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Left Continuous Moderate Markov Processes

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1. The purpose of this paper is to formulate and prove several basic results for a left continuous moderate Markov process, which are analogues of well-known results for right continuous strong Markov processes. It turns out that the first such result in our development is that about the limit or infimum of excessive functions. This was given by H. Cartan in his celebrated papers on Newtonian potentials, extended by Brelot to general potential theory, and proved by Doob by probabilistic methods. The left version of this result with certain ramifications is given in Theorems 1, 2 and 3 below. Several consequences are then drawn. In particular, Hunt's result about the regular points of a set, and Dellacherie's result on semipolar sets are given respectively in Theorems 4 and 5. Naturally, proofs of these results in the left setting follow certain well-trodden paths in the right setting, but several not so obvious detours are necessary in order to avoid the pitfalls. Some of these pitfalls are: there may be branch points; there is no zero-one law; excessive functions need not be right or left continuous on paths; the minimum of two excessive functions need not be excessive. We illustrate these pathologies by a trivial example at the end of the paper. Nevertheless, our results are as good as their right counterparts, which may or may not be surprising to the *conoscenti*. (No co-fine topology!)

Let us begin by giving a definition of a moderate Markov process. Let $(E, \mathbf{E})$ be a Lusin topological space together with its Borel field, and let $(P_t)_{t>0}$ be a Markovian semigroup on E. We set $P_0(x,\cdot)=\varepsilon_x(\cdot)$. Let $(X_t)_{t\geqq 0}$ be a process with values in E, having left limits everywhere in $(0,\infty)$, defined on a measurable space $(\Omega, \mathfrak{F})$, and adapted to a filtration $(\mathfrak{F}_t)_{t\geqq 0}$ with each $\mathfrak{F}_t\subset\mathfrak{F}$. We assume that $(P^x)_{x\in E}$ is a family of probability measures on $(\Omega,\mathfrak{F})$ which depend measurably on x and that $P^x\{X_0=x\}=1$ for each x in E. The process X is said to be a moderate Markov process with semigroup $(P_t)_{t\geqq 0}$ if for each predictable stopping time T, for each positive measurable function f, and for each $t>0$,

$$E^x\{f(X_{T+t})|\mathfrak{F}_{T-}]=P_t f(X_{T-}) \quad \text{a.s. on } \{T<\infty\}.$$

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If X is left continuous a.s., then we may replace the right-hand side of the equation with $P_t f(X_T)$.

The class of left continuous moderate Markov processes is at least as extensive as Hunt processes. Indeed, every Hunt process (hence every Feller process) has a left continuous standard modification which is a moderate Markov process. Set $\tilde{X}_0 = X_0$, $\tilde{X}_t = X_{t-}$ for $t>0$. Then $\tilde{X}_t$ is a left continuous process which is adapted to the filtration $(\mathfrak{F}_t)$ of the Hunt process, where $(\mathfrak{F}_t)$ is right continuous. Since $T+t$ is predictable, the quasi-left continuity of X implies that for each $t \geqq 0$, $X_{T+t} = \tilde{X}_{T+t}$ a.s. Thus in the statement of the strong Markov property for X,

$$E^x\{f(X_{T+t}) | \mathfrak{F}_T\} = P_t f(X_T),$$

we may replace X with $\tilde{X}$ to obtain

$$E^x\{f(\tilde{X}_{T+t}) | \mathfrak{F}_T\} = P_t f(\tilde{X}_T).$$

Since $P_t f(\tilde{X}_T)$ is $\mathfrak{F}_{T-}$ measurable, we may replace $\mathfrak{F}_T$ with $\mathfrak{F}_{T-}$ in the above to obtain

$$E^x\{f(\tilde{X}_{T+t}) | \mathfrak{F}_{T-}\} = P_t f(\tilde{X}_T) \quad \text{on } \{T<\infty\}.$$

Thus $\tilde{X}$ has the moderate Markov property.

2. Let $(X_t, t \geqq 0)$ be a moderate Markov process with Borelian (P_t) (where P_0 = the identity) as transition semigroup, and left continuous paths in $(0, \infty)$. By definition φ is *superaveraging* iff $\varphi \geqq 0$ and $\varphi \geqq P_t \varphi$ for every $t>0$; and is *excessive* if in addition $\lim_{t \downarrow \downarrow 0} P_t \varphi = \varphi$. It follows then that for each $\alpha > 0$, there exists $g_n \in \mathscr{E}_+$ such that

$$\varphi = \lim_{n \to \infty} \uparrow U^\alpha g_n. \tag{1}$$

Lemma 1. *For $\alpha > 0$, $g \in b\mathscr{E}_+$, $t \to U^\alpha g(X_t)$ is left continuous.*

This is stated in [3]; here is the proof. Write

$$h(t) = e^{-\alpha t} U^\alpha g(X_t). \tag{2}$$

Then $\{h(t), t \geqq 0\}$ is a positive supermartingale. We have $\forall x$:

$$E^x\{h(t)\} = E^x\left\{\int_t^\infty e^{-\alpha s} g(X_s)\, ds\right\}. \tag{3}$$

By martingale theory, h restricted to Q (rationals) has right and left limits in $(0, \infty)$. Put

$$\varphi(t) = \lim_{Q \ni q \uparrow \uparrow t} h(q).$$

For $q<t$,

$$h(q) \geqq E^x\{h(t) | \mathfrak{F}_{q-}\};$$

hence

$$\varphi(t) \geqq E^x\{h(t) | \mathfrak{F}_{t-}\} = h(t)$$

since $h(t)$ is $\mathfrak{F}_{t-}$ measurable by left continuity of X. On the other hand, we have by bounded convergence and (3):

$$E^x\{\varphi(t)\} = \lim_{Q \ni q \uparrow\uparrow t} E^x\{h(q)\} = E^x\{h(t)\}.$$

Thus for each $t > 0$:

$$P^x\{\varphi(t) = h(t)\} = 1; \tag{4}$$

namely, $\{\varphi(t), t > 0\}$ is a standard modification of $\{h(t), t > 0\}$. By definition $t \to \varphi(t)$ is left continuous. Consider now

$$\Gamma = \{(t, w) | \varphi(t, w) \neq h(t, w)\}.$$

This is a predictable set since φ and X are left continuous. By [4; p. 72], if $P^x(\pi_\Omega \Gamma) > 0$, where π_Ω is the projection on Ω, then there exists a predictable T such that $P^x\{T < \infty\} > 0$, $[T] \subset \Gamma$ and so

$$\varphi(T) \neq h(T) \quad \text{on } \{T < \infty\}. \tag{5}$$

Let $\{T_n\}$ announce T. We may take T_n to be Q-valued (see [3]). We have by (4), P^x-a.s. for all n:

$$\varphi(T_n) = h(T_n) \quad \text{on } \{T_n < \infty\}. \tag{6}$$

By Theorem 1 of [3], we have P^x-a.s.

$$h(T_n) \to h(T) \quad \text{on } \{T < \infty\}. \tag{7}$$

Since φ is left continuous, we have also

$$\varphi(T_n) \to \varphi(T) \quad \text{on } \{T < \infty\}. \tag{8}$$

Thus $\varphi(T) = h(T)$, P^x-a.s. This contradicts (5) and so $P^x\{\pi_\Omega \Gamma\} = 0$, for every x. Hence φ and h are indistinguishable and Lemma 1 is proved.

Lemma 2. *If φ is excessive, then a.s.*

$$t \to \varphi(X_t) \quad \textit{has right limits on } [0, \infty) \textit{ and left limits on } (0, \infty). \tag{9}$$

The same is true if φ is the pointwise limit of a sequence of excessive functions; or the infimum of such a sequence.

For the proof compare [7; p. 150].

Proof. Let g_n be as in (1), and h_n correspond to g_n as in (2). By Lemma 1, $\{h_n(t), t > 0\}$ is a left continuous positive supermartingale. Let $M_n[a, b]$ denote the number of upcrossings by $h_n(\cdot)$ from $(-\infty, a)$ to $(b, +\infty)$. We have by [8; p.

128], $\forall x$:

$$E^x\{M_n[a,b]\} \leqq \frac{b}{b-a}, \tag{10}$$

and consequently if $L[a,b] = \varliminf_n M_n[a,b]$, then by Fatou's lemma

$$E^x\{L[a,b]\} \leqq \frac{b}{b-a}. \tag{11}$$

Let $M[a,b]$ denote the corresponding number of upcrossings by a path of the process $\{e^{-\alpha t}\varphi(X_t), t>0\}$. Since $e^{-\alpha t}\varphi(X_t) = \lim_{n\to\infty} h_n(t)$ for each t, trivial counting shows that

$$M[a,b] \leqq L[a,b]. \tag{12}$$

It follows from (11) and (12) and the completeness of (Ω, F, P) that

$$P^x\{M[a,b] = \infty\} = 0. \tag{13}$$

This being true for every $a<b$, the paths of $e^{-\alpha t}\varphi(X_t)$ have a.s. no oscillatory discontinuities and so (9) is true.

Next suppose $\varphi = \lim_n \varphi_n$ where each φ_n is excessive. Let M and M_n denote the number of upcrossings associated with $\varphi(X_t)$ and $\varphi_n(X_t)$, respectively. For each n, we have just proved that there is a random variable L_n such that

$$M_n[a,b] \leqq L_n[a,b],$$

and

$$E^x\{L_n[a,b]\} \leqq \frac{b}{b-a}.$$

As before, we have in our present connotation:

$$M[a,b] \leqq \varliminf_n M_n[a,b] \leqq \varliminf_n L_n[a,b].$$

It follows that (13) is again true and so the result (9) holds for φ. Finally, let $\varphi = \inf_n \varphi_n$ where each φ_n is excessive. Then $\varphi = \lim_{n\to\infty} \psi_n$ where $\psi_n = \inf_{1\leqq m\leqq n} \varphi_m$. Since (9) is true when $\varphi = \varphi_n$, it is also true for ψ_n trivially, hence for φ as just proved. Lemma 2 is proved.

We did not prove nor need the measurability of $M[a,b]$. It would follow if we could prove that $\varphi(X_t)$ is a separable process. The same remark seems to apply to the argument in [7].

Definition. A Borel set A is thin iff $\forall x$:

$$P^x\{T_A = 0\} = 0, \qquad \text{where } T_A = \inf\{t>0 \mid X_t \in A\}. \tag{14}$$

A set is semipolar iff it is contained in a countable union of thin sets. A set is polar if $P^x\{T_A < \infty\} = 0$, $\forall x$.

For a left continuous process there is no $0-1$ law to assert that (14) is equivalent to $P^x\{T_A = 0\} < 1$.

Theorem 1. *Let f be superaveraging and $f^* = \lim_{t \downarrow \downarrow 0} P_t f$. Suppose $t \to f(X_t)$ has right and left limits on $[0, \infty)$. Then*

$$\{f > f^*\} \quad \textit{is semipolar.} \tag{15}$$

Furthermore, for each $\varepsilon > 0$:

$$\{f > f^* + \varepsilon\} \quad \textit{is thin.} \tag{16}$$

Proof. Since f^* is excessive, by Lemma 2 and our hypothesis, both limits below exist for all $t > 0$:

$$f^*(X_t)_- = \lim_{s \uparrow \uparrow t} f^*(X_s), \qquad f(X_t)_- = \lim_{s \uparrow \uparrow t} f(X_s). \tag{17}$$

Now we assume f bounded. Then we have by bounded convergence for $s \geqq 0$:

$$P_{s+} f = \lim_{t \downarrow \downarrow 0} P_{s+t} f = P_s(\lim_{t \downarrow \downarrow 0} P_t f) = P_s f^*. \tag{18}$$

Furthermore, for each x and $t > 0$:

$$\begin{aligned} E^x\{f(X_t)_-\} &= \lim_{s \uparrow \uparrow t} P_s f(x) = \lim_{s \uparrow \uparrow t} P_{s+} f(x) \\ &= \lim_{s \uparrow \uparrow t} P_s f^*(x) = E^x\{f^*(X_t)_-\} \end{aligned} \tag{19}$$

where bounded convergence is used in the first and last equation; the second equation is trivial and the third by (18). For a general superaveraging f, (19) implies that for each t and each positive constant m:

$$E^x\{(f \wedge m)(X_t)_-\} = E^x\{(f \wedge m)^*(X_t)_-\} \leqq E^x\{(f^* \wedge m)(X_t)_-\}. \tag{20}$$

But $f \geqq f^*$, so we must have equality above. Since m is arbitrary and both functions in (17) are left continuous in t, it follows that

$$P^x\{\forall\, t > 0: f(X_t)_- = f^*(X_t)_-\} = 1. \tag{21}$$

Let $\varphi = f - f^*$, where we set $\infty - \infty = 0$, then $t \to \varphi(X_t)$ has right and left limits in $[0, \infty)$. For such a function, it is an elementary fact that

$$\{t \,|\, |\varphi(X_t) - \varphi(X_t)_-| > \varepsilon\}$$

is finite in each finite $(0, t_0)$. By (21), $\varphi(X_t)_- = 0$ for all t in $(0, \infty)$, P^x-a.s. Hence if we write

$$A_\varepsilon = \{x \,|\, \varphi(x) > \varepsilon\}.$$

we have

$$P^x\{T_{A_\varepsilon} = 0\} \leqq P^x\{X_t \in A_\varepsilon \text{ for infinitely many } t \in (0, 1)\} = 0.$$

This proves (16).

We did not use Dellacherie's theorem on semipolar sets.

The following result is the generalization of the classical theorem due to Cartan, Brelot and Doob to the present setting. Let us remark that in the right continuous, strong Markov case, a very short proof was given by Chung in [2].

It is not known whether the method used there has a left-handed modification. Such a proof would be very interesting indeed. Smythe [9] gave a proof for the reverse of a right continuous, strong Markov process. Easy examples show that there are left continuous moderate Markov processes which are not such reverses. Here we give a proof for the general left case. The method reverts to Doob's old idea of supermartingale upcrossing (see Meyer [7]), but does not use Dellacherie's deep result on semipolar sets. The final results are somewhat more precise than a quick application of the latter would yield.

Theorem 2. *If f is the limit or infimum of a sequence of excessive functions, then* (15) *and* (16) *are true. Under the "hypothesis of absolute continuity" (Meyer's condition (L)), the conclusions remain true for the infimum of an arbitrary set of excessive functions.*

Proof. The f in the first sentence of the theorem is superaveraging, and (9) is true when $\varphi = f$ by Lemma 2. Hence the first assertion is a special case of Theorem 1. The second assertion is proved in the same way as in Meyer [7, p. 163], except for the following observation. For two superaveraging functions f and g, it is not necessarily true that $(f \wedge g)^* = f^* \wedge g^*$. But it is true that for any sequence of superaveraging f_n and any positive constant m, we have

$$(\inf_n (f_n \wedge m))^* = (\inf_n f_n)^* \wedge m \tag{22}$$

except on a semipolar set. To see this, observe that by the first assertion of Theorem 2, the left member of (22) is not smaller than the right member except on a semipolar set. On the other hand, it is not greater because for every $t > 0$, we have

$$P_t(\inf_n (f_n \wedge m)) \leqq P_t(\inf_n f_n) \wedge m.$$

Now an inspection of Meyer's proof loc. cit. shows that (22) is sufficient for the conclusions.

Lemma 3. *Let φ be excessive, then for any predictable S and T such that $S \leqq T$ we have $\forall x$:*

$$E^x\{\varphi(X_T);\, T < \infty \mid \mathfrak{F}_S\} \leqq \varphi(X_S) \quad \text{on } \{S < \infty\}. \tag{23}$$

Moreover, if ψ is also excessive, then

$$E^x\{(\varphi \wedge \psi)(X_T);\, T < \infty \mid \mathfrak{F}_S\} \leqq (\varphi \wedge \psi)(X_S) \quad \text{on } \{S < \infty\}. \tag{24}$$

Proof. By the moderate Markov property, we have for each x:

$$e^{-\alpha T} U^\alpha g(X_T) = E^x\left\{\int_T^\infty e^{-\alpha t} g(X_t)\, dt \mid \mathfrak{F}_T\right\}.$$

Hence

$$\begin{aligned} E^x\{e^{-\alpha T} U^\alpha g(X_T) \mid \mathfrak{F}_S\} &= E^x\left\{\int_T^\infty e^{-\alpha t} g(X_t)\, dt \mid \mathfrak{F}_S\right\} \\ &\leqq E^x\left\{\int_S^\infty e^{-\alpha t} g(X_t)\, dt \mid \mathfrak{F}_S\right\} = e^{-\alpha S} U^\alpha g(X_S), \end{aligned} \tag{25}$$

where we have used the convention that $\varphi(X_\infty)=0$ for any function φ. Using (25) for $g=g_n$ as in (1), we obtain (23) by first letting $n\to\infty$ and then $\alpha\downarrow 0$, involving monotone conergence both times. Now (24) is a trivial (but useful) consequence of (23) applied to φ and ψ separately.

Theorem 3. *Suppose that the f in Theorem 2 is such that $\{f>0\}$ is a set of potential zero. Then it is polar.*

Proof. Let $f=\lim_n f_n$ where each f_n is excessive. The hypothesis amounts to $U^\alpha f=0$ for some $\alpha>0$. It follows that for each x, $P_t f(x)=0$ for (Lebesgue) a.e. t. Hence

$$f^*(x)=\lim_{t\downarrow\downarrow 0} P_t f(x)=0,$$

and it follows from (20) that

$$P^x\{f(X_t)_-=0 \quad \text{for all } t\in(0,\infty)\}=1. \tag{26}$$

Now $\{f(X_t), t>0\}$ is a predictable process because X is left continuous. If it is not evanescent under P^x, then by [4; p. 72] there exists a predictable T such that $P^x\{0<T<\infty\}>0$, and

$$f(X_T)>0 \quad \text{on } \{T<\infty\}. \tag{27}$$

Let $\{T_k\}$ announce T, where each T_k is predictable. Since each f_n is excessive, it follows from (24) with $\psi=m$, a constant, that

$$E^x\{f_n(X_T)\wedge m;\ T<\infty\}\leqq E^x\{f_n(X_{T_k})\wedge m;\ T_k<\infty\}. \tag{28}$$

Hence by bounded convergence we have

$$E^x\{f(X_T)\wedge m;\ T<\infty\}\leqq E^x\{f(X_{T_k})\wedge m;\ T_k<\infty\}. \tag{29}$$

Since $T_k\uparrow\uparrow T$, we have $f(X_{T_k})\to f(X_T)_-=0$ by (26); hence the right side of (29) converges to zero. But for large enough m the left side cannot be zero by (27). This contradiction proves that $f(X_t)$ is an evanescent process and so $\{f>0\}$ is polar. The proof for $f=\inf_n f_n$ is similar by use of (24).

Remark. Unlike Theorem 1, Theorem 3 is not true for a superaveraging f satisfying the condition (9). Example: let b be a nonsticky boundary point in a diffusion on R^1 (or a Markov chain), and $f=1_{\{b\}}$. Then $P_t f=0$ for every $t>0$; and (9) holds when $\varphi=f$ because $\{t: X(t)=b\}$ is a discrete sequence. But $\{f>0\}=\{b\}$ is not polar.

Remark. Some of the results given above have versions in the general theory of stochastic processes. Let $(\Omega, \mathfrak{F}, P)$ be a probability space with a filtration $(\mathfrak{F}_t)$, and let M_t be a nonnegative predictable process with $E[M_0]<\infty$. Then M is said to be a predictable strong supermartingale if for any pair of predictable stopping times $S\leqq T$, $E[M_T|\mathfrak{F}_{S-}]\leqq M_S$ a.s. on $\{S<\infty\}$. Mertens [6] has proved versions of the following results for optional strong supermartingales, and his proofs apply to the predictable case with no change.

Theorem. *Let M_t be a predictable strong supermartingale.*

(i) *Then M has left limits on $(0, \infty)$.*

(ii) *If $\lim_{n\to\infty} M_{T_n} = M_T$ whenever (T_n) is a sequence of predictable stopping times announcing T, then M is left continuous. If M belongs to the class (D), this is equivalent to $\lim_{n\to\infty} E[M_{T_n}] = E[M_T]$.*

For example, if φ is an excessive funtion, choose (g_n) so that $U^\alpha g_n$ increases to φ. Since $e^{-\alpha t}\varphi(X_t) = \lim_{n\to\infty} e^{-\alpha t} U^\alpha g_n(X_t)$ is a predictable strong supermartingale, we may apply (i) above to conclude that $t \to \varphi(X_t)$ has left limits a.s. However, Mertens's techniques do not seem to yield that $t \to \varphi(X_t)$ has right limits a.s.

For an exposition of Mertens's result, see the forthcoming book by Dellacherie and Meyer, "Probabilités et potentiel", vol. 2 (Hermann, Paris).

3. For the right continuous strong Markov case, it is an essential fact that an excessive function composed with the process has right continuous sample paths. This is not the case in our situation, in general. We single out two classes of excessive functions where regularity does occur.

Propositions 1 and 2 below follow also from Merten's result (ii) in the remark above and related results.

Proposition 1. *Let f be an excessive function and T a predictable time with $P_T f(x) = f(x)$. Then $f(X_t)$ is left continuous on* $]0, T]$ a.s. P^x

Proof. Let $\Gamma_1 = \{(t, \omega): f(X_t)_- > f(X_t)\}$. If $P^x\{\pi_\Omega \Gamma_1\} > 0$, there is a predictable time S with $[S] \subset \Gamma_1$, $P^x\{S \leqq T\} > 0$. Let (S_n) be a sequence of predictable times announcing S. Then

$$f(x) = E^x\{f(X_T)\} \leqq E^x\{f(X_{S\wedge T})\} < \varliminf_{n\to\infty} E^x\{f(X_{S_n\wedge T})\} \leqq f(x)$$

by Lemma 3. Thus, $P^x\{S \leqq T\} = 0$. Now letting $\Gamma_2 = \{(t, \omega): f(X_t)_- < f(X_t)\}$ and choosing S and (S_n) as before, we have for a sufficiently large constant $R > 0$

$$E^x\{\lim_n f(X_{S_n}) \wedge R\} < E^x\{f(X_S) \wedge R\}.$$

By dominated convergence, the limit may be taken out of the expectation, and the second statement in Lemma 3 implies that

$$E^x\{f(X_S) \wedge R\} \leqq \lim_n E^x\{f(X_{S_n}) \wedge R\} < E^x\{f(X_S) \wedge R\}.$$

Thus, $P^x\{S \leqq T\} = 0$.

For Proposition 2 we assume that Ω is equipped with a family of shift operators $(\theta_t)_{t\geqq 0}$ such that for each s, θ_s is a map of Ω into Ω satisfying $X_t \circ \theta_s = X_{t+s}$ a.s. for all t.

Proposition 2. *Let A be a continuous additive functional of X with potential $f(x) = E^x\{A_\infty\}$. If $f(x) < \infty$, then $f(X_t)$ is left continuous P^x a.s.*

Proof. Let Γ_1, S and (S_n) be as in Proposition 1. Then $E^x\{h(X_S)_-\} > E^x\{h(X_S)\}$. But

$$\begin{aligned} E^x\{h(X_S)_-\} &\leqq \varliminf_{n\to\infty} E^x\{h(X_{S_n})\} = \varliminf_{n\to\infty} \mathrm{E}^x\{A_\infty - A_{S_n}\} \\ &= E^x\{A_\infty - A_S\} = E^x\{h(X_S)\}. \end{aligned}$$

Therefore, $P^x\{S<\infty\}=0$. Now, looking at the case for Γ_2, S and (S_n), we have that

$$E^x\{h(X_{S_n})\wedge R\} \geqq E^x\{h(X_S)\wedge R\} \qquad \text{for } R>0$$

by the second part of Lemma 3. Using dominated convergence we pass to the limit to get that $E^x\{h(X_S)_- \wedge R\} \geqq E^x\{h(X_S)\wedge R\}$. Letting R increase to ∞, we see that $P^x\{S<\infty\}=0$, and this completes the proof.

As remarked above, there is no useful $0-1$ law for moderate Markov processes. Thus, if we define $\Phi_A^\alpha(x)=E^x\{e^{-\alpha T_A}\}$, it is not apparent that the set $\{x\colon \Phi_A^\alpha(x)<1\}$ is semipolar. This follows, however, as a corollary of the next theorem.

Theorem 4. *Let A be Borel with* $\sup\{\Phi_A^1(x)\colon x\in A\}=a<1$. *(Such a set is said to be totally thin.) Then $\{s<r\colon X_s\in A\}$ is finite a.s. for each $r>0$.*

Proof. We first show that $\{s\in[t,t+r]\colon X_s\in A\}$ is finite for $t>0$, $r>0$. Set $T_0=t$, and recursively define times $T_{n+1}=T_n+T_A\circ\theta_{T_n}$, $n=0,1,2,\ldots$. Set $R=\lim_{n\to\infty} T_n$. Then $\{R<\infty\}=\Omega_0\cup\Omega_1$, where

$$\Omega_0=\bigcup_n \{T_n=R<\infty\},$$
$$\Omega_1=\bigcap_n \{T_n<R<\infty\}.$$

Suppose $P^x(\Omega_1)>0$. Note that the (T_n) form a strictly increasing sequence on Ω_1. Choose q so large that $P^x(\Omega_1\cap\{R<q\})>P^x(\Omega_1)-\varepsilon=c>0$. Choose S_1 predictable,

$$[S_1]\subset(T_1,T_3]\cap\{(t,\omega)\colon X_t(\omega)\in A\}\cap[0,q]$$

with $P^x\{S_1<\infty\}>c$. Set $D_1=S_1+T_A\circ\theta_{S_1}\leqq T_4$. Proceeding recursively, choose S_n predictable,

$$[S_n]\subset(T_{4n-3},T_{4n-1}]\cap\{(t,\omega)\colon X_t(\omega)\in A\}\cap[0,q],$$

with $P^x\{S_n<\infty\}>c$. Set $D_n=S_n+T_A\circ\theta_{S_n}\leqq T_{4n}$. Then

$$\begin{aligned} ce^{-q} &\leqq E^x\{e^{-D_n}\}=E^x\{e^{-S_n}\Phi_A^1(X_{S_n})\} \\ &\leqq aE^x\{e^{-S_n}\}\leqq aE^x\{e^{-D_{n-1}}\}\leqq\cdots\leqq a^n. \end{aligned}$$

Since $a<1$, and we may take n arbitrarily large, this is a contradiction. Hence $P^x(\Omega_1)=0$.

Suppose $P^x(\Omega_0)>0$. For each $\omega\in\Omega_0$, there is a sequence $(t_n(\omega))$ decreasing to $R(\omega)$ with $X_{t_n(\omega)}(\omega)\in A$. Set $S_1=(R+\varepsilon)\wedge q$ where q is chosen so large that $P^x\{(R+\varepsilon)\wedge q=R+\varepsilon\}>P^x(\Omega_0)-\delta=c>0$. Then there is an $\varepsilon_1<\varepsilon$ so that

$$P^x\{X_t(\omega)\in A \text{ for some } t\in((R+\varepsilon_1)\wedge q, S_1)\}>c.$$

Choose S_2 predictable,

$$[S_2]\subset(R\wedge q,(R+\varepsilon_1)\wedge q]\cap\{(t,\omega)\colon X_t(\omega)\in A\},$$

with $P^x\{S_2<\infty\}>c$. Set $D_2=S_2+T_A\circ\theta_{S_2}<S_1$. Proceeding recursively, again, there is an $\varepsilon_n<\varepsilon_{n-1}$ so that

$$P^x\{X_t(\omega)\in A \text{ for some } t\in((R+\varepsilon_n)\wedge q, S_{n-1})\}>c.$$

Choose S_n predictable,

$$[S_n]\subset(R\wedge q,(R+\varepsilon_n)\wedge q]\cap\{(t,\omega)\colon X_t(\omega)\in A\},$$

with $P^x\{S_n<\infty\}>c$. Set $D_n=S_n+T_A\circ\theta_{S_n}<S_{n-1}$. Then

$$\begin{aligned}ce^{-q}&\leqq E^x\{e^{-D_2}\}=E^x\{e^{-S_2}\Phi_A^1(X_{S_2})\}\leqq aE^x\{e^{-S_2}\}\\&\leqq aE^x\{e^{-D_3}\leqq\ldots\leqq a^nE^x\{e^{-D_{n+2}}\}.\end{aligned}$$

Since n is arbitrarily large, we conclude as before that $P^x(\Omega_0)=0$.

It remains to drop the hypothesis that $t>0$. This amounts to proving that $P^x\{T_A=0\}=0$, which is similar to the proof that $P^x(\Omega_0)=0$.

Corollary 1. *Let A be a Borel set with $\Phi_A^1(x)<1$ on A. Then A is semipolar. In particular, if B is an arbitrary Borel set and $B^r=\{x\colon \Phi_B^1(x)=1\}$, then $B-B^r$ is semipolar.*

Proof. Let $A_n=A\cap\left\{\Phi_A^1(x)\leqq 1-\frac{1}{n}\right\}$. Then $A=\bigcup_n A_n$ and $\Phi_{A_n}^1\leqq\Phi_A^1$. Therefore, $\sup\{\Phi_{A_n}^1(x)\colon x\in A_n\}\leqq 1-\frac{1}{n}$. Applying Theorem 4, it follows that A is semipolar. If $x\in B-B^r$, then $\Phi_{B-B^r}^1(x)\leqq\Phi_B^1(x)<1$. Thus $B-B^r$ is semipolar. This is Hunt's theorem (see [7, p. 148]).

For the remainder of this note, we fix X a left continuous moderate Markov process satisfying Meyer's hypothesis (L). Let λ be a reference probability measure for X (see [7, p. 158]). Recall that

$$\text{if } f \text{ is excessive and } \lambda(f)=0, \text{ then } f=0. \tag{30}$$

The proof of the following result requires no change in this situation.

Proposition 3. *Let μ be a measure on E. Then μ may be decomposed as $\mu=\mu_1+\mu_2$ where μ_1 does not charge any semipolar set and μ_2 is carried by a semipolar set.*

Using Theorems 2 and 4, the proof of the following result is valid here [7, p. 180]. Recall that a set P is finely perfect if $P=\{\Phi_P^1=1\}$.

Proposition 4. *Let A be compact. Then there exists a finely perfect set $P\subset A$ such that $A-P$ is semipolar.*

We now extend Dellacherie's proof of his characterization of semipolar sets to this situation.

Theorem 5. *Let X_t be a left continuous moderate Markov process with fundamental reference probability measure λ. Let G be Borel with $P^\lambda\{X_t\in G$ at most countably often$\}=1$. Then G is semipolar.*

Proof. We sketch Dellacherie's argument here (see [5, p. 112]). Set $\Gamma=\{(t,\omega): X_t(\omega)\in G\}$. By [4, VI-T33], $\Gamma\subset\bigcup_n [T_n]$ where (T_n) is a sequence of predictable stopping times. Define a measure μ on E by setting

$$\mu(H)=E^{\lambda}\left\{\sum_{n=1}^{\infty} 2^{-n} 1_H\circ X_{T_n}\right\}.$$

If $H\subset G$ and $\mu(H)=0$, then H is polar by (30). By Proposition 3, $\mu=\mu_1+\mu_2$, where μ_1 does not charge any semipolar set, and μ_2 is carried by a semipolar set. We show $\mu_1\equiv 0$. It suffices to show that every compact $K\subset G$ is semipolar. But K may be written as $P\cup(K-P)$ where P is a finely perfect set and $K-P$ is semipolar by Proposition 4. We prove below in Theorem 8 that $P=\emptyset$. Thus $\mu=\mu_2$, and μ_2 is therefore carried by a semipolar set $L\subset G$. But $\mu(G-L)=0$ implies $G-L$ is polar. Thus G is semipolar.

The next proof is a modification of one given in [7, p. 182].

Lemma 4. *Let P be finely perfect, and satisfy the hypothesis of Theorem 5. Then $P=\emptyset$.*

Proof. $\Gamma=\{(t,\omega): X_t(\omega)\in P\}\subset\bigcup_n [T_n]$ is predictable.

$$P^{\lambda}\{X_{T_n}\in P;\ T_P\circ\theta_{T_n}=0\}=E^{\lambda}\{X_{T_n}\in P;\ P^{X_{T_n}}[T_P=0]\}=P^{\lambda}\{X_{T_n}\in P\}.$$

Thus $T_n(\omega)$, whenever $X_{T_n}\in P$, is a limit from the right of times $(t_n(\omega))$ with $X_{t_n(\omega)}(\omega)\in P$. Set $\Gamma(\omega)=\{t: X_t(\omega)\in P\}$. It follows from the preceding sentence that $\bar{\Gamma}(\omega)$ has no isolated point and hence is perfect and therefore uncountable. We show $\bar{\Gamma}(\omega)-\Gamma(\omega)$ is countable. Recall $\Gamma(\omega)=\{t: \Phi_P^1(X_t(\omega))=1\}$. Now $t\in\bar{\Gamma}(\omega)-\Gamma(\omega)$ exactly when there exist $t_k(\omega)\to t(\omega)$ such that $\Phi_P^1(X_{t_k(\omega)}(\omega))=1$ and $\Phi_P^1(X_{t(\omega)}(\omega))\neq 1$. But $e^{-t}\Phi_P^1(X_t)$ has only finitely many upcrossings over any level (a,b) by Lemma 2. If there were an uncountable number of points in $\bar{\Gamma}(\omega)-\Gamma(\omega)$, there would exist $0<a<b<1$ such that $e^{-t}\Phi_P^1(X_t)$ had an infinite number of upcrossings over (a,b), which is impossible. Thus $\Gamma(\omega)$ is a.s. uncountable, which contradicts $X_t\in G$ only countably often. Therefore, $P=\emptyset$.

We conclude with an example which, although trivial, illustrates much of the pathology associated with moderate Markov processes.

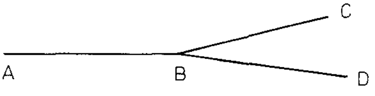

Fig. 1

Let X_t be the process uniform motion to the right on the state space given in Fig. 1. Upon reaching B, the process moves toward C with probability $\frac{1}{2}$ and moes toward D with probability $\frac{1}{2}$. Then X_t is a normal continuous process. However, the strong Markov property fails to hold at B because the $0-1$ law does not hold for the hitting time of $(B, C]$. Let $[A, B]$ (resp. $(B, C]$) denote the points between A and B, including A and B (resp. the points between B and C, including C and excluding B). Then $P^B[T_{(B,C]}=0]=\frac{1}{2}$. Thus the $0-1$ law does not hold at B for this hitting time.

If we let

$$f(x)=\begin{cases} 1 & \text{on } [A,B) \\ \frac{1}{2} & \text{on } B \\ 1 & \text{on } (B,C] \\ 0 & \text{on } (B,D] \end{cases}$$

then $f(x)$ is excessive, but $t\to f(X_t)$ is neither right or left continuous. Moreover, $f\wedge \frac{1}{2}$ is not excessive.

Finally, we give a trivial example of a continuous moderate Markov process with state space given in Fig. 2 which is not the reverse of a strong Markov process.

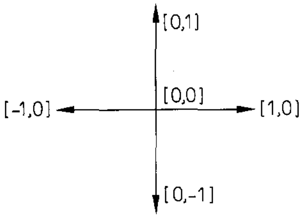

Fig. 2

On $((0,-1),(0,0))$ the process is uniform motion up; on $((-1,0),(0,0))$ the process is uniform motion to the right. At $(0,0)$, the process proceeds up toward (0,1) with probability $\frac{1}{2}$ and toward $(1,0)$ with probability $\frac{1}{2}$. This process is moderate but not strong Markov, and so is the reverse by symmetry of the state space.

References

1. Chung, K.L., Doob, J.L.: Fields, optionality, and measurability. Amer. J. Math. **87**, 397–424 (1965)
2. Chung, K.L.: A simple proof of Doob's convergence theorem, Sem. de Probabilités V. Lecture Notes in Math. **191**. Berlin-Heidelberg-New York: Springer 1970
3. Chung, K.L.: On the fundamental hypotheses of Hunt processes. Symposia Mathematica, Istituto Nazionale di Alta Matematica, Vol. IX, 43–52. New York-London: Academic Press 1972
4. Dellacherie, C.: Capacités et processus stochastiques. Ergebnisse der Mathematik, Band **67**. Berlin-Heidelberg-New York: Springer 1972
5. Dellacherie, C.: Ensembles aléatoires, Sem. de Probabilités III. Lecture Notes in Math. **88**. Berlin-Heidelberg-New York: Springer 1969
6. Mertens, J.F.: Théorie des processus stochastiques généraux applications aux surmartingales. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **22**, 45–68 (1972)
7. Meyer, P.A.: Processus de Markov. Lecture Notes in Math. **26**. Berlin-Heidelberg-New York: Springer 1967
8. Meyer, P.A.: Probabilités et potentiel. Paris: Hermann 1966
9. Smythe, R.T.: Remarks on the hypotheses of duality, Sem. de Probabilités VIII. Lecture Notes in Mathematics **381**. Berlin-Heidelberg-New York: Springer 1972/1973

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