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On Discounted Subfair Primitive Casino

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Summary. As is well known, in a subfair primitive casino a gambler with an initial fortune f, $0<f<1$, desiring to reach 1 (his goal) should play boldly since there is no other strategy that can provide him with a higher utility (the probability of reaching his goal). Now suppose the game is modified by adding a discount factor which is used to motivate the gambler to recognize the time value of his goal and complete the game as quickly as is reasonably consistent with reaching his goal. Then one would intuitively suspect that again the bold play would be optimal. We will show in this paper that for certain subfair or fair primitive casinos the bold play is always optimal regardless of the discount factor; however, for some subfair or fair primitive casinos, there exist some discount factors for which the bold play is no longer optimal.

1. Discounted Primitive Casino

Consider the following gambling problem: A gambler has an initial fortune in $(0, 1)$ and wishes to reach 1 (his goal). He may stake any amount, s, of his current fortune in each game, winning $s(1-r)/r$ with probability w and losing s with probability $1-w$, where w and r are two constants in $[0, 1]$. He receives a utility β^n if he reaches his goal on the n^{th} game, where $0<\beta\leqq 1$ is the discount factor. Then what is the optimal strategy which provides the gambler with the highest expected utility?

To make this new gambling problem fit more clearly and easily into the gambling framework of Dubins and Savage (1965), we consider it as one whose set of fortunes, utility function, and set of available gambles are as follows (although the game itself is unchanged): $F=[0, \infty)$; $u(f)=0$ or 1 according as $0\leqq f<1$ or $f\geqq 1$; $\Gamma_{\beta,w,r}(f)=\Gamma(f)=\{\gamma(f,s)\mid\gamma(f,s)=\bar{\beta}\delta(0)+\beta w\delta(f+s[\bar{r}/r]) +\beta\bar{w}\delta(f-s);\ 0<s\leqq f\}$ if $f<1$ and $\Gamma_{\beta,w,r}(f)=\Gamma(f)=\{\delta(f)\}$ if $f\geqq 1$. Here β, w, r are three constants such that $0<\beta\leqq 1$ $(\bar{\beta}=1-\beta)$, $0\leqq w\leqq 1$ $(\bar{w}=1-w)$, $0\leqq r\leqq 1$ ($\bar{r}$

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$=1-r$), and $\delta(f)$ denotes the probability measure which assigns probability one to $\{f\}$. The reason that we set $\Gamma(f)=\{\delta(f)\}$ for $f \geqq 1$ is that when the gambler has a fortune $f \geqq 1$, he has reached his goal already and he can leave the gambling house.

This new gambling problem is a modification of the primitive casino considered by Dubins and Savage (1965) and is also a generalization of the discounted red-and-black considered by Klugman (1977). The modification is designed to motivate the gambler to recognize the time value of his goal and complete the game as quickly as is reasonably consistent with reaching his goal. To distinguish it from the primitive casino game considered by Dubins and Savage, this new gambling problem will be simply called "discounted primitive casino".

In [4], Dubins and Savage showed that the bold strategy (the strategy which stakes as much as possible without risk of overshooting the goal) is optimal for a subfair primitive casino (Theorem 6.3.1. of [4]). In [6], Klugman also showed that the bold strategy is still optimal for a discounted subfair red-and-black (Theorem 2.4 of [6]). Based on these results about the optimality of the bold strategy, one would intuitively suspect that the bold strategy should also be optimal for a discounted subfair primitive casino. However, this intuitive conjecture is not always true and, in [3], Chen has shown that the bold strategy is not necessarily optimal for a discounted subfair primitive casino if $0<w \leqq r<\frac{1}{2}$ or $\frac{1}{2}<w \leqq r<(\sqrt{5}-1)/2$ (Theorem 3 of [3]). In this paper, we show that the bold strategy is again optimal for a discounted primitive casino if $0 \leqq w \leqq \frac{1}{2} \leqq r \leqq 1$ or $\frac{1}{2} \leqq w \leqq r \leqq 1$ and $r \geqq (\sqrt{5}-1)/2$. The results in this paper and [3] combined with the results in [4] (Chap. 6 of [4]) provide us with a better understanding of subfair primitive casinos (discounted or non-discounted). Since this paper is a continuation work of [3], all notation will follow that in [3]. Since the possibilities $w=0$ or 1, or $r=0$ or 1, would not be interesting, we will always assume that $0<w$, $r<1$ in this paper.

2. The Utility of Bold Strategies

In [3], Chen showed that, in a discounted primitive casino $\Gamma_{\beta,w,r}$, the utility function of the bold strategy is the unique bounded solution of the following functional equation:

$$R_{\beta,w,r}(f)=\begin{cases} \beta w R_{\beta,w,r}^{-1}(f/r) & \text{if } 0 \leqq f<r, \\ \beta w+\beta \bar{w} R_{\beta,w,r}^{-1}[(f-r)/\bar{r}] & \text{if } r \leqq f<1, \\ 1 & \text{if } f \geqq 1. \end{cases} \tag{1}$$

Furthermore, $R_{\beta,w,r}$ is right continuous and strictly increasing on the interval $[0,1]$ and satisfies the following identities:

$$R_{\beta,w,r}(f)=R_{\beta,w,\frac{1}{2}}[R_{1,r,\frac{1}{2}}^{-1}(f)] \quad \text{if } 0 \leqq f \leqq 1 \text{ and } 0<r<1, \tag{2}$$

$$R_{1,r,\frac{1}{2}}^{-1}(f)=R_{1,\frac{1}{2},r}(f) \quad \text{if } 0 \leqq f \leqq 1 \text{ and } 0<r<1. \tag{3}$$

($R_{1,w,r}$ is continuous and strictly increasing if $0<w$, $r<1$; see pp. 99 of [4] too).

For simplicity, we will hereafter write R_β for $R_{\beta,w,r}$ whenever w and r are fixed.

3. The Optimality and Non-Optimality of the Bold Strategy

As defined in [4], an available strategy (in a discounted primitive casino) for the gambler is a sequence $\sigma=(\sigma_0,\sigma_1,\sigma_2,---)$ such that σ_0 is a gamble in the set $\Gamma_{\beta,w,r}(f)$ of available gambles, and, for each integer $n\geqq 1$, σ_n is a gamble in the set $\Gamma_{\beta,w,r}(f_n)$ of available gambles, where f is the gambler's initial fortune and f_n is the gambler's fortune after n games

The worth of a particular available strategy (in a discounted primitive casino) σ is given by its utility, $u(\sigma)$, i.e., the probability that the gamber reaches his goal by using the strategy σ. An available strategy for the gambler is optimal if no other available strategy has a higher utility. In this section, we show that the bold strategy is optimal for a discounted subfair or fair primitive casino if $0\leqq w\leqq\frac{1}{2}\leqq r\leqq 1$ or $\frac{1}{2}\leqq w\leqq r$ and $(\sqrt{5}-1)/2\leqq r\leqq 1$. We also show that the bold strategy is not optimal for a discounted superfair primitive casino even if $\beta w\leqq r$. With the result about the non-optimality of the bold strategy in [3], now we are able to tell a gambler when he should play boldly and when he should not play boldly in a discounted primitive casino. As in [3], let $V_\beta(f,g)=R_\beta(rf+\bar{r}g)-\beta wR_\beta(f)-\beta\bar{w}R_\beta(g)$ for $0\leqq g\leqq f\leqq 1$. Let $V'_\beta(f,g)=R_\beta(f+g)-R_\beta(f)-R_\beta(g)$ if $0\leqq f,g,f+g\leqq 1$, and $V'_\beta(f,g)=R_\beta(f+g-1)-R_\beta(f)-R_\beta(g)+(1-\beta\bar{w})(\beta w)^{-1}$ if $0\leqq f,g\leqq 1$ and $1<f+g\leqq 2$.

From [3] and [6] we have the following lemmas.

Lemma 1. *$V_\beta(f,g)\geqq 0$ for all $0\leqq g\leqq f\leqq 1$ if and only if $V'_\beta(f,g)\geqq 0$ for all $0\leqq f$, $g\leqq 1$ and $0\leqq f+g\leqq 2$.*

Proof. See p. 173 of [3].

Lemma 2. *If $r=\frac{1}{2}$, $0\leqq w\leqq\frac{1}{2}$, and $0<\beta\leqq 1$, then the bold strategy is optimal.*

Proof. See p. 737 of [6].

The next lemma is elementary, but it is included for the sake of completeness.

Lemma 3. *If $\{d_i|i=0,1,2\ldots\}$ is a non-decreasing sequence of non-negative integers and $\frac{1}{2}\leqq w\leqq 1$, then $\sum_{i=0}^{\infty} w^{d_i}\bar{w}^{i+1}(i+d_i)\leqq 1$.*

Proof. (i) If $d_0=d_1=\ldots=0$, then

$$\sum_{i=0}^{\infty}(w)^{d_i}(\bar{w})^{i+1}(i+d_i)=\sum_{i=1}^{\infty}\bar{w}^{i+1}(i)=\bar{w}^2/w^2\leqq 1.$$

(ii) If $d_0=d_1=\ldots=d_p=0<1\leqq d_{p+1}\leqq\ldots$ for some integer $p\geqq 0$, then

$$\sum_{i=0}^{\infty} w^{d_i}\bar{w}^{i+1}(d_i+i)=\sum_{i=0}^{p}\bar{w}^{i+1}(i)+\sum_{i=p+1}^{\infty} w^{d_i}\bar{w}^{i+1}(d_i+i)$$
$$\leqq \sum_{i=1}^{\infty} w\bar{w}^{i+1}(i)+\sum_{i=0}^{p}\bar{w}^{i+2}(i)+\sum_{i=p+1}^{\infty} w^{d_i}\bar{w}^{i+1}(d_i)$$
$$=(\bar{w}^2/w^2)\{1-\bar{w}^{p+1}(1+pw)\}+\sum_{i=p+1}^{\infty} w^{d_i}\bar{w}^{i+1}(d_i).$$

(a) If $w\geqq\frac{2}{3}$ then

$$\sum_{i=p+1}^{\infty} w^{d_i}\bar{w}^{i+1}(d_i)\leqq w^{(w/\bar{w})}\sum_{i=p+1}^{\infty}\bar{w}^i\leqq w\bar{w}$$

and

$$\sum_{i=0}^{\infty} w^{d_i}\bar{w}^{i+1}(d_i+i)\leqq(\bar{w}^2/w^2)+w\bar{w}\leqq 1.$$

(b) If $\frac{1}{2}\leqq w<\frac{2}{3}$ then

$$\sum_{i=p+1}^{\infty} w^{d_i}\bar{w}^{i+1}(d_i)\leqq 2w\bar{w}^{p+2}$$

and

$$\sum_{i=0}^{\infty} w^{d_i}\bar{w}^{i+1}(d_i+i)\leqq(\bar{w}^2/w^2)\{1-\bar{w}^{p+1}(1+pw)+2w^3\bar{w}^p\}\leqq 1.$$

(iii) If $d_0\geqq 1$ then

$$\sum_{i=0}^{\infty} w^{d_i}\bar{w}^{i+1}(d_i+i)\leqq(\bar{w}^2/w)+w^{(w/\bar{w})-1}\leqq 1 \quad \text{if } w\geqq\tfrac{2}{3},$$

and

$$\sum_{i=0}^{\infty} w^{d_i}\bar{w}^{i+1}(d_i+i)\leqq(\bar{w}^2/w)+2w\bar{w}\leqq 1 \quad \text{if } \tfrac{1}{2}\leqq w<\tfrac{2}{3}.$$

Lemma 4. *If $\frac{1}{2}\leqq w\leqq r<1$, then $V_\beta'(f,g)\geqq 0$ for all $0<\beta\leqq 1$, $0\leqq f$, $g\leqq 1$, and $1\leqq f+g\leqq 2$.*

Proof. Since $V_1'(f,g)\geqq 0$ (Theorem 6.3.1 of [4]) and $V_\beta'(f,g)=V_1'(f,g)-[V_1'(f,g)-V_\beta'(f,g)]$, it suffices to show $V_1'(f,g)-V_\beta'(f,g)\leqq 0$ for all $0<\beta\leqq 1$, $0\leqq f$, $g\leqq 1$, and $1\leqq f+g\leqq 2$. Now

$$V_1'(f,g)-V_\beta'(f,g)=[1+R_1(f+g-1)-R_1(f)-R_1(g)]-[(1-\beta w)/(\beta\bar{w})+R_\beta(f+g-1)-R_\beta(f)-R_\beta(g)]$$
$$\leqq[R_1(f+g-1)-R_\beta(f+g-1)]-(1-\beta)/(\beta\bar{w})$$

for all $0<\beta\leqq 1$, $0\leqq f$, $g\leqq 1$, and $1\leqq f+g\leqq 2$ since $R_1(f)\geqq R_\beta(f)$ and $R_1(g)\geqq R_\beta(g)$ for all $0<\beta\leqq 1$. Since R_β is right continuous, it suffices to show that $(1-\beta)/(\beta\bar{w})\geqq R_1(h)-R_\beta(h)$ for all $h=\sum_{i=0}^{n} r^{d_i}\bar{r}^i$ where $\{d_0,d_1,\ldots,d_n\}$ is a non-decreasing, finite sequence of positive integers. $\left(\text{We will always write } r^d \text{ for } \sum_{j=0}^{\infty} r^{d+1}\bar{r}^j \text{ for any integer } d\geqq 0\right)$. Since, by the functional equation (1) on p. 2, if $h=\sum_{i=0}^{n} r^{d_i}\bar{r}^i$, then $R_1(h)=\sum_{i=0}^{n} w^{d_i}\bar{w}^i$ and $R_\beta(h)=\sum_{i=0}^{n}(\beta w)^{d_i}(\beta\bar{w})^i$,

$$R_1(h)-R_\beta(h)=\sum_{i=0}^{n}(w)^{d_i}(\bar{w})^i\{1-(\beta)^{d_i+i}\}$$

$$\leqq(1-\beta)\sum_{i=0}^{n} w^{d_i}\bar{w}^i(d_i+i)\leqq(1-\beta)/(\bar{w})\leqq(1-\beta)/(\beta\bar{w})$$

by Lemma 3 and the fact that $0<\beta\leqq 1$.

Lemma 5. *If* $\frac{1}{2}\leqq w\leqq r$ and $(\sqrt{5}-1)/2\leqq r<1$, *then* $V_\beta'(f,g)\geqq 0$ *for all* $0<\beta\leqq 1$, $0\leqq f$, $g\leqq 1$, *and* $0\leqq f+g<1$.

Proof. Without loss of generality, we can and do assume that $0<g\leqq f<1$. Since R_β is right continuous and strictly increasing on the interval $[0,1]$, it suffices to show that $V_\beta'(f,g)\geqq 0$ for $f=\sum_{i=0}^{l} r^{a_i}\bar{r}^i$ and $g=\sum_{i=0}^{m} r^{b_i}\bar{r}^i$ where $\{a_i\}$ and $\{b_i\}$ are two finite, non-decreasing sequences of positive integers. Since $g\leqq f$, $b_0\geqq a_0$. Since $0<f+g<1$, $f+g$ can be expressed as $\sum_{i=0}^{\infty} r^{c_i}\bar{r}^i$ or $\sum_{i=0}^{n} r^{c_i}\bar{r}^i$ $\left(\text{we will always write } r^d \text{ for } \sum_{j=0}^{\infty} r^{d+1}\bar{r}^j \text{ for any integer } d\geqq 0\right)$ where $\{c_i\}$ (possibly finite) is a non-decreasing sequence of positive integers. Now we assume that $f+g=\sum_{i=0}^{\infty} r^{c_i}\bar{r}^i$ $\left(\text{the case that } f+g=\sum_{i=0}^{n} r^{c_i}\bar{r}^i \text{ can be proved similarly}\right)$. Let $k=\inf\{i\,|\,a_i>c_i\}$ if there is a such i, and $=l+1$ if there is no a such i. Since $c_k=c_{k+1}=\ldots=c_{k+p}<c_{k+p+1}\leqq\ldots$ for some integer $p\geqq 0$,

$$\sum_{j=k}^{\infty} r^{c_j}\bar{r}^j<\sum_{j=k}^{\infty} r^{c_k}\bar{r}^j=r^{c_k-1}\bar{r}^k.$$

Now if $k=0$ then $c_k+k=c_0<a_0\leqq b_0$ and if $k\geqq 1$ then

$$r^{c_k-1}\bar{r}^k>\sum_{j=k}^{\infty} r^{c_j}\bar{r}^j\geqq g\geqq r^{b_0}$$

and $c_k+k<b_0$ since $0<\bar{r}\leqq r^2<1$. Now

$$\begin{aligned}V'_\beta(f,g)&=R_\beta(f+g)-R_\beta(f)-R_\beta(g)\\&=\sum_{j=0}^{\infty}(\beta w)^{c_j}(\beta\bar{w})^j-\left[\sum_{j=0}^{l}(\beta w)^{a_j}(\beta\bar{w})^j+\sum_{j=0}^{m}(\beta w)^{b_j}(\beta\bar{w})^j\right]\\&=\sum_{j=k}^{\infty}(\beta w)^{c_j}(\beta\bar{w})^j-\left[\sum_{j=k}^{l}(\beta w)^{a_j}(\beta\bar{w})^j+\sum_{j=0}^{m}(\beta w)^{b_j}(\beta\bar{w})^j\right].\end{aligned}$$

It is easy to see that $V'_\beta(f,g)\geqq 0$ if

$$w^{c_k}\bar{w}^k\geqq\sum_{j=k}^{l}(w)^{a_j}(\bar{w})^j+\sum_{j=0}^{m}(w)^{b_j}(\bar{w})^j$$

since $0<\beta\leqq 1$, $c_k+k<a_k+k$, and $c_k+k<b_0$. Hence we can and do assume that

$$w^{c_k}\bar{w}^k<\sum_{j=k}^{l}(w)^{a_j}\bar{w}^j+\sum_{j=0}^{m}w^{b_j}\bar{w}^j.$$

By Theorem 6.3.1 of [4], we have

$$\sum_{j=k}^{\infty}(w)^{c_j}(\bar{w})^j\geqq\sum_{j=k}^{l}w^{a_j}\bar{w}^j+\sum_{j=0}^{m}w^{b_j}\bar{w}^j.$$

Hence there are two cases to be considered.

"*Case I*" $\quad w^{c_k}\bar{w}^k<\sum_{j=k}^{l}w^{a_j}\bar{w}^j+\sum_{j=0}^{m}w^{b_j}\bar{w}^j=\sum_{j=k}^{\infty}w^{c_j}\bar{w}^j.$

In this case, we let $G(\beta)=\sum_{j=k}^{\infty}w^{c_j-c_k}(\bar{w})^{j-k}(\beta)^{(c_j-c_k)+(j-k)}$ and

$$H(\beta)=\sum_{j=k}^{l}(w)^{a_j-c_k}(\bar{w})^{(j-k)}(\beta)^{(a_j-c_k)+(j-k)}+\sum_{j=0}^{m}(w)^{b_j-c_k-k}(\bar{w})^j(\beta)^{b_j-c_k-k+j},$$

then $V'_\beta(f,g)=w^{c_k}\bar{w}^k(\beta)^{c_k+k}\{G(\beta)-H(\beta)\}$. To show that $V'_\beta(f,g)\geqq 0$, it suffices to show that $G(1)-G(\beta)\leqq H(1)-H(\beta)$ since $G(1)=H(1)$. Now if $0<\beta<1$, then

$$\{H(1)-H(\beta)\}/(1-\beta)-\left[\sum_{j=k+1}^{\infty}(w)^{c_j-c_k}(\bar{w})^{j-k}\right]\geqq H(1)-\sum_{j=k+1}^{\infty}(w)^{c_j-c_k}(\bar{w})^{j-k}=1$$

and

$$\begin{aligned}&\{G(1)-G(\beta)\}/(1-\beta)-\sum_{j=k+1}^{\infty}(w)^{c_j-c_k}(\bar{w})^{j-k}\\&\quad\leqq\sum_{j=k+1}^{\infty}(w)^{c_j-c_k}(\bar{w})^{j-k}\{(c_j-c_k)+(j-k-1)\}\leqq 1\qquad\text{(by Lemma 3)}\end{aligned}$$

since $G(1)=H(1)=1+\sum_{j=k+1}^{\infty}(w)^{c_j-c_k}(\bar{w})^{j-k}$. Hence $H(1)-H(\beta)\geqq G(1)-G(\beta)$.

Therefore $V'_\beta(f,g)\geqq 0$ for all $0<\beta\leqq 1$ and the proof of "Case I" is complete.

"*Case II*" $(w)^{c_k}(\bar{w}^k)<\sum_{j=k}^{l}(w)^{a_j}(\bar{w})^j+\sum_{j=0}^{m}(w)^{b_j}(\bar{w})^j<\sum_{j=k}^{\infty}(w)^{c_j}(\bar{w})^j.$

In this case, we choose g', $0<g'<g$, such that $f+g'=\sum_{j=0}^{k}r^{c_j}\bar{r}^j+\sum_{j=k+1}r^{d_j}\bar{r}^j$ for some non-decreasing sequence $\{d_i\}$ of positive integers (possibly finite) such that $d_{k+1}\geqq c_{k+1}$ and

$$w^{c_k}\bar{w}^k+\sum_{j=k+1}(w)^{d_i}(\bar{w})^j=\sum_{j=k}^{l}w^{a_j}\bar{w}^j+\sum_{j=0}^{m}w^{b_j}w^j.$$

Now let

$$G(\beta)=1+\sum_{j=k+1}(w)^{d_j-c_k}(\bar{w})^{j-k}(\beta)^{(d_j-c_k)+(j-k)}$$

and

$$H(\beta)=\sum_{j=k}^{l}(w)^{a_j-c_k}(\bar{w})^{(j-k)}(\beta)^{(a_j-c_k)+(j-k)}+\sum_{j=0}^{m}(w)^{b_j-c_k-k}(\bar{w})^j(\beta)^{b_j-c_k-k+j}.$$

Follow the proof of "Case I", we get $G(\beta)\geqq H(\beta)$ for all $0<\beta\leqq 1$. Now

$$R_\beta(f+g)>R_\beta(f+g')=\sum_{j=0}^{k}(\beta w)^{c_j}(\beta\bar{w})^j+\sum_{j=k+1}(\beta w)^{d_j}(\beta\bar{w})^j$$

since R_β is strictly increasing on $[0,1]$. Therefore $R_\beta(f+g)-R_\beta(f)-R_\beta(g)=V'_\beta(f,g)\geqq 0$ for all $0<\beta\leqq 1$ and the proof of "Case II" now is complete.

Lemma 6. *If* $0<w\leqq\frac{1}{2}\leqq r<1$, *then the bold strategy is optimal.*

Proof. By Theorem 2.12.1 of [4], it is sufficent to show that $R_\beta(rf+\bar{r}g)\geqq\beta wR_\beta(f)+\beta\bar{w}R_\beta(g)$ for all $0\leqq g\leqq f\leqq 1$. By Identities (2) and (3) in Sect. 2,

$$R_\beta(rf+\bar{r}g)=R_{\beta,w,\frac{1}{2}}(R^{-1}_{1,r,\frac{1}{2}}(rf+\bar{r}g))=R_{\beta,w,\frac{1}{2}}(R_{1,\frac{1}{2},r}(rf+\bar{r}g)).$$

Since $\frac{1}{2}\leqq r$, $R_{1,\frac{1}{2},r}(rf+\bar{r}g)\geqq\frac{1}{2}R_{1,\frac{1}{2},r}(f)+\frac{1}{2}R_{1,\frac{1}{2},r}(g)$ (Theorem 6.3.1 of [4]). By Lemma 2,

$$\begin{aligned}R_{\beta,w,\frac{1}{2}}&[\tfrac{1}{2}R_{1,\frac{1}{2},r}(f)+\tfrac{1}{2}R_{1,\frac{1}{2},r}(g)]\\&\geqq\beta wR_{\beta,w,\frac{1}{2}}[R_{1,\frac{1}{2},r}(f)]+\beta\bar{w}R_{\beta,w,\frac{1}{2}}[R_{1,\frac{1}{2},r}(g)].\end{aligned}$$

Hence

$$\begin{aligned}R_\beta(rf+rg)&\geqq\beta wR_{\beta,w,\frac{1}{2}}[R_{1,\frac{1}{2},r}(f)]+\beta\bar{w}R_{\beta,w,\frac{1}{2}}[R_{1,\frac{1}{2},r}(g)]\\&=\beta wR_{\beta,w,\frac{1}{2}}[R^{-1}_{1,r,\frac{1}{2}}(f)]+\beta\bar{w}R_{\beta,w,\frac{1}{2}}[R^{-1}_{1,r,\frac{1}{2}}(g)]\\&=\beta wR_{\beta,w,r}(f)+\beta\bar{w}R_{\beta,w,r}(g)\qquad\text{if }0\leqq g\leqq f\leqq 1.\end{aligned}$$

Therefore the bold strategy is optimal.

By Lemmas 4, 5, and 6, we have the following theorem about the optimality of the bold strategy in a discounted subfair primitive casino if $0 \leqq w \leqq \frac{1}{2} \leqq r \leqq 1$ or $\frac{1}{2} \leqq w \leqq r$ and $(\sqrt{5}-1)/2 \leqq r \leqq 1$.

Theorem 1. *In a discounted subfair primitive casino* $\Gamma_{\beta,w,r}$, *if* $0 \leqq w \leqq \frac{1}{2} \leqq r \leqq 1$ *or* $\frac{1}{2} \leqq w \leqq r$ *and* $(\sqrt{5}-1)/2 \leqq r \leqq 1$, *then the bold strategy is optimal.*

Remark. As in [4], if the bold strategy is optimal, then there are some nonbold strategies which are optimal too.

The next theorem about the non-optimality of the bold strategy in a discounted primitive casino is due to Chen. For the sake of comparison, we include it as Theorem 2.

Theorem 2. *In a discounted subfair primitive casino* $\Gamma_{\beta,w,r}$, *if* $0<w \leqq r<\frac{1}{2}$ *or* $\frac{1}{2}<w \leqq r<(\sqrt{5}-1)/2$, *then the bold strategy is not necessarily optimal, i.e., for some discount factors* β *in* $(0,1)$, *the bold strategy is not optimal if* $0<w \leqq r<\frac{1}{2}$ *or* $\frac{1}{2}<w \leqq r<(\sqrt{5}-1)/2$.

It is known that if $\beta=1$ and $0<r<w<1$ then the bold strategy is no longr optimal. But when $\beta w \leqq r$ the process of the gambler's fortune is a supermartingale since

$$E(f_{n+1}|f_n)=\beta w\left(f_n+\frac{\bar{r}}{r}s\right)+\beta\bar{w}(f_n-s)<f_n$$

and the optimal sampling theorem (Theorem 5.10 of [2]) gives the result $U_\beta(f)<f$ for all $0<f<1$ indicating that the game is subfair (see p. 74 of [4]), where U_β is the utility function of the game (see p. 25 of [4]). One would intuitively suspect that for any superfair primitive casino there exists some discount factors for which the bold is optimal. However, this intuitive conjecture is false even if $\beta w<r$. The next two theorems tell us about the non-optimality of the bold strategy for discounted superfair primitive casinos.

Theorem 3. *In a discounted superfair primitive casino* $\Gamma_{\beta,w,r}$ *if* $0<r<w<1$ *and* $0<r<(\sqrt{5}-1)/2$ *then the bold strategy is not optimal, i.e., for any discount factor* β *in* $(0,1]$, *the bold strategy is not optimal if* $0<r<w<1$ *and* $0<r<(\sqrt{5}-1)/2$ *(even if* $\beta w \leqq r$*).*

Proof. In view of Theorem 2.14.1 of [4], it suffices to show that $V_\beta(f,g)<0$ for some $0 \leqq g<f \leqq 1$ which by Lemma 1, is equivalent to showing that $V'_\beta(f,g)<0$ for some $0 \leqq f$, $g \leqq 1$. Since $0<r<(\sqrt{5}-1)/2$, $0<r+r^2<1$. Now we let $f=r$, $g=r^2$, then $f+g=r+r^2=r+\sum_{j=1} r^{c_j}\bar{r}^j$ for some non-decreasing sequence $\{c_j\}$ of positive integers (possibly finite). Hence $R_\beta(f+g)=(\beta w)+\sum_{j=1}(\beta w)^{c_j}(\beta\bar{w})^j$, $R_\beta(f)=\beta w$, $R_\beta(g)=(\beta w)^2$. By Theorem 6.5.1 of [4], $\sum_j w^{c_j}\bar{w}^j<w^2$. Since $0<\beta \leqq 1$ and $c_j \geqq 1$ for all $j \geqq 1$, $\sum_j(\beta w)^{c_j}(\beta\bar{w})^j<(\beta w)^2$.

Theorem 4. *In a discounted superfair primitive casino* $\Gamma_{\beta,w,r}$, if $(\sqrt{5}-1)/2 \leqq r < w < 1$, *then the bold strategy is not necessairly optimal. Actually there exists a* β_0 *in* $[0, r/w)$ *such that the bold strategy is not optimal if the discount factor* $\beta > \beta_0$.

Proof. As in the proof of Theorem 3, it suffices to show that $V'_\beta(f, g) < 0$ for some $0 \leqq f$, $g \leqq 1$. Since $0 < r < 1$, there exists a positive integer k such that $r < r + r^k < 1$. Now let $f = r$ and $g = r^k$. Then $f + g = r + r^k = r + \sum_{j=1} r^{c_j} \bar{r}^j$ for some non-decreasing sequence $\{c_j\}$ of positive integers (possibly finite). We assume that $\{c_j\}$ is infinite since the case that $\{c_j\}$ is finite can be proved similarly.

(a) If $c_1 + 1 \geqq k$, then

$$V'_\beta(f+g) = (\beta w) + \sum_{j=1}^{\infty} (\beta w)^{c_j} (\beta \bar{w})^j < V'_\beta(f) + V'_\beta(g) = (\beta w) + (\beta w)^k$$

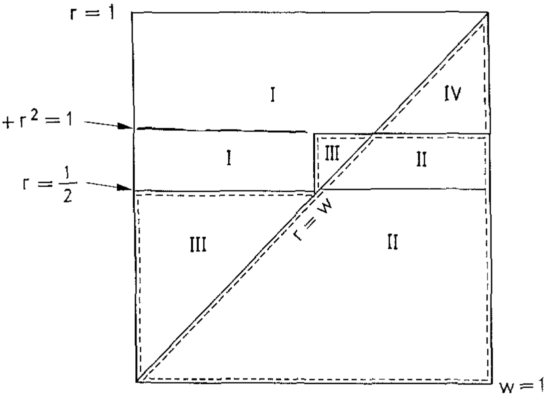
for all $0 < \beta \leqq 1$ since $\sum_{j=1}^{\infty} (w)^{c_j} (\bar{w})^j < w^k$ (by Theorem 6.5.1 of [4]).

(b) If $c_1 + 1 < k$, then

$$V'_\beta(f+g) - V'_\beta(f) - V'_\beta(g) = \sum_{j=1}^{\infty} (\beta w)^{c_j} (\beta \bar{w})^j - (\beta w)^k < 0$$

for all $\beta \geqq \beta_0$ for some β_0 in $[0, r/w)$.

By Theorems 1, 2, 3, and 4, we have the following figure.



I) The bold strategy is always optimal for any discount factor β in $(0, 1]$.

II) The bold strategy is always not optimal for any discount factor β in $(0, 1]$.

III) There is a β_0 in $(0, 1]$ such that the bold strategy is not optimal if the discount factor β is in $(0, \beta_0)$.

IV) There is a β_0 in $\left[0, \frac{r}{w}\right)$ such that the bold strategy is not optimal if the discount factor β is in $(\beta_0, 1]$.

Remarks. 1. We do not know about the existence of an optimal strategy when the bold strategy is not optimal.

2. The discount factor in this paper is used to motivate the gambler to recognize the time value of his fortune (his goal) and to handle inflation. We recently used a more direct method to handle inflation (the method is to discount the gambler's fortune step by step by the fixed discount rate $(1+\alpha)^{-1}$, here α is the inflation rate). We have obtained some results about the optimality and non-optimality of the bold strategy. The proofs of these results will appear somewhere else.

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