$ be a lattice model, with vertices $\mathscr{L}$, probability spaces $(X_i, \mathfrak{B}_i, v_i)_{i \in \mathscr{L}}$, hypergraph $\mathscr{G}$ on $\mathscr{L}$, integration weights $\{p_E\}_{E \in \mathscr{G}}$, and Gibbs factors $g_E \in L^{p_E}$, $E \in \mathscr{G}$. (See Sect. 2 for definitions.) Let $A \subset \mathscr{L}$, $|A| < \infty$, and choose a weight $r_A$ on $A$ conformable for integration with the weights $\{p_E\}$. Let $\Lambda$ be a finite subhypergraph of $\mathscr{G}$ with $A \subset V\Lambda$. By (2.9), the expectation $\mathfrak{E}(\cdot)_0$ (with respect to the product measure $\Pi v_i$) of $f_A \cdot \prod_{E \in \Lambda} g_E$ has the expansion

$$\mathfrak{E}(f_A \cdot \prod_\Lambda g_E)_0 = \sum_{\Gamma \subset \Lambda} u(f_A, \{g_E\}_{E \in \Gamma})_0 \cdot \mathfrak{E}(\prod_{F \in \Lambda - \Gamma} g_F)_0. \tag{3.1}$$

Although the sum in (3.1) is over all subhypergraphs $\Gamma\subset\Lambda$, by (2.7) only those graphs $\Gamma$ such that $\Gamma\cup\{A\}$ is connected make a nonvanishing contribution. (By $\Gamma\cup\{A\}$, we mean the graph obtained from $\Gamma$ by adding the edge $A$.) We separate out the $\Gamma=\emptyset$ term and divide through formally by the normalization factor $Z(\Lambda)\equiv\mathfrak{E}(\prod_{\Lambda} g_E)_0$ to obtain

$$\mathfrak{E}(f_A)_\Lambda-\mathfrak{E}(f_A)_0=\sum_{\Gamma\subset\Lambda} u(f_A,\{g_E\}_{E\in\Gamma})_0\cdot\frac{Z(\Lambda-\Gamma)}{Z(\Lambda)} \tag{3.2}$$

$\Gamma\neq\emptyset,\quad \Gamma\cup\{A\}$ connected.

Equation (3.2) for the Gibbs expectation $\mathfrak{E}(f_A)_\Lambda\equiv Z(\Lambda)^{-1}\int f_A\prod_{\Lambda} g_E\,dv$ is our basic expansion for a simple expectation. We next loosely sketch the ideas used to control it, and later make the modifications appropriate for truncated expectations.

Convergence in (3.2) is derived from the factors $u(f_A,\{g_E\}_{E\in\Gamma})_0$. They would vanish identically if the random variables $f_A$, $\{g_E\}_{E\in\Gamma}$ could be partitioned into mutually independent families. Although the connectedness of $\Gamma\cup\{A\}$ prevents such a partition, the condition $d_{\mathscr{G}}<\infty$ ensures that a significant subset of these random variables does indeed split into independent families. Lemma 3.1 exploits this idea to show that $|u(f_A,\{g_E\}_{E\in\Gamma})_0|$ is bounded above by $e^{-K_1|\Gamma|}$ where $K_1$ becomes arbitrarily large as the Gibbs factors $g_E$ approach 1. Lemma 3.2 next given a bound $e^{K_2|\Gamma|}$ on the number of subgraphs $\Gamma$ such that $\Gamma\cup\{A\}$ is connected which have a fixed value of $|\Gamma|$. The remaining factor $Z(\Lambda-\Gamma)/Z(\Lambda)$ is controlled inductively in Lemma 3.3 with an upper bound $e^{K_3|\Gamma|}$. ($K_3>0$ becomes small as the Gibbs factors tend to 1.) Combining these three estimates, we find

$$|\mathfrak{E}(f_A)_\Lambda-\mathfrak{E}(f_A)_0|\leqq\text{const}\sum_{|\Gamma|=1}^{\infty} e^{-(K_1-K_2-K_3)|\Gamma|}, \tag{3.3}$$

with the right-hand side of (3.3) tending to zero as the Gibbs factors approach one. This is the prototype of our key bound.

We may apply the same argument to derive uniform decay estimates for truncated Gibbs expectations $u(f_{A_1},f_{A_2},\dots,f_{A_n})_\Lambda$ after using the method of duplicate variables to write a truncated expectation as an ordinary expectation on a larger space. We briefly review this combinatoric device, which is set forth in detail in [14, 15].

Let $\{X_i: i=1,\dots n\}$ be a family of random variables on some probability space, and let $X_i^\alpha$, $\alpha\in\{1,\dots,n\}$ be $n$ independent and identically distributed copies of the original family $\{X_i\}$. Let $\omega$ be a primitive $n^{\text{th}}$ root of unity, and define

$$\tilde{X}_i=\sum_{\alpha=1}^{n}\omega^\alpha X_i^\alpha. \tag{3.4}$$

One may show [14, 15] that

$$u(X_1,\dots,X_n)=\frac{1}{n}\mathfrak{E}(\tilde{X}_1\tilde{X}_2\dots\tilde{X}_n). \tag{3.5}$$

In our present framework, (3.5) relates the truncated Gibbs expectation of a family of $n$ random variables in a lattice model $\mathfrak{M}$ to the ordinary Gibbs expectation of a product in an enlarged lattice $\mathfrak{M}^n$ obtained from $\mathfrak{M}$ by replacing each probability space $(X_i, \mathfrak{B}_i, \nu_i)$ with its $n$-fold product $\prod_{\alpha=1}^{n} (X_i, \nu_i)$. We find

$$u(\{f_{A_i}\})_A = \frac{1}{n} \mathfrak{E}(\tilde{f}_{A_1} \dots \tilde{f}_{A_n})_A = \frac{1}{n} \frac{\mathfrak{E}(\tilde{f}_{A_1} \dots \tilde{f}_{A_n} \cdot \prod_{E \in A} G_E)_0}{\mathfrak{E}(\prod_{E \in A} G_E)_0}. \tag{3.6}$$

Here $G_E$ is the product $G_E = \prod_{\alpha=1}^{n} g_E^{\alpha}$ of the $n$ copies in $\mathfrak{M}^n$ of our original Gibbs factor $g_E$ in $\mathfrak{M}$ and of course $\tilde{f}_{A_i}$ is defined by (3.4).

We apply expansion (3.2) to the quotient in (3.6). It follows from the methods of [14, 15] that

$$u(\prod_1^n \tilde{f}_{A_i} \{G_E\}_{E \in \Gamma})_0 = u(\tilde{f}_{A_1}, \tilde{f}_{A_2}, \dots, \tilde{f}_{A_n}, \{G_E\}_{E \in \Gamma})_0. \tag{3.7}$$

Consequently, nonzero contributions in (3.2) (as applied here) come from only those graphs $\Gamma \subset A$ such that the graph $\Gamma \cup \{A_i, i=1, \dots n\}$ obtained by adding to $\Gamma$ the $n$ edges $A_i$ is connected. Note that this is a more stringent requirement than mere connectedness of $\Gamma \cup \left\{ \bigcup_{i=1}^{n} A_i \right\}$ which is all one could infer without (3.7). With these preparations, we find that the truncated Gibbs expectation $u(f_{A_1}, \dots, f_{A_n})_A$ has the expansion

$$u(\{f_{A_i}\})_A - u(\{f_{A_i}\})_0 = \sum_{\Gamma \subset A} u(\tilde{f}_{A_1}, \dots, \tilde{f}_{A_n}, \{G_E\}_{E \in \Gamma})_0 \cdot \frac{Z(A-\Gamma)^n}{Z(A)^n},$$
$$\Gamma \neq \emptyset, \quad \Gamma \cup \{A_i \colon i=1, \dots n\} \text{ connected.} \tag{3.8}$$

(Here $Z(A-\Gamma)$ and $Z(\Gamma)$ are taken in the original model $\mathfrak{M}$.) Exponential decay follows from the bound (3.3) after noting that the first term appearing in the bounding series has $|\Gamma| = \rho_{\mathscr{G}}(\{A_i\})$, the spread in $\mathscr{G}$ of the family $\{A_i\}$.

We now implement the program just described.

**Lemma 3.1.** *Let $\mathfrak{M}$ be a finite lattice model with hypergraph $\mathfrak{H}$ having edges $E_1, E_2, \dots, E_m$ and integration weights $p_{E_i}$.*

*For any $m$ functions $F_i \in L^{p_{E_i}}$, the truncated expectation with respect to the product measure obeys the estimate*

$$|u(F_1, \dots, F_m)_0| \leqq 3^{\frac{1}{2}\sum d(E_i)} \prod_{i=1}^{m} \|F_i\|_{p_{E_i}} \tag{3.9a}$$

*where*

$$d(E_i) = |\{j \neq i \colon E_i \cap E_j \neq \emptyset\}|.$$

*Remark.* If $m \geqq 2$ and $c_1, c_2, \ldots c_m$ are constants, then by (2.7) $u(\{F_i - c_i\})_0 = u(\{F_i\})_0$. Thus (3.9a) may be replaced by

$$|u(F_1, \ldots F_m)_0| \leqq 3^{\frac{1}{2}\sum_i d(E_i)} \inf_{\{c_i\}} \prod_{i=1}^{m} \|F_i - c_i\|_{p_{E_i}}. \tag{3.9b}$$

*Proof.* The truncated expectation is a sum with coefficients of products of ordinary expectations. However, if the functions appearing in one of the ordinary expectations can be grouped into independent families, then this expectation factors, and the term containing it may be partially cancelled with a subsequent term in the sum. We derive the bound (3.9) by estimating these cancellations with the help of the combinatoric method of Möbius functions [2, 12].

We introduce some notation. Let $\coprod^m$ be the set of all partitions of $\{1, \ldots, m\}$, partially ordered by refinement: $\mathfrak{P} \leqq \mathfrak{Q}$ if and only if every set $Q \in \mathfrak{Q}$ is contained by some set $P \in \mathfrak{P}$. For $\mathfrak{P}, \mathfrak{Q} \in \coprod^m$ set

$$\mathfrak{E}_{\mathfrak{P}} = \prod_{P \in \mathfrak{P}} \mathfrak{E}(\prod_{i \in P} F_i)_0$$

$$u_{\mathfrak{Q}} = \prod_{Q \in \mathfrak{Q}} u(\{F_j : j \in Q\})_0. \tag{3.10}$$

Note in particular that when $\mathfrak{Q}$ is the maximal partition $1 = \{\{1, \ldots, m\}\}$, $u_1 = u(F_1, \ldots, F_m)_0$. It follows from (2.8) that

$$\mathfrak{E}_{\mathfrak{P}} = \sum_{\mathfrak{Q} \leqq \mathfrak{P}} u_{\mathfrak{Q}} \qquad \forall \mathfrak{P} \in \coprod^m, \tag{3.11}$$

so that by Möbius inversion,

$$u_{\mathfrak{Q}} = \sum_{\mathfrak{R} \leqq \mathfrak{Q}} \mathfrak{E}_{\mathfrak{R}} \cdot \mu_1(\mathfrak{R}, \mathfrak{Q}) \qquad \forall \mathfrak{Q} \in \coprod^m. \tag{3.12}$$

As a special case of (3.12), we have

$$u(F_1, \ldots, F_m)_0 \equiv u_1 = \sum_{\mathfrak{R} \leqq 1} \mathfrak{E}_{\mathfrak{R}} \cdot \mu_1(\mathfrak{R}, 1). \tag{3.13}$$

Here of course $\mu_1$ is the Möbius function of the partially ordered set $\coprod^m$.

To perform the cancellations in equations (3.12), we eliminate some redundancy in their antecedent equations (3.11). By (2.7), $u_{\mathfrak{Q}} \neq 0$ only if the subgraphs $\mathfrak{H}_Q = \{E_i : i \in Q\}$ are connected for all blocks $Q$ in the partition $\mathfrak{Q}$. Let $\coprod^m_{\mathfrak{H}} \subset \coprod^m$ be the set of all partitions which are so connected, with the induced ordering. Any partition $\mathfrak{P} \in \coprod^m$ has a unique maximal connected refinement $\mathfrak{P}_c \leqq \mathfrak{P}$, $\mathfrak{P}_c \in \coprod^m_{\mathfrak{H}}$ and one may readily see that the equations in (3.11) for all $\mathfrak{P}$ having the same $\mathfrak{P}_c$ are identical. Thus (3.11) reduces to a family of equations over the smaller partially ordered set $\coprod_{\mathfrak{H}}$ and inverting we find

$$u_{\mathfrak{Q}} = \sum_{\mathfrak{R} \leqq \mathfrak{Q}} \mathfrak{E}_{\mathfrak{R}} \cdot \mu_2(\mathfrak{R}, \mathfrak{Q}) \qquad \forall \mathfrak{R}, \mathfrak{Q} \in \coprod^m_{\mathfrak{H}}. \tag{3.14}$$

Here $\mu_2$ is the Möbius function of $\coprod_{\mathfrak{H}}^m$. The expectations $\mathfrak{E}_{\mathfrak{R}}$ in (3.14) do not factor further, so we now estimate the sum term by term.

We bound $|\mathfrak{E}_{\mathfrak{R}}|$ by $\prod_{i=1}^{m} \|F_i\|_{p_{E_i}}$ immediately. To control the Möbius coefficients $\mu_2$, it is convenient to embed $\coprod_{\mathfrak{H}}$ in the set of all subgraphs of the line graph $L(\mathfrak{H})$ (a true graph defined in Sect. 2). Define the map $\iota: \coprod_{\mathfrak{H}}^m \to 2^{L(\mathfrak{H})}$ by

$$\iota(\mathfrak{P}) = \bigcup_{P \in \mathfrak{P}} L(\mathfrak{H}) \upharpoonright P, \tag{3.15}$$

where we have identified a subset $P \subset \{1, \ldots, m\}$ with the vertices it labels in $L(\mathfrak{H})$. Thus, an edge in $L(\mathfrak{H})$ lies in $\iota(\mathfrak{P})$ when both its endpoints are labelled by the same set $P \in \mathfrak{P}$. With $2^{L(\mathfrak{H})}$ ordered by containment, we see that the identification $\iota$ preserves the ordering.

The image of $\iota$ is the set of all subgraphs $\mathfrak{K} \subset L(\mathfrak{H})$ which are maximally connected in the sense that addition of any new edge from $L(\mathfrak{H})$ to $\mathfrak{K}$ will decrease the number of connected components of $\mathfrak{K}$. (This image is of considerable interest in the study of graph coloring problems and finite geometries, where it is called the bond lattice [12].) If $\mathfrak{K}$ is an arbitrary subgraph of $L(\mathfrak{H})$, let $\overline{\mathfrak{K}}$ be the smallest maximally connected subgraph larger than $\mathfrak{K}$. The map $^{-}$: $2^{L(\mathfrak{H})} \to \iota(\coprod_{\mathfrak{H}}^m)$ is a closure relation ($\overline{\mathfrak{K}} \geqq \mathfrak{K}$, $\overline{\overline{\mathfrak{K}}} = \overline{\mathfrak{K}}$), so if $\mu_3$ is the Möbius function of $2^{L(\mathfrak{H})}$,

$$\mu_2(\mathfrak{R}, \mathfrak{Q}) = \sum_{\{\mathfrak{K}:\, \overline{\mathfrak{K}} = \iota(\mathfrak{Q})\}} \mu_3(\iota(\mathfrak{R}), \mathfrak{K}), \tag{3.16}$$

as one may readily verify ([3]).

It is well known that $|\mu_3| \leqq 1$ ($\mu_3$ can be computed explicitly [2, 12]); thus

$$|\mu_2(\mathfrak{R}, \mathfrak{Q})| \leqq 2^{|\iota(\mathfrak{Q})| - |\iota(\mathfrak{R})|}. \tag{3.17}$$

Applying (3.17) to the special case $\mathfrak{Q} = 1 = \{\{1, \ldots, \mathrm{m}\}\}$ of (3.14). we find

$$|u(\{F_i\})_0| \leqq \prod_1^m \|F_i\| \cdot \sum_{k=0}^{|L(\mathfrak{H})|} \binom{|L(\mathfrak{H})|}{k} \cdot 2^{|L(\mathfrak{H})| - k} = 3^{|L(\mathfrak{H})|} \prod_1^m \|F_i\|. \tag{3.18}$$

Since $|L(\mathfrak{H})| = 1/2 \sum_{i=1}^{m} d(E_i)$, the proof is complete. QED

The graph $\mathfrak{H}$ of Lemma 3.1 is usually composed of two pieces, a basic graph $\Gamma$ and some edges $A_1, \ldots, A_n \subset V\Gamma$ added to it. We would like to estimate the exponent $\sum_{E \in \mathfrak{H}} d_{\mathfrak{H}}(E)$ of (3.9) in terms of the overall degree of $d_\Gamma$ of $\Gamma$. Since

$$\sum_{E \in \Gamma} d_{\mathfrak{H}}(E) \leqq \sum_{E \in \Gamma} d_\Gamma(E) + d_\Gamma \sum_{j=1}^{n} |A_j| \tag{3.19}$$

and

$$\sum_{A_j \in \mathfrak{H} - \Gamma} d_{\mathfrak{H}}(A_j) \leqq d_\Gamma \sum_{j=1}^{n} |A_j| + n(n-1), \tag{3.20}$$

we find

$$\tfrac{1}{2}\sum_{E\in\mathfrak{H}} d_{\mathfrak{H}}(E) \leqq d_\Gamma \sum_1^n |A_j| + \frac{n^2}{2} + \frac{d_\Gamma}{2}|\Gamma|. \tag{3.21}$$

The term $\frac{n^2}{2}$ in (3.21) accounts for possible overlap of the sets $A_i \subset V\Gamma$, and can be omitted if they are mutually disjoint.

**Lemma 3.2.** *Let $\mathscr{G}$ be a hypergraph with $d_{\mathscr{G}} < \infty$. Given a finite set $A \subset V\mathscr{G}$ and a positive integer $\gamma$, let $N_{\mathscr{G}}(\gamma, A)$ be the number of subgraphs $\Gamma \subset \mathscr{G}$ with $|\Gamma| = \gamma$ edges such that $\Gamma \cup \{A\}$ is connected. Then*

$$N_{\mathscr{G}}(\gamma, A) \leqq (2d_{\mathscr{G}})^{|A|+2\gamma}. \tag{3.22}$$

*Proof.* Enumerate the elements $a_1, a_2, \ldots, a_{|A|}$ of $A$, and introduce the line graph $L(\mathscr{G})$.

Let $B_i \subset VL(\mathscr{G})$ be the set of vertices in $L(\mathscr{G})$ which when regarded as edges in $\mathscr{G}$ contain $a_i$. Interpreting the problem in $L(\mathscr{G})$, we must bound the number of subsets $V \subset VL(\mathscr{G})$, $|V| = \gamma$, such that every connected component of the restriction $L(\mathscr{G}) \restriction V$ meets some $B_i$. With this interpretation, we may use a method of [4] to obtain a suitable estimate.

Associate a connected component of $L(\mathscr{G}) \restriction V$ with the smallest index $i$ such that $B_i$ meets the component, and make the convention that the components of all remaining indices are empty. (Note that by the definition of $B_i$, at most one component of $L(\mathscr{G}) \restriction V$ may meet it.) Let $\gamma_i$ be the number of vertices of $V$ in the $i^{\text{th}}$ component. Fix the $\gamma_i$, $i \in \{1, \ldots, n\}$, while otherwise permitting $V$ to vary.

The $i^{\text{th}}$ component of $L(\mathscr{G}) \restriction V$ admits a spanning tree, with $\gamma_i - 1$ edges. This tree may be traversed by a continuous chain of $2(\gamma_i - 1)$ edges, each edge appearing twice. There are at most $d_{\mathscr{G}}^{2(\gamma_i-1)}$ such chains emanating from a specific initial vertex $b_i \in B_i$. Letting $b_i$ range over $B_i$ and multiplying over all the $B_i$, we find the number of families $V$ such that every component of $L(\mathscr{G}) \restriction V$ meets some $B_i$ and such that the $i^{\text{th}}$ component has $\gamma_i$ vertices is bounded by $d_{\mathscr{G}}^{2\gamma} \cdot \prod_1^{|A|} d_{\mathscr{G}}(a_i)$. Since the number of possible choices for the $\gamma_i$ is at most $2^{|A|+\gamma}$, we obtain

$$N_{\mathscr{G}}(\gamma, A) \leqq (2d_{\mathscr{G}}^2)^{\gamma} \cdot 2^{|A|} \prod_1^{|A|} d_{\mathscr{G}}(a_i) \leqq (2d_{\mathscr{G}})^{|A|+2\gamma}. \quad \text{QED} \tag{3.23}$$

In proving Lemma 3.3 we shall need to estimate the number $N_{\mathscr{G}}(\gamma)$ of connected subgraphs $\Gamma \subset \mathscr{G}$ with $\gamma$ edges which contain a given edge $E$ of $\mathscr{G}$. The bound

$$N_{\mathscr{G}}(\gamma) \leqq d_{\mathscr{G}}^{2\gamma} \tag{3.24}$$

follows from the argument just given.

**Lemma 3.3.** *Let $\mathfrak{M}$ be a lattice model with vertices $\mathscr{L}$, hypergraph $\mathscr{G}$, weights $\{p_E\}_{E\in\mathscr{G}}$, and Gibbs factors $\{g_E\}_{E\in\mathscr{G}}$. For all $C > 1$ there exists $\delta > 0$ depending only* *on $C$ and $d_{\mathscr{G}}$ such that if*

$$\|g_E - 1\|_{p_E} < \delta \qquad \forall E \in \mathscr{G} \tag{3.25}$$

*then $Z(\Lambda) \neq 0$ for all finite $\Lambda \subset \mathscr{G}$, and moreover*

$$\left|\frac{Z(\Lambda - \Gamma)}{Z(\Lambda)}\right| \leqq C^{|\Gamma|} \qquad \forall \Gamma \subset \Lambda \subset \mathscr{G}. \tag{3.26}$$

*Proof.* By subtracting the edges of $\Gamma$ from $\Lambda$ one at a time, we see it suffices to prove (3.26) when $|\Gamma| = |\Lambda| - 1$. We proceed by induction on $|\Lambda|$, showing that if a $\delta$ can be produced such that (3.26) holds when $|\Lambda| < l$, it also holds when $|\Lambda| = l$. The special case $|\Lambda| = 1$ may be trivially verified.

Select $\Lambda, |\Lambda| = \ell$, let $E_1$ be a distinguished edge of $\Lambda$, and set $\Gamma = \Lambda - \{E_1\}$. We may assume inductively that $Z(\Gamma) \neq 0$, and so apply the expansion (3.2) to $\mathfrak{E}(g_{E_1} \cdot \prod_{E \in \Gamma} g_E)_0 \equiv Z(\Lambda)$:

$$\frac{Z(\Lambda)}{Z(\Gamma)} - 1 = \mathfrak{E}(g_{E_1} - 1)_0 + \sum_{\mathfrak{H} \subset \Gamma} u(g_{E_1}, \{g_E\}_{E \in \mathfrak{H}})_0 \frac{Z(\Gamma - \mathfrak{H})}{Z(\Gamma)} \tag{3.27}$$

where the sum is over those subgraphs $\mathfrak{H}$ such that $\mathfrak{H} \cup \{E_1\}$ is connected. By Lemma 3.1, the bound (3.24), and the inductive assumption, we estimate

$$\begin{aligned} \left|\frac{Z(\Lambda)}{Z(\Gamma)} - 1\right| &\leqq \delta + \sum_{|\mathfrak{H}| - 1}^{|\Gamma|} \delta^{|\mathfrak{H}|+1} \, 3^{\frac{1}{2}(1+|\mathfrak{H}|)d_{\mathscr{G}}} \, d_{\mathscr{G}}^{2|\mathfrak{H}|} C^{|\mathfrak{H}|} \\ &< \delta + \delta^2 \, 3^{\frac{1}{2}d_{\mathscr{G}}} \frac{C \cdot 3^{d_{\mathscr{G}}} \cdot d_{\mathscr{G}}^2}{1 - \delta \cdot C \cdot 3^{d_{\mathscr{G}}} \cdot d_{\mathscr{G}}^2} \equiv \eta(\delta, C). \end{aligned} \tag{3.28}$$

It is clear by inspection of (3.28) that, by decreasing $\delta$ if necessary, we make $\eta(\delta, C)$ small enough to ensure

$$\left|\frac{Z(\Lambda)}{Z(\Gamma)} - 1\right| \leqq \eta(\delta, C) \Rightarrow \left|\frac{Z(\Gamma)}{Z(\Lambda)}\right| \leqq C, \tag{3.29}$$

and that the requisite value of $\delta$ depends only on $C$ and $d_{\mathscr{G}}$. Thus, the inductive step is achieved. QED

These three lemmas give control of the expansion (3.8), which we now use to prove

**Theorem 3.4.** *Let $\mathfrak{M}$ be a lattice model with vertices $\mathscr{L}$, probability spaces $(X_i, \mathfrak{B}_i, \nu_i)_{i \in \mathscr{L}}$ hypergraph $\mathscr{G}$ on $\mathscr{L}$, integration weights $\{p_E\}_{E \in \mathscr{G}}$, and Gibbs factors $g_E \in \mathscr{L}^{p_E}(\prod_{i \in E} X_i, \prod_{i \in E} \nu_i)$. Let $A_1, \ldots, A_n \subset \mathscr{L}$ be $n$ finite subsets, and choose for them integration weights $r_{A_1}, \ldots, r_{A_n}$ conformable with each other and the weights $p_E$ of $\mathscr{G}$. There exists a constant $D > 1$ depending only on $d_{\mathscr{G}}$ such that $\forall K > 0$, $\exists \delta > 0$ depending only on $K$, $d_{\mathscr{G}}$ and $n$ so that if*

$$\|g_E - 1\|_{p_E} < \delta \qquad \forall E \in \mathscr{G} \tag{3.30}$$

*then*

$$|u(f_{A_1},\dots,f_{A_n})_\Lambda - u(f_{A_1},\dots,f_{A_n})_0| < n^n\, 3^{\frac{n^2}{2}}\, D^{\sum_1^n |A_i|} \prod_{i=1}^{n} \|f_{A_i}\|_{r_{A_i}} e^{-K\rho(\{A_i\})} \tag{3.31}$$

*for all finite* $\Lambda \subset \mathcal{G}$ *and all functions* $f_{A_i} \in \mathcal{L}^{r_{A_i}}(\prod_{j\in A_i} X_j, \prod_{j\in A_i} v_j)$*. Here the spread* $\rho(\{A_i\})$ *of the family* $\{A_i\}$ *in* $\mathcal{G}$ *is by definition*

$$\rho(\{A_i\}) = \inf_{\Gamma\subset\mathcal{G}} \{|\Gamma| : \Gamma \cup \{A_i, i=1,\dots n\} \text{ is connected}\}.$$

*Remark.* Although the $\delta$ we require to achieve a given decay rate $K$ in $n^{\text{th}}$ order truncated expectations depends on $n$, it is independent of $|A_i|$ and $f_{A_i}$.

*Proof.* Apply the expansion (3.8):

$$u(\{f_{A_i}\})_\Lambda - u(\{f_{A_i}\})_0 = \frac{1}{n}\sum_{\Gamma\subset\Lambda} u(\tilde f_{A_i},\dots,\tilde f_{A_n}, \{\mathcal{G}_E\}_{E\in\Gamma})_0 \cdot \frac{Z(\Lambda-\Gamma)^n}{Z(\Lambda)^n} \tag{3.32}$$

where the sum is over those $\Gamma \neq \emptyset$ such that $\Gamma \cup \{A_1,\dots,A_n\}$ is connected. By (3.21)

$$u(\{f_{A_i}\},\{\mathcal{G}_E\}_{E\in\Gamma})_0| \leqq 3^{\frac{n^2}{2} + d_{\mathcal{G}}\cdot\sum_1^n |A_i| + \frac12 d_{\mathcal{G}}|\Gamma|} \cdot \left(\prod_1^n \|\tilde f_{A_i}\|\right)\left(\prod_{E\in\Gamma} \|\mathcal{G}_E - 1\|\right). \tag{3.33}$$

Since

$$\|\tilde f_{A_i}\| \leqq n\|\tilde f_{A_i}\| \,\&\, \|\mathcal{G}_E - 1\| \leqq (1+\delta)^n - 1 \equiv \delta_n, \tag{3.34}$$

inequality (3.33) implies

$$|u(\{f_{A_i}\},\{\mathcal{G}_E\})_0| \leqq 3^{\frac{n^2}{2} + d_{\mathcal{G}}\cdot\Sigma|A_i| + \frac12 d_{\mathcal{G}}|\Gamma|} \cdot n^n \cdot \prod_1^n \|f_{A_i}\| \cdot \delta_n^{|\Gamma|}. \tag{3.35}$$

By Lemma 3.3, for small $\delta$,

$$\left[\frac{Z(\Lambda-\Gamma)}{Z(\Lambda)}\right]^n \leqq C^{n|\Gamma|}, \qquad C \geqq 1. \tag{3.36}$$

By Lemma 3.2, there are at most $(2d_{\mathcal{G}})^{\sum_1^n |A_i| + 2|\Gamma|}$ terms in (3.32) having a fixed value of $|\Gamma|$. Combining these estimates, we find

$$\begin{aligned}|u_\Lambda - u_0| \leqq \frac{1}{n}[n^n 3^{\frac{n^2}{2}}][3^{d_{\mathcal{G}}}\cdot 2d_{\mathcal{G}}]^{\sum_1^n |A_i|}\left[\prod_1^n \|f_{A_i}\|\right]\\ \cdot [\delta_n 3^{\frac12 d_{\mathcal{G}}} C^n 4^{d_{\mathcal{G}}^2}]^{\rho(\{A_i\})} \cdot \sum_{\gamma=0}^{\infty} [\delta_n 3^{\frac12 d_{\mathcal{G}}} C^n 4 d_{\mathcal{G}}^2]^\gamma\end{aligned} \tag{3.37}$$

Take $D = 3^{d_{\mathcal{G}}} 2d_{\mathcal{G}}$, and choose $\delta$ so small that

$$\delta_n \cdot 3^{\frac12 d_{\mathcal{G}}} \cdot C^n \cdot 4 d_{\mathcal{G}}^2 \leqq e^{-K} \tag{3.38}$$

and

$$\sum_0^\infty [\delta_n 3^{\frac{1}{2}d_{\mathscr{G}}} C^n 2d_{\mathscr{G}}^2]^\gamma \leqq n. \tag{3.39}$$

(If $n=1$ the sum in (3.39) starts at $\gamma=1$ so that the inequality still may be satisfied.) The theorem now follows from (3.37). QED

Recall that the factor $3^{n^2/2}$ in (3.31) arises from the possibility in Lemma 3.1 that the sets $A_i$ might overlap, and can be omitted if $A_i \cap A_j = \emptyset \, \forall i \neq j$. In this case, the term $u(\{F_{A_i}\})_0$ also vanishes. We formalize these comments in a corollary:

**Corollary 3.5.** *If the hypothesis of Theorem* 3.4 *is strengthened by assuming further that* $A_i \cap A_j = \emptyset \, \forall i \neq j$, *the uniform bound* (3.31) *may be replaced by*

$$|u(f_{A_1}, \ldots, f_{A_n})_A| \leqq n^n D^{\left|\bigcup_1^n A_i\right|} \prod_1^n \|f_{A_i}\|_{r_{A_i}} e^{-K\rho(\{A_i\})}. \tag{3.40}$$

These bounds (3.31), (3.40) on truncated expectations are our central technical results.

## Section 4: Applications

In this section we utilize the decay estimates of Section 3 to construct and analyze the Gibbs measure $\mu = \lim_{\Lambda \to \infty} \mu_\Lambda$ in weakly coupled lattice models. We shall find that this limit is very well-behaved: it is approached uniformly over the small-coupling region, correlations decay exponentially, expectations are analytic in the Gibbs factors, translation-invariant models have translation-invariant Gibbs measures, and the central limit theorem holds. Since much of the reasoning needed to derive these properties from the uniform bounds of the preceding section is somewhat standard, we give brief proofs.

As a preliminary to construction of the infinite-volume Gibbs measure $\mu$ we control the change in $\mathfrak{E}(f_A)_\Lambda$ when a single edge is added to $\Lambda$.

**Lemma 4.1.** *Let* $\mathfrak{M}$ *be a lattice model with vertices* $\mathscr{L}$, *probability spaces* $(X_i, \mathfrak{B}_i, \nu_i)_{i \in \mathscr{L}}$, *hypergraph* $\mathscr{G}$, *integration weights* $\{p_E\}_{E \in \mathscr{G}}$, *and Gibbs factors* $g_E \in \mathscr{L}^{p_E}$. *Let* $A \subset \mathscr{L}$, $|A| < \infty$, *let* $r_A$ *be a conformable weight for* $A$, *and let* $\Lambda \subset \mathscr{G}$ *be a finite subgraph with* $A \subset V\Lambda$. *There exists a constant* $D > 1$ *depending only on* $d_{\mathscr{G}}$ *such that* $\forall K > 0 \, \exists \delta > 0$, $\delta$ *depending only on* $K$ *and* $d_{\mathscr{G}}$, *so that if*

$$\|g_E - 1\|_{p_E} < \delta \qquad \forall E \in \mathscr{G}$$

*then*

$$|\mathfrak{E}(f_A)_{\Lambda \cup \{E\}} - \mathfrak{E}(f_A)_\Lambda| \leqq \|f_A\|_{r_A} D^{|A|} \exp[-K \rho_{\mathscr{G}}(A, E)]. \tag{4.1}$$

*Proof.* Write

$$\mathfrak{E}(f_A)_{\Lambda \cup \{E\}} - \mathfrak{E}(f_A)_\Lambda = \frac{Z(\Lambda)}{Z(\Lambda \cup \{E\})} \cdot [\mathfrak{E}(f_A \cdot [\bar{g}_E - 1])_\Lambda - \mathfrak{E}(f_A)_\Lambda \mathfrak{E}(\bar{g}_E - 1)_\Lambda] \tag{4.2}$$

where

$$\bar{g}_E = \int g_E \cdot \prod_{i \in E - VA} dv_i. \tag{4.3}$$

The lemma now follows by applying Lemma 3.3 and Theorem 3.44. QED

With Lemma 4.1 in hand, we construct the infinite-volume limit by adding one edge at a time. Let $B_r(A)$ and $S_r(A)$ be the ball and sphere of radius $r$ about $A$:

$$\begin{aligned} B_r(A) &= \{E \in \mathscr{G}: \rho(A,E) \leqq r\} \\ S_r(A) &= \{E \in \mathscr{G}: \rho(A,E) = r\}. \end{aligned} \tag{4.4}$$

Let $\Lambda \subset \mathscr{G}$ be a subgraph trapped between two balls: $B_r(A) \subset \Lambda \subset B_R(A)$. Order the edges $E_1, E_2, \ldots$ of $\Lambda = B_r(A)$ so that the separation $\rho(A, E_i)$ increases with $i$. Let $\Lambda_i = B_r(A) \cup \{E_j : j \leqq i\}$ and write $\mathfrak{E}(f_A)_\Lambda - \mathfrak{E}(f_A)_{B_r(A)}$ as the telescoping sum

$$\mathfrak{E}(f_A)_\Lambda - \mathfrak{E}(f_A)_{B_r(A)} = \sum_{i=0}^{|\Lambda - B_r| - 1} [\mathfrak{E}(f_A)_{\Lambda_i \cup \{E_{i+1}\}} - \mathfrak{E}(f_A)_{\Lambda_i}]. \tag{4.5}$$

By the lemma, we have

$$|\mathfrak{E}(f_A)_\Lambda - \mathfrak{E}(f_A)_{B_r(A)}| \leqq \|f_A\|_{r_A} \cdot D^{|A|} \cdot \sum_{\rho=r+1}^{R} |S_\rho(A)| \, e^{-K\rho}. \tag{4.6}$$

If $\mathscr{G}$ is a translation-invariant hypergraph on $Z^N$, $|S_\rho(A)| \approx |A| \rho^{N-1}$, and existence of the limit $\lim\limits_{\Lambda \to \infty} \mathfrak{E}(f_A)_\Lambda$ is immediate from (4.6) for any exponential decay rate $K > 0$. However, power law growth of the sphere surface area $|S_\rho(A)|$ in the radius does not follow from the single assumption $d_{\mathscr{G}} < \infty$. Exponential increase appears in Cayley trees and similar examples, correctly suggesting unusual behavior [8, 16]. Fortunately, the growth is no worse than exponential, since a simple path-counting argument shows

$$|S_\rho(A)| \leqq |A| \, d_{\mathscr{G}}^\rho. \tag{4.7}$$

Convergence thus follows from (4.6) by choosing $\delta$ small enough to ensure

$$\kappa \equiv d_{\mathscr{G}} \, e^{-K} < 1. \tag{4.8}$$

We summarize in a theorem these conclusions concerning the existence of the infinite-volume limit.

**Theorem 4.2.** *Let $\mathfrak{M}$ be a lattice model with vertices $\mathscr{L}$, hypergraph $\mathscr{G}$, integration weights $\{p_E\}_{E \in \mathscr{G}}$, and Gibbs factors $g_E \in L^{p_E}$. Let $A \subset \mathscr{L}$ with $|A| < \infty$, let $r_A$ be a weight for $A$ conformable with $p_E$, let $B_r(A) \subset \mathscr{G}$ be the ball of radius r about A, and let $\Lambda \supset B_r(A)$, $|\Lambda| < \infty$. There exists a constant $D > 1$ such that $\forall \kappa \in (0,1) \exists \delta > 0$, $\delta$ depending only on $\kappa$ and $d_{\mathscr{G}}$, such that*

$$|\mathfrak{E}(f_A)_\Lambda - \mathfrak{E}(f_A)_{B_r(A)}| \leqq \|f_A\|_{r_A} \cdot |A| \cdot D^{|A|} \cdot \frac{\kappa^{r+1}}{1-\kappa} \, \forall f_A \in L^{r_A}. \tag{4.9}$$

*Thus, the net* $\{\mathfrak{E}(f_A)_\Lambda\}$ *of finite-volume Gibbs expectations is a Cauchy set converging to the limit* $\mathfrak{E}(f_A)_\infty = \lim_{\Lambda\to\infty} \mathfrak{E}(f_A)_\Lambda$ *uniformly in the region* $\|g_E - 1\| < \delta$.

*Proof.* The proof is immediate from the discussion of the previous paragraph. QED

Exponential decay of correlations in the infinite-volume limit now follows from Theorem 3.4 by passing to the $\Lambda\to\infty$ limit:

**Theorem 4.3.** *Let* $\mathfrak{M}$ *be the lattice model of Theorem* 4.2. *Let* $A_1, \ldots, A_n \subset \mathscr{L}$ *be* $n$ *finite subsets, having integration weights* $r_{A_1}, \ldots, r_{A_n}$ *conformable with those of* $\mathfrak{M}$. *There exists a constant* $D > 1$ *depending only on* $d_{\mathscr{G}}$ *such that* $\forall K > 0$, $\exists \delta > 0$ *depending only on* $K$, $d_{\mathscr{G}}$, *and* $n$ *such that if*

$$\|g_E - 1\|_{p_E} < \delta \qquad \forall E \in \mathscr{G}$$

*then* $\forall f_{A_i} \in L^{r_{A_i}}$,

$$|u(f_{A_1}, \ldots, f_{A_n})_\infty - u(f_{A_1}, \ldots, f_{A_n})_0| \leqq n^n 3^{\frac{n^2}{2}} D^{\sum_1^n |A_i|} \prod_1^n \|f_{A_i}\|_{r_{A_i}} e^{-K\rho(\{A_i\})}. \tag{4.10}$$

*Here the spread* $\rho(\{A_i\})$ *in* $\mathscr{G}$ *of the family* $\{A_i\}$ *is by definition*

$$\rho(\{A_i\}) = \inf\{|\Gamma| : \Gamma \cup \{A_1, \ldots, A_n\} \text{ is connected}\}. \tag{4.11}$$

**Corollary 4.4.** *If the sets* $A_i$ *in Theorem* 4.3 *are mutually disjoint, the bound* (4.10) *may be replaced by*

$$|u(f_{A_1}, \ldots, f_{A_n})_\infty| \leqq n^n D^{|\cup A_i|} \prod_1^n \|f_{A_i}\|_{r_{A_i}} e^{-K\rho(\{A_i\})}. \tag{4.12}$$

*Proof.* There results are immediate by passage to the limit in Theorem 3.4 and Corollary 3.5. QED

*Remark.* We emphasize that although the $\delta$ we require for a given exponential decay rate $K$ is dependent on the order $n$ of truncation, it is independent of the cardinalities $|A_i|$ and functions $f_{A_i}$. Moreover, by taking a weaker measure of the spread in $\mathscr{G}$ of $\{A_i\}$, one may eliminate the $n$-dependence of $\delta$ as well [7].

It is evident from the uniform approach to the limit over the weak-coupling region $\|g_E - 1\| < \delta$ that an infinite-volume Gibbs expectation $\mathfrak{E}(f_A)_\infty$ is analytic in each Gibbs factor $g_E \in L^{p_E}$. This conclusion may be stated more strongly for translation-invariant models.

**Theorem 4.5.** *Let* $\mathfrak{M}$ *be a translation-invariant lattice model on* $Z^N$ *with fundamental edges* $E_1, E_2, \ldots, E_M \subset Z^N$, *integration weights* $\{p_{E_i}\}$, $1 \leqq i \leqq M$, *and Gibbs factors* $g_{E_i} \in L_C^{p_{E_i}}$. *Then the infinite-volume limit is translation-invariant in the polydisc of convergence*

$$\Delta = \left\{ (g_{E_1}, \ldots, g_{E_M}) \in \mathop{\times}_{i=1}^{M} L^{p_{E_i}} : \|g_{E_i} - 1\| < \delta \right\} \tag{4.13}$$

*guaranteed by Theorem 4.2. Moreover, for any conformable $f_A \in L_C^{r_A}$ the map*

$$\mathfrak{E}(f_A)_\infty : \Delta \to C \tag{4.14}$$

*defined by considering $\mathfrak{E}(f_A)_\infty$ as a function of its Gibbs factors is analytic.*

*Proof.* Translation invariance is immediate from Theorem 4.2. Analyticity follows because $\mathfrak{E}(f_A)_\infty$ is the limit of a sequence $\mathfrak{E}(f_A)_{B_r(A)}$ of quotients of continuous polynomials which converges uniformly in $\Delta$. QED

In many applications the Gibbs factors $g_E$ depend analytically on several complex parameters $z_1, \ldots, z_k \in C$ representing continuations into the complex plane of temperature, magnetic field, coupling strength, etc. Of course, when this is so we have analyticity of $\mathfrak{E}(f_A)_\infty$ for suitable parameter values by composition.

We conclude our study of weakly coupled lattice models with the central limit theorem. One common criterion yielding this theorem for families of dependent random variables is that of strong mixing [11]. Unfortunately, the factor $D^{\left|\bigcup\limits_1^n A_i\right|}$ in the bound (4.12) prevents direct verification of strong mixing. On the other hand, the exponential decay in (4.12) is much better than strong mixing requires, and we shall use it to obtain the central limit theorem in another way.

We introduce some notation. Let $\mathfrak{M}$ be a translation-invariant lattice model with probability space $(X, \mathfrak{B}, \nu)$ vertices $Z^N$, hypergraph $\mathscr{G}$ generated by the fundamental edges $E_1, \ldots, E_M \subset Z^N$, integration weights $\{p_{E_i}\}_{i\in\{1,\ldots,M\}}$ and Gibbs factors $g_{E_i} \in L^{p_{E_i}}$. Let $A \subset Z^N$, $|A| < \infty$, and choose a weight $r_A$ for $A$ so that the enlarged translation-invariant model $\mathfrak{M}^+$ with fundamental edges $A$, $\{E_i\}$ having weights $r_A$, $\{p_{E_i}\}$ is conformable for integration. Denote by $\mathscr{G}^+$ the translation-invariant hypergraph generated by the edges $A, \{E_i\}$. Select $f: X^A \to C$ such that for some $\eta > 0$,

$$\eta^{|f|} \in L^{r_A}. \tag{4.15}$$

For $i \in Z^N$ let $f_i$ be the function obtained by translating $f$ to act on the space $X^{A+i}$. If $V \subset Z^N$, $|V| < \infty$, formally define the mean-subtracted moment generating function

$$\Phi_V(z) = \mathfrak{E}\left[\exp\left(\frac{z}{\sqrt{|V|}} \cdot \sum_{i\in V} [f_i - \mathfrak{E}(f)_{\infty,\mathfrak{M}}]\right)\right]_{\infty,\mathfrak{M}}. \tag{4.16}$$

Direct the finite subsets $V \subset Z^n$ by containment, and adjoint a greatest element $\infty$. With reference to this notation, we have

**Theorem 4.6.** *There exists $\delta > 0$ (depending only on $d_{\mathscr{G}}^+$) such that if*

$$\|g_{E_i} - 1\|_{p_{E_i}} < \delta \qquad \forall i \in \{1, \ldots, M\} \tag{4.17}$$

*then*

$$\Phi_V(z) \xrightarrow[V\to\infty]{} e^{\frac{\sigma^2 z^2}{2}} \tag{4.18}$$

*uniformly on compacts, where*

$$\sigma^2 \equiv \sum_{i \in Z^N} [\mathfrak{E}(ff_i)_{\infty,\mathfrak{M}} - \mathfrak{E}(f)^2_{\infty,\mathfrak{M}}]. \tag{4.19}$$

*(Note that the Gibbs factors need not be real.)*

*Proof.* We first show that for suitable $\delta$ the moment-generating functions $\Phi_V$ are well-defined and nowhere zero. Take $\delta$ sufficiently small to ensure that if in the enlarged model $\mathfrak{M}^+$ we have

$$\|g_A - 1\|_{r_A} < \delta \,\&\, \|g_{E_i} - 1\|_{p_{E_i}} < \delta \qquad \forall i \in \{1, \ldots, M\}, \tag{4.20}$$

then:

(i) there is uniform convergence to the infinite-volume limit in $\mathfrak{M}^+$ (Theorem 4.2); and,

(ii) third-order truncated expectations in $\mathfrak{M}^+$ decay exponentially (Theorem 4.3).

By decreasing $\eta$ if necessary, we may suppose further that

$$\|e^{zf} - 1\|_{r_A} < \delta \qquad \forall |z| < \eta. \tag{4.21}$$

It follows from (i) by (4.9) that $\Phi_V(z)$ is well-defined and analytic in the disc $\{|z| < \eta \cdot \sqrt{|V|}\}$. We claim further that $\Phi_V$ never vanishes in this region. To see this, recall that $\Phi_V(z)$ is by definition the uniform (on compacts) limit of the finite-volume expectations

$$\mathfrak{E}_A \equiv \left[\frac{e^{-z\sqrt{|V|}\,\mathfrak{E}(f)_\infty}}{Z(A)}\right] \times \mathfrak{E}[e^{\frac{z}{\sqrt{|V|}}\sum_{i\in V} f_i} \cdot \prod_{E \in A} g_E]_0. \tag{4.22}$$

The first factor in (4.22) clearly never vanishes in $\{|z| < \eta |V|^{1/2}\}$. The second factor also has no zeroes there, because by (4.21) we may regard the factors $\exp(zf_i/\sqrt{|V|})$ as Gibbs factors for the enlarged model $\mathfrak{M}^+$ and then invoke Lemma 3.3. Taking the limit $A \to \infty$, the Hurwitz Theorem implies that $\Phi_V(z)$ is never zero in $\{|z| < \eta \sqrt{|V|}\}$. (Since $\Phi_V(0) = 1$, the possibility that $\Phi_V \equiv 0$ does not arise.)

We turn now to the question of convergence as $V \to \infty$. Since $\Phi_V$ is nowhere zero in $\{|z| < \eta \sqrt{|V|}\}$ we may introduce logarithms in (4.18). Thus we must prove

$$\left|\log \Phi_V(z) - \frac{\sigma^2}{2} z^2\right| \xrightarrow[V \to \infty]{} 0 \qquad \text{unif. on cpcts.} \tag{4.23}$$

(Note that the series (4.19) for $\sigma^2$ is absolutely convergent by (ii).)

Consider the Maclaurin series with remainder for $\log \Phi_V$:

$$\log \Phi_V(z) = a_0(V) + a_1(V)\, z + \frac{a_2(V)}{2!} z^2 + R_V(z). \tag{4.24}$$

Without loss of generality, we may take

$$\mathfrak{E}(f)_\infty = 0$$

in order to simplify the expressions we now give for the coefficients $a_i(V)$:

$$a_0(V)=a_1(V)=0; \qquad a_2(V)=\frac{1}{|V|}\mathfrak{E}([\sum_{i\in V} f_i]^2)_{\infty,\mathfrak{M}}. \tag{4.25}$$

By translation invariance, $\lim_{V\to\infty} a_2(V)=\sigma^2$, so (4.23) will follow from (4.24) by disposing of the remainder $R_V(z)$. If $|z|<\eta\sqrt{|V|}$ we have

$$|R_V(z)|\leqq\frac{|z|^3}{6}\sup_{|\zeta|<|z|}|R_V(\xi)|. \tag{4.26}$$

Computing, we find

$$R_V'''(\zeta)=\frac{1}{\sqrt{|V|}}\left[\frac{\mathfrak{E}(S^3\prod_{i\in V} h_i)_{\infty,\mathfrak{M}}}{\mathfrak{E}(\prod_{i\in V} h_i)_{\infty,\mathfrak{M}}}-3\frac{\mathfrak{E}(S^2\prod_{i\in V} h_i)_{\infty,\mathfrak{M}}\,\mathfrak{E}(S\prod_{i\in V} h_i)_{\infty,\mathfrak{M}}}{\mathfrak{E}(\prod_{i\in V} h_i)^2_{\infty,\mathfrak{M}}}\right.$$
$$\left.+2\frac{\mathfrak{E}(S\prod_{i\in V} h_i)^3_{\infty,\mathfrak{M}}}{\mathfrak{E}(\prod_{i\in V} h_i)^3_{\infty,\mathfrak{M}}}\right] \tag{4.27}$$

where

$$S=\sum_{i\in V} f_i, \qquad h_i=e^{\zeta f_i/\sqrt{|V|}}. \tag{4.28}$$

Regard the factors $h_i=\exp(\zeta f_i/|V|^{1/2})$ in (4.27) as Gibbs factors for the enlarged model $\mathfrak{M}^+$. By (4.21), we may apply assumption (ii) to show the part of (4.27) in curly brackets is uniformly bounded in $\zeta$, $|\zeta|\leqq|z|$. Thus, $R_V(z)$ converges to zero uniformly on compacts as $V\to\infty$ at least as fast as $1/|V|^{1/2}$, and the theorem is proved. QED

*Acknowledgments.* I would like to thank Joel Lebowitz for suggesting and encouraging the direction of research taken in this paper. I am grateful to Mark Kac, John Riordan, and J. Laurie Snell for pointing out some helpful references. I thank Hervé Kunz and Errico Presutti for kindly informing me of their results. Finally, I am indebted to James Glimm and Thomas Spencer for extensive discussions and strong encouragement in the writing of this paper.

## References

1. Berge, C.: Graphs and Hypergraphs. Amsterdam: North-Holland 1973
2. Comtet, L.: Advanced Combinatorics. Dortrecht: D. Reidel 1974
3. Crapo, H.: Möbius Inversion in Lattices. Arch. Math. **XIX**, 595–607 (1968)
4. Glimm, J., Jaffe, A., Spencer, T.: The Particle Structure of the Weakly Coupled $P(\Phi)_2$ Model and Other Applications of High Temperature Expansions, in *Constructive Quantum Field Theory*, ed. G. Velo and A.S. Wightman, pp. 133–264. Berlin-Heidelberg-New York: Springer 1973
5. Israel, R.B.: High Temperature Analyticity in Classical Lattice Systems. Comm. Math. Phys. **50**, 245 (1976)

6. Kunz. H.: Private communication
7. Martin-Löf, A.: Mixing Properties, Differentiability of the Free Energy, and the Central Limit Theorem for a Pure Phase in the Ising Model at Low Temperature, Comm. Math. Phys. **32**, 75–92 (1975)
8. Moore, T., Snell, J.L.: Gibbs Measures on Cayley Trees. Dartmouth preprint (1976)
9. Nachbin, L.: Lectures on the Theory of Distributions. University of Rochester (1964)
10. Presutti, E.: Private communication
11. Rosenblatt, M.: Random Processes. Berlin-Heidelberg-New York: Springer 1974
12. Rota, G.-C.: On the Foundations of Combinatorial Theory I: Theory of Möbius Functions. Z. f. Wahrscheinlichkeitstheorie verw. Gebiete **2**, 340–368 (1964)
13. Ruelle, D.: Statistical Mechanics. New York: Benjamin 1969
14. Sylvester, G.: Representations and Inequalities for Ising Model Ursell Functions. Comm. Math. Phys. **42**, 209–220 (1975)
15. Sylvester, G.: Continuous-Spin Ising Ferromagnets. M.I.T. Thesis (1976)
16. Zittartz, J.: Phase Transitions of Continuous Order, in *International Symposium on Mathematical Problems in Theoretical Physics*, ed. H. Araki, pp. 330–335. Berlin-Heidelberg-New York: Springer 1975
17. Malyshev, V.A.: Soviet Math. Dokl. **16**, pp. 1141–1145 (Amer. Math. Soc. Transl. 1975)

Received October 3, 1977