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# Weakly and Strongly Mixing Locally Compact Abelian Groups of Measure Preserving Transformations with Application to Abelian Groups of Gaussian Automorphisms*

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## Introduction

In this paper we introduce weakly and strongly mixing locally compact abelian groups of measure preserving transformations. For this purpose we generalise the limit statement of the mean ergodic theorem for a single contraction, to abelian groups of contractions in a Hilbert space, and we define a mean for functions on a locally compact abelian group which plays the role of the strong Cesàro mean for functions defined on the integers. In the second part we consider ergodic, weakly and strongly mixing abelian groups of Gaussian automorphisms. We give a spectral representation for weakly stationary processes having a locally compact abelian group as parameter group, and we generalise Ito's complex multiple Wiener integral to abelian groups of Gaussian automorphisms.

## § 1. Weakly and Strongly Mixing Groups of Transformations

We begin with the generalisation of the limit statement of the mean ergodic theorem.

Let $H$ be a Hilbert space, $G$ an abelian group – in the multiplicative notation –, $\{V_t: t\in G\}$ a necessarily abelian group of contractions in $H$ with the composition $V_s V_t = V_{st}\,(s, t\in G)$, and $\mathscr{K}$ the convex hull of $\{V_t: t\in G\}$. For $t_1, \ldots, t_k\in G$, $(t_1, \ldots, t_k)$ denotes the subgroup of $G$ generated by $\{t_1, \ldots, t_k\}$.

$$\mathscr{I}=\left\{M_l(t_1, \ldots, t_n)=l^{-n} \sum_{i_1, \ldots, i_n=0}^{l-1} V_{t_1^{i_1} \ldots t_n^{i_n}}: l, n\in\mathbb{N}, t_1, \ldots, t_n\in G \text{ such that } (t_1, \ldots, t_n) \text{ is the direct product of } (t_1), \ldots, (t_n)\right\}.$$

$\forall f\in H$ let $\mathscr{K}(f)$ be the norm closure or the

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weak closure of $\{Mf\colon M\in\mathscr{K}\}$, which are equal in this case, and $\mathscr{I}(f)$ the norm closure of $\{Mf\colon f\in\mathscr{I}\}$, and $F=\{h\in H\colon V_t h=h\ \forall t\in G\}$. By the mean ergodic theorem, for any $f\in H$, $\mathscr{K}(f)\cap F$ contains exactly one element $f^*$. (Jacobs [4], **1**, pages 87ff).

**Theorem 1.** $\forall f\in H, \forall\varepsilon>0\ \exists M_{l_0}(t_1,\dots,t_n)\in\mathscr{I}$ *such that* $\|M_l(t_1,\dots,t_n)f-f^*\|<\varepsilon$ $\forall l\geqq l_0$. *In particular* $\mathscr{I}(f)\cap F=\mathscr{K}(f)\cap F$.

*Proof.* Let $f\in H$, $\varepsilon>0$. There is nothing to prove if $f=0$. Let $f\neq 0$. $\exists M\in\mathscr{K}$ such that $\|Mf-f^*\|<\frac{\varepsilon}{2}$ and therefore we can find an $M'=\sum_{k_1,\dots,k_n=0}^{m}\alpha_{k_1,\dots,k_n}V_{t_1^{k_1}\dots t_n^{k_n}}$, where $m\in\mathbb{N}$, $\alpha_{k_1,\dots,k_n}\geqq 0$, $\sum_{k_1,\dots,k_n=0}^{m}\alpha_{k_1,\dots,k_n}=1$, and $t_1,\dots,t_n\in G$ such that $(t_1,\dots,t_n)$ is the direct product of $(t_1),\dots,(t_n)$, for which we also have $\|M'f-f^*\|<\frac{\varepsilon}{2}$. Abbreviating $M_l=M_l(t_1,\dots,t_n)$ we get for $l>m$ $\|M_l-M'M_l\|\leqq 2nml^{-1}$ by a cancellation argument. Thus we get:

$$\exists l_0\in\mathbb{N}\quad \forall l\geqq l_0\colon \|M_l-M'M_l\|<\frac{\varepsilon}{2\|f\|}$$

and

$$\forall l\geqq l_0\ \ \|M_lf-f^*\|\leqq\|M_lf-M'M_lf\|+\|M'M_lf-f^*\|$$
$$\leqq\|M_l-M'M_l\|\cdot\|f\|+\|M'f-f^*\|<\varepsilon.\quad\square$$

If $G$ is a finitely generated abelian group, the choice of $t_1,\dots,t_n$ in the proof of theorem 1 does not depend on $\varepsilon$. Therefore we have the

**Corollary.** *If $G$ is a finitely generated abelian group, i.e. there exist $t_1,\dots,t_n\in G$, such that $G$ is the direct product of $(t_1),\dots,(t_n)$, then for every $f\in H$,* $\lim_{l\to\infty}\|M_l(t_1,\dots,t_n)f-f^*\|=0$.

In the following, let $G$ be a locally compact abelian group and $\Gamma$ the dual group of $G$. $\forall t\in G\ \forall\gamma\in\Gamma$ let $(t,\gamma)=\gamma(t)$. For a topological space $E$, $\mathscr{B}(E)$ denotes the Borel-$\sigma$-algebra in $E$. $M(\Gamma)$ is the set of finite regular Borel measures on $\Gamma$. $\hat{\varphi}$ denotes the Fourier transform of a complex $\varphi$, integrable w.r. to the Haar measure on $\Gamma$, and $\hat{\mu}$ is the Fourier transform of $\mu\in M(\Gamma)$, where a fixed Haar measure on $\Gamma$ was chosen. $dt$ denotes that Haar measure on $G$, which is normed with regard to the fixed Haar measure on $\Gamma$, such that the inversion formula holds (Rudin [8], page 22).

Let $(W_\alpha)_{\alpha\in I}$ be a neighbourhood basis of 1 in $\Gamma$ and $(\varphi_\alpha)_{\alpha\in I}$ a family of real continuous non-negative-definite functions on $\Gamma$, such that each $\varphi_\alpha$ has compact support in $W_\alpha$, $\varphi_\alpha(1)=1$, $\hat{\varphi}_\alpha$ is real and $\hat{\varphi}_\alpha\geqq 0$. (For the existence of such a family see Rudin [8], page 23). From the inversion theorem we get $\int_G\hat{\varphi}_\alpha(t)\,dt=\varphi_\alpha(1)=1$ $\forall\alpha\in I$.

Thus for all real measurable nonnegative functions $\psi$ on $G$

$$M_\alpha\psi=\int_G\hat{\varphi}_\alpha(t)\,\psi(t)\,dt$$

can be looked at as a mean. Furthermore let $a \in \mathbb{R}$ and $\psi$ be a real measurable nonnegative function on $G$. The statement $\lim_\alpha M_\alpha \psi = a$ is defined by: $\forall \varepsilon > 0$ $\exists$ neighbourhood $W$ of 1 in $\Gamma$, such that $\forall \alpha \in I$ for which $W_\alpha \subset W$ we have $|M_\alpha \psi - a| < \varepsilon$.

*Definition 1.* Let $B \in \mathscr{B}(G)$. If $\lim_\alpha M_\alpha 1_B$ exists, then the limit $d(B) \in [0, 1]$ is called the density of $B$.

The mean has the following properties:

1. $\forall \mu \in M(\Gamma)$, $\forall s \in G$ we have

$$\lim_\alpha \int_G \hat{\varphi}_\alpha(t)\, |\hat{\mu}(st)|^2\, dt = \sum_{\gamma \in \Gamma} \mu(\{\gamma\})^2.$$

2. Let $\psi$ be a real bounded measurable nonnegative function on $G$. Then

$$\lim_\alpha M_\alpha \psi = 0 \Leftrightarrow \lim_\alpha M_\alpha \psi^2 = 0 \Leftrightarrow \forall \varepsilon > 0 \ \exists B \in \mathscr{B}(G)$$

for which $d(B) = 0$, such that $\psi|_{B^c} < \varepsilon$.

Properties 1 and 2 evidence the analogy between the mean $\lim_\alpha M_\alpha$ and the strong Cesàro mean when $G = \mathbb{Z}$. Rudin ([8], page 118) considered the particular case, when $\psi = |\hat{\mu}|^2$ for $\mu \in M(\Gamma)$. Let $\{U_t : t \in G\}$ be a necessarily abelian group of isometries in $H$. For $H' \subset H$, let Span $H'$ be the complex linear space which is spanned by $H'$, and $\overline{\text{Span}}\, H'$ the subspace of $H$ which is spanned by $H'$, i.e. the closure of Span H′. We define $H_d = \overline{\text{Span}}\, \{h \in H : \forall t \in G \ \exists c_t \in \mathbb{C}$ such that $U_t h = c_t h\}$ and $H_c = H_d^\perp$. $H_c$ and $H_d$ are invariant under $\{U_t : t \in G\}$. The map $t \mapsto (U_t f, f)$ is non-negative-definite for every $f \in H$. If $G$ is a locally compact abelian group and the map $t \mapsto U_t$ continuous in the weak operator topology or in the strong operator topology which is the same in the case of isometries, then $t \mapsto (U_t f, f)$ is continuous for every $f \in H$, and by Bochner's theorem (Rudin [8], page 19) for every $f \in H$ there is a uniquely determined $\mu_f \in M(\Gamma)$, such that

$$(U_t f, f) = \int_\Gamma (t, \gamma^{-1})\, \mu_f(d\gamma) = \hat{\mu}_f(t) \qquad (t \in G).$$

**Theorem 2.** *Let $H$ be a Hilbert space, $G$ a locally compact abelian group, $\{U_t : t \in G\}$ a group of isometries in $H$, and the map $t \mapsto U_t$ continuous in the weak operator topology. Then for all $f \in H$ the following statements are equivalent:*

1) $f \in H_c$
2) $\mu_f$ *is continuous*
3) $\lim_\alpha M_\alpha |(U_t f, f)| = 0$
4) $\lim_\alpha M_\alpha |(U_t f, g)| = 0 \qquad \forall g \in H$

*If $H$ is separable or if $G$ is countable, then 5) and 6) are also equivalent:*

5) $\exists (t_n)_{n \in \mathbb{N}} \in G$ *such that* $U_{t_n} f \to 0$ $(n \to \infty)$ *weakly.*
6) $\exists (t_n)_{n \in \mathbb{N}} \in G$ $\forall \varepsilon > 0$ $\forall g \in H$ $\exists p \in \mathbb{N}$ $\exists \alpha_1, \dots, \alpha_p \geqq 0$ *for which* $\sum_{n=1}^p \alpha_n = 1$, *such that* $\forall U \in \overline{\{U_t : t \in G\}}$ *we have* $\sum_{n=1}^p \alpha_n |(U U_{t_n} f, g)| < \varepsilon$.

($\bar{\mathscr{U}}=\overline{\{U_t\colon t\in G\}}$ is the closure of $\{U_t\colon t\in G\}$ in the set of bounded operators with respect to the weak operator topology (Jacobs [4], 1, pages 97ff)).

The statements 1), 5), and 6) do not depend on a topology on $G$, on the other hand $G$ can always be equipped with the discrete topology. Then $G$ is locally compact and $t\mapsto U_t$ is continuous in the weak operator topology. Therefore we have the

**Corollary.** *Let $G$ be an abelian group and $\{U_t\colon t\in G\}$ a group of isometries in $H$, such that $H$ is separable or $G$ is countable. Then for all $f\in H$ 1), 5) and 6) are equivalent.*

*Proof of Theorem 2.* The following implications will be proved:

$$4)\Rightarrow 1)\Rightarrow 2)\Rightarrow 3)\Rightarrow 4)\Rightarrow 5)\Rightarrow 6)\Rightarrow 2).$$

If $f\in H$ and $f=0$ everything is evident. Let $f\neq 0$.

4)$\Rightarrow$1): Let $g\in H_d$ such that $\forall\, t\in G\ \exists\, c_t\in\mathbb{C}$ (necessarily $|c_t|=1$) with $U_t g=c_t g$, thus we have $|(f,g)|=|(U_t f,g)|$ and $|(f,g)|=\lim\limits_{\alpha} M_\alpha |(U_t f,g)|=0 \Rightarrow f\in H_c$.

1)$\Rightarrow$2): Let $f\in H_c$ and let us assume that $\exists\,\gamma_0\in\Gamma$ such that $\mu_f(\{\gamma_0\})>0$. $\forall\, t\in G$ let $V_t=(t,\gamma_0)\,U_t$. $\{V_t\colon t\in G\}$ is a group of isometries.

$f\in H_c\Rightarrow f\perp F$, and $f\perp F\Leftrightarrow 0\in\mathscr{K}(f)$ by the mean ergodic theorem. Therefore we have $0\in\mathscr{K}(f)\cap F$. And according to theorem 1 $\exists\, t_1,\ldots,t_n\in G\ \exists\, l_0\in\mathbb{N}$ such that $\forall\, l\geqq l_0$ (using $M_l=M_l(t_1,\ldots,t_n)$)

$$\|M_l f\|<\frac{\mu_f(\{\gamma_0\})}{2\|f\|},$$

and this implies

$$|(M_l f,f)|\leqq \|M_l f\|\,\|f\|<\tfrac{1}{2}\mu_f(\{\gamma_0\}).$$

On the other hand we obtain $\forall\, l\in\mathbb{N}$

$$(M_l f,f)=\int_\Gamma \prod_{k=1}^{n} l^{-1}\sum_{i=0}^{l-1}(t_k^i,\gamma_0\gamma^{-1})\,\mu_f(d\gamma),$$

and $\forall\,\gamma\in\Gamma$ we have

$$\lim_{l\to\infty}\prod_{k=1}^{n} l^{-1}\sum_{i=0}^{l-1}(t_k^i,\gamma_0\gamma^{-1})=\begin{cases}1 & \text{if } \forall\, k\in\{1,\ldots,n\}\\ 0 & (t_k,\gamma_0\gamma^{-1})=1 \text{ otherwise.}\end{cases}$$

By bounded convergence we get

$$\lim_{l\to\infty}(M_f f,f)=\mu_f(\{\gamma\in\Gamma\colon (t_k,\gamma_0\gamma^{-1})=1\ \forall\, k\in\{1,\ldots,n\}\})\geqq\mu_f(\{\gamma_0\}).$$

and this is in contradiction with the above result.

2)$\Rightarrow$3): Use properties 1 and 2 from page 315.

3)⇒4) is implied by the following: Let $H'$, $H''\subset H$. If $\forall f\in H'$ and $\forall g\in H''$ $\lim_\alpha M_\alpha |(U_t f, g)|=0$, then $\forall f\in\overline{\text{Span}\, H'}$ and $\forall g\in\overline{\text{Span}\, H''}$ $\lim_\alpha M_\alpha |(U_t f, g)|=0$. Moreover, if $H'\subset H''$, then $\forall f\in\overline{\text{Span}\, H'}$ and $\forall g\in H$ $\lim_\alpha M_\alpha |(U_t f, g)|=0$.

4)⇒5): If $H$ is separable or if $G$ is countable, there exists a sequence $(g_l)_{l\in\mathbb{N}}\in H$, such that $\{U_t f\colon t\in G\}\subset\overline{\text{Span}}\,\{g_l\colon l\in\mathbb{N}\}$. $\forall l\in\mathbb{N}$ $\forall n\in\mathbb{N}$ $\exists B_n^l\in\mathscr{B}(G)$ for which $d(B_n^l)=1$ such that $\forall t\in B_n^l$

$$|(U_t f, g_l)|<\frac{1}{n}$$

holds (property 2).

Define $\forall n\in\mathbb{N}$ $B_n=\bigcap_{j=1}^{n} B_n^j$. We have $d(B_n)=1$. Let $t_n\in B_n$. $\forall l\in\mathbb{N}$ we have $\lim_{n\to\infty}(U_{t_n} f, g_l)=0$, because of the following: Let $l\in\mathbb{N}$, $\varepsilon>0$ and let $N\in\mathbb{N}$ be as large that $\frac{1}{N}\leqq\varepsilon$ and $N\geqq l$, then $\forall n\geqq N$ $|(U_{t_n} f, g_l)|<\varepsilon$ holds. This implies $U_{t_n} f\to 0$ $(n\to\infty)$ weakly.

5)⇒6): cf. Jones [5].

6)⇒2): Let $(t_n)_{n\in\mathbb{N}}\in G$, such that $\forall\varepsilon>0$ $\exists p\in\mathbb{N}$ $\exists\alpha_1,\ldots,\alpha_p\geqq 0$ such that $\sum_{n=1}^{p}\alpha_n=1$, and

$$\sum_{n=1}^{p}\alpha_n |(U_{tt_n} f, f)|<\frac{\varepsilon}{\|f\|^2}$$

holds for all $t\in G$.

$$\Rightarrow\forall t\in G\colon \sum_{n=1}^{p}\alpha_n |(U_{tt_n} f, f)|^2<\varepsilon$$

$$\Rightarrow\sum_{\gamma\in\Gamma}\mu_f(\{\gamma\})^2=\lim_\alpha M_\alpha\sum_{n=1}^{p}\alpha_n |(U_{tt_n} f, f)|^2\leqq\varepsilon$$

(using property 1)

$\Rightarrow\mu_f$ is continuous.

This completes the proof of theorem 2.

Let $(\Omega,\mathscr{F},P)$ be a probability space, $G$ an abelian group and $\{T_t\colon t\in G\}$ a group of measure preserving transformations in $(\Omega,\mathscr{F},P)$. $\forall t\in G$ the isometry $U_t$ in $L_2(P)$ is defined by $U_t f=f\circ T_t$ $(f\in L_2(P))$. $\{U_t\colon t\in G\}$ is an abelian group of isometries in $L_2(P)$.

*Definition 2.* $\{T_t\colon t\in G\}$ is called weakly mixing if $L_2(P)_c=\{1\}^\perp$.

*Remark.* The definition of weakly mixing for $G=\mathbb{R}$ given by Rohlin [7] and the usual definition for $G=\mathbb{Z}$ are equivalent to the above definition. If $G$ can be equipped with a topology in such a way that $G$ is a compact topological group and that $t \mapsto U_t$ is continuous in the weak operator topology, then we have $L_2(P)_c = \{0\}$. Then $\{T_t: t \in G\}$ cannot be weakly mixing except in trivial cases. Weakly mixing implies ergodicity.

A consequence of theorem 2 is

**Theorem 3.** *The following statements are equivalent:*

1) $\{T_t: t \in G\}$ *is weakly mixing.*

*And, if* $L_2(P)$ *is separable or if* $G$ *is countable,* 2) *and* 3).

2) $\forall f \in \{1\}^{\perp}\ \exists (t_n)_{n \in \mathbb{N}} \in G$ *such that* $U_{t_n} f \underset{n \to \infty}{\to} 0$ *weakly.*

3) $\forall f \in \{1\}^{\perp}\ \exists (t_n)_{n \in \mathbb{N}} \in G\ \forall \varepsilon > 0\ \forall g \in L_2(P)\ \exists p \in \mathbb{N}\ \exists \alpha_1, \ldots, \alpha_p \geqq 0$ *where* $\sum_{n=1}^{p} \alpha_n = 1$ *such that* $\forall U \in \bar{\mathscr{U}}$: $\sum_{n=1}^{p} \alpha_n |(U U_{t_n} f, g)| < \varepsilon$.

*If* $G$ *is a locally compact abelian group and if* $t \mapsto U_t$ *is continuous in the weak operator topology, then* 4), 5), 6), 7) *and* 8) *are equivalent to* 1):

4) $\lim_\alpha M_\alpha |P(B \cap T_t^{-1} C) - P(B) P(C)| = 0 \quad \forall B, C \in \mathscr{F}$.

5) $\lim_\alpha M_\alpha |P(B \cap T_t^{-1} B) - P(B)^2| = 0 \quad \forall B \in \mathscr{F}$.

6) $\{T_t: t \in G\}$ *has a continuous spectrum in* $\{1\}^{\perp}$, *i.e.* $\forall f \in \{1\}^{\perp}$ $\mu_f$ *is continuous.*

7) $\lim_\alpha M_\alpha |(U_t f, f)| = 0 \quad \forall f \in \{1\}^{\perp}$.

8) $\lim_\alpha M_\alpha |(U_t f, g)| = 0 \quad \forall f \in \{1\}^{\perp}\ \forall g \in L_2(P)$.

*Remark.* $\{T_t: t \in G\}$ is weakly mixing if and only if $\{T_t \times T_t: t \in G\}$ is ergodic.

The proof is the same as for $G = \mathbb{Z}$ (see e.g. Halmos [2], page 39) using the mean $\lim_\alpha M_\alpha$ instead of the strong Cesàro mean, thereby $G$ is equipped with the discrete topology.

Again let $G$ be a locally compact abelian group and let $t \mapsto U_t$ be continuous in the weak operator topology. $C_0(G)$ is the set of the continuous complex functions on $G$ which vanish at infinity.

*Definition 3.* $\{T_t: t \in G\}$ is called (strongly) mixing if $P(B \cap T_t^{-1} C) - P(B) P(C) \in C_0(G) \quad \forall B, C \in \mathscr{F}$.

*Remark.* The notion of strong mixing depends on the topology of $G$. If $G$ is compact then $\{T_t: t \in G\}$ is strongly mixing by definition. Thus the definition is not useful for compact groups (cf. remark after definition 2).

**Theorem 4.** *The following statements are equivalent:*

1) $\{T_t: t \in G\}$ *is mixing.*
2) $P(B \cap T_t^{-1} B) - P(B)^2 \in C_0(G) \quad \forall B \in \mathscr{F}$.
3) $\hat{\mu}_f \in C_0(G) \quad \forall f \in \{1\}^{\perp}$.
4) $(U_t f, g) \in C_0(G) \quad \forall f \in \{1\}^{\perp}\ \forall g \in L_2(P)$.

*Remark.* If $G$ is a locally compact and not compact abelian group and $\mu \in M(\Gamma)$, then $\hat{\mu} \in C_0(G)$ implies that $\mu$ is continuous. For the proof see Rudin [8], pages 118–119, but $G$ may not be compact and there, that is overlooked. Thus we have: If $G$ is a locally compact and not compact abelian group and if $t \mapsto U_t$ is continuous in the weak operator topology, then from $\{T_t: t \in G\}$ mixing follows that $\{T_t: t \in G\}$ is weakly mixing.

## §2. Ergodic, Weakly and Strongly Mixing Gaussian Processes

Again let $(\Omega, \mathscr{F}, P)$ be the underlying probability space. All the stochastic processes mentioned in this section are complex processes, unless special mention is made. Second order processes are processes which have finite second moments, normalised processes have expectation 0. Let $G$ be a locally compact abelian group and $\Gamma$ its dual group. Second order processes $(Z_t)_{t \in G}$ are called weakly stationary if $\forall s, t \in G$: $E Z_s \bar{Z}_t = E Z_{st^{-1}} \bar{Z}_1$. Stationary second order processes are weakly stationary. If $(Z_t)_{t \in G}$ is weakly stationary, then the function $t \mapsto E Z_t \bar{Z}_1$ is non-negative-definite. If the map $t \mapsto Z_t$ is continuous in the quadratic mean, then according to Bochner's theorem (Rudin [8], page 13) there exists a uniquely determined measure $\mu \in M(\Gamma)$, the spectral measure of $(Z_t)_{t \in G}$, such that $E Z_t \bar{Z}_1 = \int_\Gamma (t, \gamma^{-1})\, \mu(d\gamma) = \hat{\mu}(t)$ $(t \in G)$. The spectral measure is symmetric for real processes.

**Theorem 5** (Spectral representation). *Let $G$ be a locally compact abelian group and $\Gamma$ its dual group, $(Z_t)_{t \in G}$ a weakly stationary process, $t \mapsto Z_t$ continuous in the quadratic mean, $\mu$ the spectral measure of $(Z_t)_{t \in G}$, and let the underlying probability space be rich (fine) enough. Then there is a subspace $L$ of $L_2(P)$ which contains* $\overline{\text{Span}}\,\{Z_t: t \in G\}$, *and there is an orthogonal valued measure $M: \mathscr{B}(\Gamma) \to L$* (see Urbanik [10], pages 6ff for $G = \mathbb{Z}$) *such that*

1) $\int \cdot\, dM: L_2(\mu) \to L$ *is an isomorphism, in particular* $\forall B \in \mathscr{B}(\Gamma)$: $\mu(B) = \|M(B)\|^2$.
2) $\forall t \in G$: $Z_t = \int (t^{-1}, \cdot)\, dM$ *P-a.s.*
3) *if* $(Z_t)_{t \in G}$ *is normalised, then* $\forall \varphi \in L_2(\mu)$: $E \int \varphi\, dM = 0$.
4) *if* $(Z_t)_{t \in G}$ *is Gaussian, then* $\int \varphi_1\, dM, \ldots, \int \varphi_n\, dM$ *are jointly Gaussian* $\forall \varphi_1, \ldots, \varphi_n \in L_2(\mu)$.
5) *if $G$ is discrete (i.e. equipped with the discrete topology), then* $L = \overline{\text{Span}}\,\{Z_t: t \in G\}$.

*$M$ has the following properties, if $G$ is discrete.*

6) *If* $\{\tilde{U}_t: t \in G\}$ *is the abelian group of unitary operators on $L$ for which* $\tilde{U}_s Z_t = Z_{ts}$ $(s, t \in G)$, *then*
$\forall \varphi \in L_2(\mu)$: $\tilde{U}_t \int \varphi\, dM = \int (t^{-1}, \cdot)\, \varphi\, dM$ *P-a.s.*

*And if we also assume* $(Z_t)_{t \in G}$ *to be real, then*

7) $\forall \varphi \in L_2(\mu)$: $\int \varphi\, dM = \int \bar{\varphi}(\cdot^{-1})\, dM$ *P-a.s., in particular* $\forall B \in \mathscr{B}(\Gamma)$: $M(B) = M(B^{-1})$.
8) *We have* $\forall B, C \in \mathscr{B}(\Gamma)$
   i) $\operatorname{Re} M(B) = \operatorname{Re} M(B^{-1})$, $\operatorname{Im} M(B) = -\operatorname{Im} M(B^{-1})$
   ii) $E(\operatorname{Re} M(B) \operatorname{Im} M(C)) = 0$
   iii) $E(\operatorname{Re} M(B) \operatorname{Re} M(C)) = \frac{1}{2}(\mu(B \cap C) + \mu(B \cap C^{-1}))$
   iv) $E(\operatorname{Im} M(B) \operatorname{Im} M(C)) = \frac{1}{2}(\mu(B \cap C) - \mu(B \cap C^{-1}))$

9) *If* $(Z_t)_{t\in G}$ *is normalised and Gaussian, then*

i) $(\operatorname{Re} M(B))_{B\in\mathscr{B}(\Gamma)}$ *and* $(\operatorname{Im} M(B))_{B\in\mathscr{B}(\Gamma)}$ *are normalised, Gaussian and independent,*

ii) $\forall B_1,\dots,B_n\in\mathscr{B}(\Gamma)$, *for which* $B_1,\dots,B_n,B_1^{-1},\dots,B_n^{-1}$ *are pairwise disjoint* $M(B_1),\dots,M(B_n)$ *are independent.*

Theorem 5 is a generalisation of Cramér's spectral representation (Cramér [1]). The proof is similar to the case where $G=\mathbb{Z}$.

The following statement: - Let $\mu\in M(\Gamma)$ and $B\in\mathscr{B}(\Gamma)$ such that $B=B^{-1}$, $\{\gamma\in\Gamma:\gamma^2=1\}\subset B$ and $\mu(\Gamma\setminus B)>0$. Then there is a compact $K\subset\Gamma\setminus B$ such that $K\cap K^{-1}=\emptyset$ and $\mu(K\cup K^{-1})>0$ - together with an exhaustion argument gives

**Theorem 6.** *Let* $\mu\in M(\Gamma)$. *Then there exists* $\tilde{\Gamma}\in\mathscr{B}(\Gamma)$ *such that* $\tilde{\Gamma}\cap\tilde{\Gamma}^{-1}=\emptyset$ *and* $\mu(\tilde{\Gamma}\cup\tilde{\Gamma}^{-1})=\mu(\Gamma\setminus\{\gamma\in\Gamma:\gamma^2=1\})$.

Theorem 9 is valid for all Hausdorff topological groups.

Now we extend the multiple Wiener integral to real normalised stationary Gaussian processes with distribution $R$ having an abelian group $G$ as parameterset. The spectral measure $\mu$ of the process, which is obtained, when $G$ is equipped with the discrete topology, is assumed to be continuous and $\mu(\{\gamma\in\Gamma:\gamma^2=1\})=0$.

Let $K_t\colon \mathbb{R}^G\to\mathbb{R}$ be defined by $K_t(x)=x_t$ and $T_t\colon\mathbb{R}^G\to\mathbb{R}^G$ by $(T_t(x))_s=x_{ts}$ $(s,t\in G;\ x=(x_s)_{s\in G}\in\mathbb{R}^G)$. $\{T_t\colon t\in G\}$ is an abelian group of measure preserving invertible transformations on $(\mathbb{R}^G,\mathscr{B}^G,R)$. Let $\{U_t\colon t\in G\}$ be the abelian group of unitary operators in $L_2(R)$, which was defined for $\{T_t\colon t\in G\}$ in section 1.

The definition of the multiple Wiener integral can be done by repeating the arguments from Totoki [9], pages 48–57 (cf. Ito [3]), if one replaces the set $[-\frac{1}{2},0]$, which is used there for $G=\mathbb{Z}$, by $\tilde{\Gamma}$ from theorem 6, using the measure with orthogonal values $M$ which belongs to the process $(K_t)_{t\in G}$ according to theorem 5. For $p\in\mathbb{N}\cup\{0\}$ let $I_p$ be the $p$-th complex multiple Wiener integral,

$$H_0=\{I_p(c)\colon c\in\mathbb{C}\}=\mathbb{C}$$

$$H_p=\{I_p(\varphi_p)\colon \varphi_p\in L_2(\Gamma^p,\mu^p)\}\qquad (p\in\mathbb{N}).$$

The $H_p (p\in\mathbb{N}\cup\{0\})$ are pairwise orthogonal and invariant under $\{U_t\colon t\in G\}$ and $L_2(R)=\bigoplus_{p=0}^{\infty} H_p$.

For $\varphi_p\in L_2(\Gamma^p,\mu^p)$ let $\tilde{\varphi}_p=\frac{1}{p!}\sum_{\pi\in\mathscr{S}_p}\varphi_p\circ\pi$, where $\mathscr{S}_p$ is the $p$-th permutation group. We have $I_p(\varphi_p)=I_p(\tilde{\varphi}_p)$ and

$$U_t I_p(\tilde{\varphi}_p)=I_p\left(\prod_{\nu=1}^{p}(t^{-1},\gamma_\nu)\,\tilde{\varphi}_p(\gamma_1,\dots,\gamma_p)\right)\qquad (t\in G).$$

**Theorem 7.** *Let G be an abelian group and R the distribution of a real normalised stationary Gaussian process with parameterset G. Let Γ be the dual group of G and μ the spectral measure of the process, which we get, when we equip G with the* *discrete topology, and let* $\mu(\{\gamma\in\Gamma:\gamma^2=1\})=0$. $\{T_t: t\in G\}$ *is the above defined group of measure preserving invertible transformations of* $(\mathbb{R}^G, \mathscr{B}^G, R)$. *Then the following statements are equivalent:*

1) $\mu$ *is continuous.*
2) $\{T_t: t\in G\}$ *is ergodic.*
3) $\{T_t: t\in G\}$ *is weakly mixing.*

*Proof.* 3)$\Rightarrow$2) is evident, and 2)$\Rightarrow$1) is similar as in the case $G=\mathbb{Z}$ (Totoki [9], page 58).

1)$\Rightarrow$3): i) Let $f\in L_2(R)$ be eigenvector of $\{U_t: t\in G\}$, then $\exists\,\gamma_0\in\Gamma$ such that $U_t f=(t,\gamma_0)f$ $(t\in G)$.

We have: $\forall\, p\in\mathbb{N}\ \exists\,\varphi_p\in L_2(\Gamma^p,\mu^p)$ and $\exists\,\varphi_0\in\mathbb{C}$ such that $f=\sum_{p=0}^{\infty} I_p(\varphi_p)$.

Now we get $(t,\gamma_0)f=\sum_{p=0}^{\infty} I_p((t,\gamma_0)\tilde{\varphi}_p)$ and

$$U_t f=\sum_{p=0}^{\infty} I_p\left(\prod_{\nu=1}^{p}(t^{-1},\gamma_\nu)\,\tilde{\varphi}_p(\gamma_1,\ldots,\gamma_p)\right)\qquad (t\in G).$$

Thus $\forall\, p\in\mathbb{N}\ \forall\, t\in G\ \exists\, N(p,t)\in\mathscr{B}(\Gamma^p)$ for which $\mu^p(N(p,t))=0$ such that $\forall(\gamma_1,\ldots,\gamma_p)\in N(p,t)^c$:

$$(t,\gamma_0)\,\tilde{\varphi}_p(\gamma_1,\ldots,\gamma_p)=\prod_{\nu=1}^{p}(t,\gamma_\nu^{-1})\,\tilde{\varphi}_p(\gamma_1,\ldots,\gamma_p),$$

and therefore

$$\tilde{\varphi}_p(\gamma_1,\ldots,\gamma_p)=\left(t,\gamma_0\prod_{\nu=1}^{p}\gamma_\nu\right)\tilde{\varphi}_p(\gamma_1,\ldots,\gamma_p).$$

ii) $\forall\, p\in\mathbb{N}\ \forall\,\varepsilon>0\ \exists\, M(p,\varepsilon)\in\mathscr{B}(\Gamma^p)$ for which $\mu^p(M(p,\varepsilon)^c)<\varepsilon$ such that $\tilde{\varphi}_p(\gamma_1,\ldots,\gamma_p)=0\ \forall(\gamma_1,\ldots,\gamma_p)\in M(p,\varepsilon)$. This is implied by the following:

For $N(p)=\left\{(\gamma_1,\ldots,\gamma_p)\in\Gamma^p:\ \gamma_0\prod_{\nu=1}^{p}\gamma_\nu=1\right\}$ we have $\mu^p(N(p))=0$ since $\mu$ is continuous.

$\forall(\gamma_1,\ldots,\gamma_p)\in N(p)^c\ \exists\, t_{\gamma_1,\ldots,\gamma_p}\in G$ such that $\left(t_{\gamma_1,\ldots,\gamma_p},\gamma_0\prod_{\nu=1}^{p}\gamma_\nu\right)\neq 1$ since $G$ separates the points of $\Gamma$ (see Rudin [8], page 24).

$\Rightarrow\forall(\gamma_1,\ldots,\gamma_p)\in N(p)^c\ \exists$ open neighbourhood $U(\gamma_1,\ldots,\gamma_p)$ of $(\gamma_1,\ldots,\gamma_p)$ such that $\forall(\bar{\gamma}_1,\ldots,\bar{\gamma}_p)\in U(\gamma_1,\ldots,\gamma_p)$

$$\left(t_{\gamma_1,\ldots,\gamma_p},\gamma_0\prod_{\nu=1}^{p}\bar{\gamma}_\nu\right)\neq 1.$$

Furthermore: $\exists$ compact $K(p,\varepsilon)\subset N(p)^c$ with $\mu^p(K(p,\varepsilon)^c)<\varepsilon$.

$\{U(\gamma_1,\ldots,\gamma_p):(\gamma_1,\ldots,\gamma_p)\in K(p,\varepsilon)\}$ is an open covering of $K(p,\varepsilon)$.

$\Rightarrow \exists(\gamma_{\kappa 1}, \ldots, \gamma_{\kappa p}) \in K(p, \varepsilon) \quad (\kappa=1, \ldots, k)$

such that

$$K(p, \varepsilon) \subset \bigcup_{\kappa=1}^{k} U(\gamma_{\kappa 1}, \ldots, \gamma_{\kappa p}).$$

$$M(p, \varepsilon)=K(p, \varepsilon) \setminus \bigcup_{\kappa=1}^{k} N(p, t_{\gamma_{\kappa 1}, \ldots, \gamma_{\kappa p}})$$

has the required properties.

$\Rightarrow \tilde{\varphi}_p=0 \quad \mu^p\text{-a.s. } \forall p \in \mathbb{N} \Rightarrow f=I_0(\varphi_0) \in \mathbb{C} \Rightarrow L_2(R)_c=\{1\}^{\perp}$

$\Rightarrow \{T_t: t \in G\}$ is weakly mixing. □

The equivalence of 1) and 2) has been proved by Maruyama [6] if $G=\mathbb{Z}$.

**Theorem 8.** *The same assumptions as in theorem 7 are made and G is assumed to be infinite. Then* 1) $\hat{\mu} \in C_0(G)$ *and* 2) $\{T_t: t \in G\}$ *is mixing, are equivalent.*

*Proof.* We have to prove 1)⇒2): $\forall p \in \mathbb{N}$ let $N_p=\{\varphi \in L_2(\Gamma^p, \mu^p): \varphi=1_{B_1 \times \ldots \times B_p}$ where $B_1, \ldots, B_p \in \mathscr{B}(\Gamma)$ and $B_\nu \subset \tilde{\Gamma}$ *or* $B_\nu^{-1} \subset \tilde{\Gamma}$ $(\nu=1, \ldots, p)$ and $B_1, \ldots, B_p$, $B_1^{-1}, \ldots, B_p^{-1}$ are pairwise disjoint$\}$. Since $G$ is not finite, $\mu$ is continuous, and we have $\overline{\text{Span}}\, N_p=L_2(\Gamma^p, \mu^p)$ (Totoki [9], page 51) and $H_p=\overline{\text{Span}}\,\{I_p(\varphi_p): \varphi_p \in N_p\}$.

For $H'=\{I_p(\varphi_p): p \in \mathbb{N}, \varphi_p \in N_p\}$ we have $\overline{\text{Span}}\, H'=\{1\}^{\perp}$, and it is enough to show that $(U_t f, f) \in C_0(G)\ \forall f \in H'$. Consequently let $p \in \mathbb{N}$, $\varphi_p \in N_p$, $\varphi_p=1_{B_1 \times \ldots \times B_p}$. We get $I_p(\varphi_p)=M(B_1) \ldots M(B_p)$ and $U_t I_p(\varphi_p)=U_t M(B_1) \ldots U_t M(B_p) \quad (t \in G)$.

We have $\forall \nu \in \{1, \ldots, p\}$ $U_t M(B_\nu)=\int \overline{(t^{-1}, \cdot)}\, 1_{B_\nu} dM$ (theorem 5).

$\Rightarrow U_t M(B_\nu)$ and hence $U_t M(B_\nu) \overline{M(B_\nu)}$ are measurable with respect to $\mathscr{A}_\nu$ $=\sigma(M(C): C \in \mathscr{B}(\Gamma), C \subset B_\nu)$.

$\mathscr{A}_1, \ldots, \mathscr{A}_p$ are independent.

$$\Rightarrow (U_t I_p(\varphi_p), I_p(\varphi_p))=\prod_{\nu=1}^{p}(U_t M(B_\nu), M(B_\nu)) \in C_0(G),$$

using $(U_t M(B_\nu), M(B_\nu)) \in C_0(G)$ since $M(B_\nu) \in \overline{\text{Span}}\,\{K_t: t \in G\}$ $(\nu=1, \ldots, p)$. Thus theorem 8 is proved.

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