Z. Wahrscheinlichkeitstheorie verw. Gebiete
57, 265–291 (1981)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Approximation of Rectangular Sums of B-valued Random Variables

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Summary. Given independent identically distributed random variables $\{x_n; n \in \mathbb{N}^q\}$ indexed by q-tuples of positive integers and taking values in a separable Banach space B we approximate the rectangular sums $\{\sum_{m \leqq n} x_m; n \in \mathbb{N}^q\}$ by a Brownian sheet. We obtain the corresponding result for random variables with values in a separable Hilbert space H while assuming an optimal moment condition. Generalized versions of the functional law of the iterated logarithm are thus derived.

1. Introduction

Let $\mathbb{N}^q$ denote the set of q-tuples of positive integers. For any $q \geqq 1$ and $n \in \mathbb{N}^q$ we define:

$$n = (n_1, \ldots, n_q), \qquad [n] = \prod_{i=1}^{q} n_i$$

and

$$a_n = (2q[n] \operatorname{loglog}^+ [n])^{1/2}.$$

Here $\log^+ r = \log(\max(r, 8))$. Set $e = (1, \ldots, 1_q)$. Also, for m, $n \in \mathbb{N}^q$, put $m \leqq n$ (resp. $m < n$) if $m_i \leqq n_i$ (resp. $m_i < n_i$) for each $i = 1, \ldots, q$.

Throughout this paper we denote generically by $\{x_n; n \in \mathbb{N}^q\}$ a collection of independent copies of a random vector x and set $S_n = \sum_{m \leqq n} x_m$.

Assume for the moment that x is real valued. If $q = 1$, it is well known that

$$x \in L^2 \quad \text{and} \quad Ex = 0$$
$$\Leftrightarrow \overline{\lim_{n \to \infty}}\, a_n^{-1} |S_n| < \infty \quad \text{a.s.} \tag{1.1}$$

However if $q > 1$, it is known that

$$x \in (L^2 \log^{q-1} L)/\operatorname{loglog} L \quad \text{and} \quad Ex = 0$$
$$\Leftrightarrow \lim_{r \to \infty} \sup_{[n] \geqq r} a_n^{-1} |S_n| < \infty \quad \text{a.s.} \tag{1.2}$$

0044-3719/81/0057/0265/$05.40

(see [16], p. 280). Moreover whenever (1.1) or (1.2) is in force a corresponding functional law of the iterated logarithm holds. A basic consequence of the work presented here is that these results extend to the case of a random variable x taking values in a separable Hilbert space (Corollary 1).

Let B denote a real separable Banach space with norm $\|\cdot\|$. If $\{z_n; n\in\mathbb{N}^q\}$ is a collection of B-valued random variables such that[1]

$$\lim_n P(\|z_n\|>\varepsilon)=0$$

for every $\varepsilon>0$, we designate this property by writing

$$P\lim_n z_n=0.$$

The covariance function $T(\cdot,\cdot)$ of a B-valued random variable x is defined by

$$T(f,g)=E\{f(x)\,g(x)\} \qquad f,g\in B^* \tag{1.3}$$

and x is said to be pregaussian if its covariance structure is realized by some Wiener measure on B. (For a construction of Wiener measure see [4].)

The following theorem generalizes a result of Philipp [13] to integers $q>1$.

Theorem 1. *Let $q\geqq 1$. Suppose that x is a random variable taking values in B. Assume that*

$$x\in L^2\log^{q-1}L,$$

$$x \text{ is pregaussian and } P\lim_n a_n^{-1}S_n=0.$$

Then there is a Brownian sheet $\{W(t);\ t\in[0,\infty)^q\}$ in B with covariance function $T(\cdot,\cdot)$ determined by (1.3) *such that*

$$\lim_n a_n^{-1}\|S_n-W(n)\|=0 \quad \text{a.s.} \tag{1.4}$$

A Brownian sheet $\{W(t); t\in[0,\infty)^q\}$ in B with covariance function $T(\cdot,\cdot)$ is a B-valued process having independent increments $W(R_1),\ldots,W(R_i)$ when $R_1,\ldots,R_i$ are disjoint rectangles in $[0,\infty)^q$ and $W(t)=0$ if any of the coordinates of t vanishes. Further, the increment $W(R)$ over a rectangle R has distribution $\mu_{|R|}$ where $|R|$ is the volume of R and, for $r>0$, μ_r is a Wiener measure on B with variance parameter r satisfying

$$\int_B f(\xi)\,g(\xi)\,\mu_r(d\xi)=r\cdot T(f,g) \qquad f,g\in B^*.$$

Here, we put $W(R)=\sum(\pm)W(v)$, the sum extending to all vertices v of R.

We have a functional law of the iterated logarithm for the Brownian sheet $\{W(t);\ t\in[0,\infty)^q\}$. Let $C_B([0,1]^q)$ denote the Banach space of B-valued continuous functions f on $[0,1]^q$ with the norm

$$\|f\|_{B,\infty}=\sup_{t\in[0,1]^q}\|f(t)\|.$$

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[1] $\lim_n$ means $\lim_{[n]\to\infty}$, $\overline{\lim}_n$ means $\lim_{r\to\infty}\sup_{[n]\geqq r}$.

For $n\in\mathbb{N}^q$ and $t=(t_1,\dots,t_q)\in[0,1]^q$ define $f_n\in C_B([0,1]^q)$ by

$$(1.5)\quad f_n(t)=\begin{cases} a_n^{-1}W(m) \text{ for } t_i=m_i/n_i;\ i=1,\dots,q,\ e\leqq m\leqq n. \\ 0 \text{ if } t_i=0 \text{ for some } i=1,\dots,q. \\ \begin{cases}\text{Lagrange interpolation in } t_1,\dots,t_q \text{ over the} \\ \text{cube } \{t\in[0,1]^q: m_i-1\leqq t_i\leqq m_i, i=1,\dots,q\}, \\ e\leqq m\leqq n.\end{cases}\end{cases}$$

Let H_T be the reproducing kernel Hilbert space in B generated by the covariance function $T=T(\cdot,\cdot)$ of $W(e)$. Let further $\{\varphi_\nu^*;\nu\geqq 1\}$ be a sequence of bounded linear functionals on B with the property that the points $\varphi_\nu=\int_B \xi\varphi_\nu^*(\xi)P\{W(e)\in d\xi\}$, $n\geqq 1$, constitute a C.O.N.S. $\{\varphi_\nu,\nu\geqq 1\}$ in H_T and $\xi=\sum_{\nu=1}^{\infty}\varphi_\nu^*(\xi)\varphi_\nu$ for $\xi\in H_T$ (see e.g. [9], Lemma 2.1). The inner product $(\cdot,\cdot)$ in H_T is given by $(\varphi_\mu,\varphi_\nu)=\int_B\varphi_\mu^*(\xi)\varphi_\nu^*(\xi)P\{W(e)\in d\xi\}$. We put

$$(1.6)\quad \mathcal{K}_T=\left\{f\in C_B([0,1]^q)\right.$$
$$\left.\cdot\left|\begin{array}{l} f(t)\in H_T \text{ for } t\in[0,1]^q, \varphi_\nu^*(f)\ll l \text{ for } \nu\geqq 1, \\ f(t)=\sum_\nu \varphi_\nu\int_0^t\{d\varphi_\nu^*(f)/dl\}\,dl \text{ and } \sum_\nu\int\{d\varphi_\nu^*(f)/dl\}^2\,dl\leqq 1.\end{array}\right.\right\}$$

Here, l denotes Lebesgue measure on $[0,1]^q$.

Theorem 2. *Let f_n and $\mathcal{K}_T$ be as given in* (1.5) *and* (1.6) *respectively. Then*

$$\lim_n \inf_{f\in\mathcal{K}_T}\|f_n-f\|=0 \quad a.s.$$

and

$$P(\{f\in C_B([0,1]^q): f \text{ is a } \|\cdot\|_{B,\infty}\text{-limit point of } \{f_n;n\in\mathbb{N}^q\}\}=\mathcal{K}_T)=1.$$

Note. Theorem 2 also holds when in the definition of f_n the R.H.S. of (1.5) is replaced by $a_n^{-1}W(n_1t_1,\dots,n_qt_q)$. Implicit in this statement is the fact that a Brownian sheet in B has continuous sample paths.

We shall prove Theorem 2 and this Note in Sect. 8.

Let us say that a mean zero B-valued random variable x having a second moment belongs to FLIL if Theorem 2 holds when f_n is replaced by g_n and $T=T(\cdot,\cdot)$ is defined by (1.3). Here, by g_n we mean the R.H.S. of (1.5) with S_m in place of $W(m)$. Let K be the closed unit ball of H_T. We say that x belongs to CLIL if

(i) $\lim_n \inf_{\xi\in K}\|\xi-a_n^{-1}S_n\|=0$ a.s.

(1.7) and

(ii) $P(\{\xi\in B: \xi \text{ is a } \|\cdot\|\text{-limit point of } \{a_n^{-1}S_n; n\in\mathbb{N}^q\}\}=K)=1.$

We also say that x belongs to BLIL if just (1.7)(i) holds. Clearly, FLIL $\subset$ CLIL $\subset$ BLIL.

Theorem 3. *Let $q \geqq 1$. Let x be a B-valued random variable with $x \in L^2 \log^{q-1} L$. Then*

$$P \lim_n a_n^{-1} S_n = 0 \Rightarrow x \in \mathrm{FLIL} \tag{1.8}$$

and

$$x \in \mathrm{BLIL} \Rightarrow \lim_n a_n^{-1} E \|S_n\| = 0. \tag{1.9}$$

Theorem 3 is proved in Sect. 9. I conjecture that the moment condition of this theorem can not be improved. Nevertheless if we make x take values in a separable Hilbert space H then we obtain a characterization of the invariance principle (1.4) with a weaker moment condition when $q \geqq 2$.

Theorem 4. *Let $q \geqq 1$. Let x be a random variable taking values in a separable Hilbert space $(H, |\cdot|)$. Then*

$$Ex = 0 \quad \text{and} \quad \begin{cases} x \in L^2, & q = 1 \\ x \in (L^2 \log^{q-1} L)/\log\log L, & q \geqq 2 \end{cases} \tag{1.10}$$

if and only if there is a Brownian sheet $\{W(t); t \in [0, \infty)^q\}$ in H with covariance function $T(\cdot,\cdot)$ given by

$$T(f,g) = E\{f(x)\, g(x)\} \qquad f, g \in H^*$$

such that

$$\lim_n a_n^{-1} |S_n - W(n)| = 0 \quad a.s. \tag{1.11}$$

Corollary 1. *Assume the hypothesis of Theorem 4. Then, (1.10) holds $\Leftrightarrow x \in \mathrm{FLIL} \Leftrightarrow x \in \mathrm{BLIL}$.*

Corollary 1 generalizes a functional law of the iterated logarithm due to Wichura [16]. To prove the corollary, notice that Theorems 2 and 4 combine to give (1.10) $\Rightarrow x \in \mathrm{FLIL}$, while the proof of the reverse implication is the same as that for $H = \mathbb{R}$ (cf. (1.2)).

Let us develop the main ideas of the proof of Theorem 1. First, the proof is reduced to the verification of Proposition 4.1 by the method illustrated in Sect. 7.

We prove a bounded law of the iterated logarithm for rectangular sums (Proposition 3.1) so a finite dimensional approximation can be effective. To do this we adapt both the Hartman and Wintner [6] truncation approach and the Kuelbs [8] approach to a Kolmogorov law of the iterated logarithm for B-valued random variables to the situation of multiparameter indexing (Sect. 2). In Sect. 5 we show that certain rectangular sums of finite dimensional random vectors obey a kind of weak law of the iterated logarithm (Proposition 5.6). Proposition 4.1 is thus obtained via an application of Theorem 3 of Philipp [13] (quoted here in Sect. 6).

We prove Theorem 4 in Sect. 10 by modifying the proof of Theorem 1.

2. Preliminary Lemmas

To begin we follow the approach of Hartman and Wintner [6]. Let x be a B-valued random variable with $x \in L^2 \log^{q-1} L$ for some $q \geqq 1$. Denote the distribution of x by σ. Let τ be a probability measure on B satisfying

(2.1) and
$$\int_B \|\xi\|^2 (\log^+ \|\xi\|)^{q-1} \tau(d\xi) < \infty$$
$$\int_{\|\xi\|>r} \|\xi\| \sigma(d\xi) \leqq \chi(r) \int_{\|\xi\|>r} \|\xi\| \tau(d\xi)$$

for some decreasing function χ: $[0, \infty) \to \mathbb{R}^+$ which tends to zero at infinity. Choose ε: $\mathbb{N} \to \mathbb{R}^+$ so that

(2.2)
(i) $\varepsilon(r) > \chi(r^{1/9})$
(ii) $\varepsilon(r) > r^{-1/6} (\log\log^+ r)^{1/2}$
(iii) $\varepsilon(r) \to 0$ as $r \to \infty$
(iv) $\alpha(r) = \varepsilon(r)(r(\log\log^+ r)^{-1})^{1/2} \uparrow \infty$ as $r \to \infty$.

Define:

(2.3)
$$w_n = \int_{\|\xi\|>\alpha([n])} \|\xi\| \sigma(d\xi) \qquad n \in \mathbb{N}^q.$$

(2.4) **Lemma.**
$$\sum_{n \in \mathbb{N}^q} a_n^{-1} w_n < \infty.$$

Proof. By (2.1)–(2.3) and partial summation,

(2.5)
$$\sum_{m \leqq n} a_m^{-1} w_m \leqq \sum_{m \leqq n} a_m^{-1} \chi(\alpha([m])) \int_{\|\xi\|>\alpha([m])} \|\xi\| \tau(d\xi)$$
$$= \int_{\|\xi\|>\alpha([n])} \|\xi\| \tau(d\xi) \sum_{i=1}^{[n]} \omega(i) a_i^{-1} \chi(\alpha(i))$$
$$+ \sum_{r=1}^{[n]-1} \int_{\alpha(r)<\|\xi\| \leqq \alpha(r+1)} \|\xi\| \tau(d\xi) \sum_{i=1}^{r} \omega(i) a_i^{-1} \chi(\alpha(i)).$$

Here, we put
$$\omega(i) = \omega_q(i) = \sum_{n_1 \dots n_q = n} 1.$$

Put also:
$$D(r) = \sum_{1 \leqq i \leqq r} \omega_q(i).$$

Then by partial summation,

(2.6)
$$\sum_{i=1}^{r} \omega(i) a_i^{-1} \chi(\alpha(i)) = a_r^{-1} \chi(\alpha(r)) D(r)$$
$$+ \sum_{i=1}^{r-1} (a_i^{-1} \chi(\alpha(i)) - a_{i+1}^{-1} \chi(\alpha(i+1))) D(i).$$

But, from [5],

(2.7) $$D(r) \sim r(\log r)^{q-1}/(q-1)! \qquad (r \to \infty).$$

So, breaking up the R.H.S. of (2.6) and using (2.2) (i)–(iv) together with (2.7) we find:

(2.8) $$\begin{aligned} \text{R.H.S. of (2.6)} &= O(a_r^{-1}\chi(r^{1/3})D(r) + D(r^{1/3})) \\ &\quad + \sum_{i=r^{1/3}}^{r-1} \{(a_i^{-1} - a_{i+1}^{-1})\chi(\alpha(i)) + a_{i+1}^{-1}(\chi(\alpha(i)) - \chi(\alpha(i+1))\} D(i) \\ &= O(a_r^{-1}\chi(r^{1/3})D(r) + D(r^{1/3})) + O(\chi(\alpha(r^{1/3})) \sum_{i=r^{1/3}}^{r} (a_i^{-1} - a_{i+1}^{-1})\, i(\log^+ i)^{q-1}) \\ &= O((\log^+ r)^{q-1}\alpha(r)). \end{aligned}$$

Hence, combining (2.5), (2.6) and (2.8),

$$\sum_{m \leqq n} a_m^{-1} w_m = O(\int_B \|\xi\|^2 (\log^+ \|\xi\|)^{q-1} \tau(d\xi)) < \infty. \quad \square$$

The following lemma is a generalization of Ottaviani's inequality ([2], p. 45).

(2.9) **Lemma.** *Let* $\{y_n; n \in \mathbb{N}^q\}$ *be independent B-valued random variables for some* $q \geqq 1$. *Set* $T_m = \sum_{k \leqq m} y_k$. *Fix* $n \in \mathbb{N}^q$. *Let* $\mathcal{D}_q(n)$ *denote the set of non-empty differences* $\Delta = R_1 \setminus R_2$ *generated by rectangles* $R_i \subset \{m \in \mathbb{N}^q : m \leqq n\}$; $i = 1, 2$, *with* R_1 *having a vertex at n. Define* $r_0(\delta) = \delta$, $r_1(\delta) = \delta/(1-\delta)$, *and* $r_{i+1}(\delta) = r_1(r_i(\delta))$ *for* $0 \leqq \delta < 1$. *Suppose that*

$$\max_{\Delta \in \mathcal{D}_q(n)} P(\|\sum_{m \in \Delta} y_m\| > A) = \delta < 1$$

(2.10) *and*

$$r_{q-1}(\delta) < 1.$$

for some $A > 0$. *Then*

(2.11) $$P(\max_{m \leqq n} \|T_m\| > 2^q A) \leqq (1 - r_{q-1}(\delta))^{-q} P(\|T_n\| > A).$$

Proof. When $q = 1$, (2.10) $\Rightarrow$ (2.11) follows by Ottaviani's inequality which is valid in the Banach space setting. We assume iductively that (2.10) $\Rightarrow$ (2.11) for $q \leqq Q - 1$.

For $Q \geqq 2$ and $m = (m_1, \ldots, m_Q) \in \mathbb{N}^Q$ we write:

$$m' = (m_2, \ldots, m_Q) \in \mathbb{N}^{Q-1}$$

and

$$T'_{m'} = \sum_{k' \leqq m'} y'_{k'} = \sum_{k' \leqq m'} (\sum_{k_1 \leqq 1} y_k, \sum_{k_1 \leqq 2} y_k, \ldots, \sum_{k_1 \leqq n_1} y_k) \in B^{n_1}.$$

Let the norm $|||\cdot|||$ on B^ν be defined for $\nu \geqq 1$ by

$$|||\xi||| = \max_{i \leqq \nu} \|\xi_i\|, \qquad \xi = (\xi_1, \ldots, \xi_\nu) \in B^\nu.$$

Clearly, we have $\max_{m \leqq n} \|T_m\| = \max_{m' \leqq n'} |||T'_{m'}|||$. Thus by our induction hypothesis we obtain

(2.12) $$P(\max_{m \leqq n} \|T_m\| \geqq 2^Q A) \leqq (1 - r_{Q-2}(\delta'))^{-Q+1} P(|||T'_{n'}||| > 2A), \qquad n \in \mathbb{N}^Q.$$

Here,

$$\delta' = \max_{\Delta' \in \mathscr{D}_{Q-1}(n')} \mathrm{P}(|||\sum_{k' \in \Delta'} y'_{k'}||| > 2A)$$

(2.13) and

$$|||\sum_{k' \in \Delta'} y'_{k'}||| = \max_{i \leqq n_1} \| \sum_{k' \in \Delta', k_1 \leqq i} \mathrm{y}_k \|.$$

Hence, by Ottaviani's inequality,

$$\delta' \leqq \max_{\Delta' \in \mathscr{D}_{Q-1}(n')} (1-\delta)^{-1} \delta = r_1(\delta).$$

Similarly one finds that

$$P(|||T'_{n'}||| > 2\alpha) \leqq (1 - \delta'')^{-1} P(\|T_n\| > A)$$

(2.14) with

$$\delta'' = \max_{i < n_1} P(\| \sum_{k' \leqq n', i < k_1 \leqq n_1} y_k \| > A) \leqq \delta.$$

The proof by induction is complete upon combining (2.12), (2.13) and (2.14) and noticing that

$$(1 - r_{Q-1}(\delta))^{-Q+1} (1-\delta)^{-1} \leqq (1 - r_{Q-1}(\delta))^{-Q}$$

for $Q \geqq 2$. □

The next lemma suits our purposes as a multiparameter analogue of Theorem 3.1 of [8].

(2.15) **Lemma.** *Let* $\{y_n; n \in \mathbb{N}^q\}$ *be independent mean zero B-valued random variables for some* $q \geqq 1$. *Put* $T_n = \sum_{m \leqq n} y_m$. *Suppose that*

$$\|y_n\| \leqq \Gamma \alpha([n])$$

with $\alpha(\cdot)$ *as defined in* (2.1)(iv) *and a constant* Γ. *Suppose also that*

$$\overline{\lim_n} E\|y_n\|^2 \leqq 1 \quad \text{and} \quad \overline{\lim_n} P(a_n^{-1} \|T_n\| > C) < \tfrac{1}{24}$$

for some positive constant C. Then there is a finite number L such that

$$\overline{\lim_n} a_n^{-1} \|T_n\| = L \quad a.s.$$

Proof. Write:

$$d_m = (d_m(1), \ldots, d_m(q)) = (2^{m_1} - 2, \ldots, 2^{m_q} - 2)$$

for $m = (m_1, \ldots, m_q) \in \mathbb{N}^q$. By the Borel-Cantelli lemma it suffices to show there exists a number M so that upon defining the events

(2.16) $$E_m = \{ \max_{d_m < n \leqq d_{m+e}} (a_n^{-1} \|T_n\|) > M \}$$

one has

$$\sum_{m\in\mathbb{N}^q} P(E_m)<\infty.$$

For, by the Kolmogorov 0–1 law it is enough to have

$$\overline{\lim_n}\, a_n^{-1}\,\|T_n\|<\infty \quad \text{a.s.}$$

By first assuming that the random vectors $\{y_n; n\in\mathbb{N}^q\}$ are symmetric and later removing this restriction one finds exactly as shown in [8] that

$$\sup_n a_n^{-1} E\,\|T_n\|<\infty.$$

Similarly, we have the following.

Remark. If one assumes here that $P\lim_n a_n^{-1}T_n=0$ then

$$\lim_n a_n^{-1} E\,\|T_n\|=0.$$

We define rectangles $\Delta_m(J)$ for each $m\in\mathbb{N}^q$ and $J\subset\{1,\dots,q\}$ by

$$\Delta_m(J)=\left\{n\in\mathbb{N}^q \,\middle|\, \begin{matrix} d_m(i)<n_i\leqq d_{m+e}(i) \text{ for } i\in J \text{ and} \\ n_i\leqq d_m(i) \text{ for } i\notin J;\ i=1,\dots,q \end{matrix}\right\}$$

Then,

$$\begin{aligned} &P(\max_{d_m<n\leqq d_{m+e}} a_n^{-1}\,\|T_n\|>M) \\ &\quad\leqq \sum_{J\subset\{1,\dots,q\}} P(\max_{d_m<n\leqq d_{m+e}} \|\sum_{k\leqq n, k\in\Delta_m(J)} y_k\|>2^{-q}a_{d_m}M). \end{aligned} \tag{2.18}$$

We choose $M>2^{3q+2}\sup_m a_{d_m}^{-1} \max_{d_m\leqq n\leqq d_{m+e}} E\,\|T_n\|$ which is possible by (2.17). Using Lemma 2.9 and elementary probability inequalities one sees that the R.H.S. of (2.18) is bounded by

$$2^q(1-r_{q-1}(2^{-q-1}))^{-q} \sum_{n\in\mathbb{N}^q: n_i=d_m(i) \text{ or } d_{m+e}(i)} P(\|T_n\|\geqq 2^{-3q}a_{d_m}M). \tag{2.19}$$

Finally, to each of the summands in (2.19) we apply Theorem 2.1 of [8] with

$$b=b^{(m)}=[d_{m+e}]^{1/2}, \qquad c=c^{(m)}=\Gamma[d_{m+e}]^{-1/2}\,\alpha(d_{m+e}])$$

and

$$\varepsilon=\varepsilon^{(m)}=2^{-3q-1}M\,a_{d_m}[d_{m+e}]^{-1/2}.$$

Then from (2.16), (2.18) and (2.19),

$$\begin{aligned} P(E_m)=O(\exp\{&-\varepsilon^2(1-(1+\varepsilon c/2)\sum_{n\leqq d_{m+e}} E\,\|y_n\|^2/[d_{m+e}] \\ &-\max_{d_m\leqq n\leqq d_{m+e}} E\,\|T_n\|/2\,[d_{m+e}]^{1/2}\,\varepsilon)\}) \end{aligned}$$

since $\lim_m \varepsilon^{(m)}c^{(m)}=0$ (by (2.2)(iii)) and $\|y_n\|\leqq c^{(m)}b^{(m)}$ for $n\leqq d_{m+e}$. Moreover, if M is sufficiently large,

$$\overline{\lim_m} \max_{d_m \leqq n \leqq d_{m+e}} E\|T_n\|/2[d_{m+e}]^{1/2}\varepsilon^{(m)} < \tfrac{1}{4}$$

while

$$\overline{\lim_m} [d_{m+e}]^{-1} \sum_{n \leqq d_{m+e}} E\|y_n\|^2 \leqq 1.$$

Hence when M is large,

$$\sum_m P(E_m) = O(\sum_m \exp(-(\varepsilon^{(m)})^2/4)) = O(\sum_m (m_1 + \ldots + m_q)^{-2q}) < \infty. \quad \square$$

3. Bounded Law of the Iterated Logarithm

The following proposition provides us with a bounded law of the iterated logarithm. We shall reduce its proof to the verification of (3.3), below.

(3.1) **Proposition.** *Let $q \geqq 1$ and x be a B-valued random variable with*

$$x \in L^2 \log^{q-1} L \quad \text{and} \quad P \lim_n a_n^{-1} S_n = 0.$$

Then we have

$$\lim_n \inf_{\xi \in K} \|a_n^{-1} S_n - \xi\| = 0 \quad a.s. \tag{3.2}$$

where K is the closed unit ball of the reproducing kernel Hilbert space H_T determined by the covariance function $T(\cdot,\cdot)$ defined in (1.3).

Proof. We may assume that x is symmetric. For, if ε takes the values ± 1 each with probability 1/2 independently of x then εx is symmetric and $E\{f(x)g(x)\} = E\{f(\varepsilon x) g(\varepsilon x)\}$. Moreover K is symmetric and $|\varepsilon| = 1$, so

$$\inf_{\xi \in K} \|a_n^{-1} \varepsilon S_n - \xi\| = \inf_{\xi \in K} \|a_n^{-1} S_n - \xi\|.$$

Assume now that in addition to our hypotheses we have

$$P(\{a_n^{-1} S_n;\, n \in \mathbb{N}^q\} \text{ is relatively compact in } B) = 1. \tag{3.3}$$

The conclusion of Proposition 3.1 then follows by the argument provided in the proof of Theorem 3.1 (I) of [7]. To see this we need only note that Wichura ([16], Theorem 5 and comments p. 280) has shown:

$$P(\overline{\lim_n} f(a_n^{-1} S_n) = E^{1/2} f^2(x)) = 1 \qquad f \in B^*. \tag{3.4}$$

Thus, by (3.4) and Lemma 2.1 of [7], for any $f \in B^*$ one gets

$$E^{1/2} f^2(x) = \sup_{\xi \in K} f(\xi).$$

Hence, by the separability of B and the Hahn-Banach Theorem,

$$P(\{\text{accumulation points of } \{a_n^{-1} S_n;\ n \in \mathbb{N}^q\}\} \not\subset K) = 0.$$

Whence (3.2) must hold, for otherwise we gain a contradiction.

Thus to complete the proof of the proposition there remains only to show that (3.3) holds. For this purpose we truncate the random variables $\{x_n;\ n \in \mathbb{N}^q\}$ by setting

$$x_n' = x_n 1_{\{\|x_n\| \leqq \alpha([n])\}} \tag{3.5}$$

with $\alpha(\cdot)$ as defined in (2.2)(iv). Put $S_n' = \sum_{m \leqq n} x_m'$.

We first notice that if $\sum_n a_n^{-1} c_n < \infty$ for some non-negative numbers $\{c_n;\ n \in \mathbb{N}^q\}$ then by partial summation,

$$\lim_{r \to \infty} (r \log\log r)^{-1/2} \sum_{[n] \leqq r} c_n = 0.$$

But, by Lemma 2.4, $\sum_n a_n^{-1} \|x_n - x_n'\| < \infty$ a.s. It therefore follows that

$$\lim_n a_n^{-1} \|S_n - S_n'\| = 0 \quad \text{a.s.} \tag{3.6}$$

Next, we define the mapping τ_δ that Kuelbs introduced in his proof of Theorem 4.1 of [8]. Namely, if $0 < \delta < 1$,

$$\tau_\delta(\xi) = E(x \mid x^{-1}(I))(\xi), \qquad \xi \in B.$$

Here I is a finite partition of B containing $\{0\}$ such that

$$A \in I \Leftrightarrow -A \in I$$

and

$$E\|\tau_\delta(x) - x\|^2 \leqq \delta.$$

Since $\{0\} \in I$ we also have $E\|\tau_\delta(x_n' - x_n'\|^2 \leqq \delta$. Put $y_n = \tau_\delta(x_n')$ and $T_n = T_n(\delta) = \sum_{m \leqq n} y_m$. Since τ_δ has finite dimensional range and

$$\tau_\delta(\xi) \leqq C_\delta(\|\xi\| + 1), \qquad \xi \in B$$

one has $\|y\|_n \leqq C_\delta(\alpha([n]) + 1)$, $\sup_n E\|y_n\|^2 \leqq C_\delta^2 E\{(1 + \|x\|)^2\} < \infty$ and

$$P \lim_n a_n^{-1} T_n = 0. \tag{3.7}$$

Therefore Lemma 2.15 implies

$$P(\{a_n^{-1} T_n;\ n \in \mathbb{N}^q\} \text{ is relatively compact in } B) = 1 \tag{3.8}$$

since bounded subsets of finite dimensional spaces are relatively compact.

Now (3.6) and (3.8) will combine to yield (3.3) if for each $\varepsilon>0$ there is a $\delta=\delta_\varepsilon>0$ so that

$$\overline{\lim_n}\, a_n^{-1}\,\|S'_n-T_n(\delta)\|<\varepsilon \quad \text{a.s.} \tag{3.9}$$

For, given (3.9), with probability one we can use a diagonalization procedure to construct from any sequence $\{a_{n(i)}^{-1}S_{n(i)},\ i\geqq 1\}$ a subsequence which is Cauchy and therefore convergent.

To obtain (3.9) we first notice that $P\lim_n a_n^{-1}S'_n=0$. (Use (3.6) and the assumption that $P\,lim_n\, a_n^{-1}S_n=0$). Therefore, by (3.7),

$$P\lim a_n^{-1}(S'_n-T_n)=0. \tag{3.10}$$

But $S'_n-T_n=\sum_{m\leqq n} x'_m-y_m$ is a sum of independent random vectors and $\|x'_n-y_n\|\leqq\text{const.}\,\alpha([n])$. Thus by (3.10) and the Remark included in the proof of Lemma 2.15,

$$\lim_n a_n^{-1}E\,\|S'_n-T_n\|=0. \tag{3.11}$$

Finally, if $\varepsilon>0$ and F_m is defined by the R.H.S. of (2.16) with ε in place of M and $S'_n-T_n(\delta)$ in place of T_n we use (3.11) and the argument of Lemma 2.15 to get

$$\sum_m P(F_m)<\infty.$$

This gives us (3.9). Whence (3.3) holds. □

4. Reduction of Theorem 1

Because we are dealing with sums of independent identically distributed random vectors, to prove Theorem 1 it is enough to establish the following proposition.

(4.1) **Proposition.** *Let $q\geqq 1$ and let x satisfy the hypothesis of Theorem 1. Then for each $\theta>0$ there is a Brownian sheet $\{W_\theta(t);\ t\in[0,\infty)^q\}$ in B with covariance function $T(\cdot,\cdot)$ defined by (1.3) such that*

$$\overline{\lim_n}\, a_n^{-1}\,\|S_n-W_\theta(n)\|\leqq\theta \quad \textit{a.s.}$$

To prove Proposition 4.1 we approximate x by a finite dimensional random vector. For this we employ the maps Π_N associated to the covariance function $T(\cdot,\cdot)$ of x, as defined in Lemma 2.1 of [7]. With the notation of the introduction we write

$$\Pi_N(\xi)=\sum_{\nu=1}^{N}\varphi_\nu^*(\xi)\,\varphi_\nu, \qquad \xi\in\text{B}. \tag{4.2}$$

(4.3) **Lemma.** *Let x be as in Proposition* 3.1. *Let* $\theta>0$. *Then there exists* N_θ *such that*

$$\overline{\lim_n}\, a_n^{-1}\,\|S_n-\Pi_{N_\theta}S_n\|\leqq\theta/3 \qquad a.s.$$

Proof. The lemma follows by Proposition 3.1. Indeed the maps $Q_N=I-\Pi_N$ are linear and continuous. Thus

$$(4.4)\qquad \begin{aligned} &\overline{\lim_n}\,\inf_{\xi\in K}\|a_n^{-1}S_n-\xi\|=0 \qquad \text{a.s.}\\ \Rightarrow\ &\overline{\lim_n}\,\inf_{\xi\in K}\|a_n^{-1}Q_N(S_n)-Q_N(\xi)\|=0 \qquad \text{a.s.}\end{aligned}$$

Moreover as shown in Theorem 3.1 of [7], given $\theta>0$ there exists N_θ such that

$$(4.5)\qquad \sup_{\xi\in K}\|Q_{N_\theta}(\xi)\|\leqq\frac{\theta}{3}.$$

(This relies on the fact that K as defined in Proposition 3.1 is compact in B.) Combining (4.4) and (4.5) we evidently have the statement of the lemma. □

We now fix $\theta>0$ and $N=N_\theta$. The space $\Pi_N(B)$ is the Euclidean space $\mathbb{R}^p$ ($p=\min(N,\dim H_T)$) equipped with the norm $|\cdot|=\|\cdot\|_T$ induced by the B-norm on $H_T\subset B$. The B-norm $\|\cdot\|$ is continuous with respect to the norm $|\cdot|$ on H_T and in fact

$$(4.6)\qquad \|\xi\|\leqq E^{1/2}\|x\|^2\,|\xi|, \qquad \xi\in H_T.$$

We define $\hat{x}=\Pi_N(x)$ with Π_N given by (4.2). Thus, $\hat{x}$ is a random variable in $\mathbb{R}^p$ having the properties:

$$(4.7)\qquad |\hat{x}|=\|\Pi_N\|\,\|x\|, \qquad E\hat{x}=0,\ E\hat{x}\hat{x}^{\top}=I_p.$$

Let $\{\hat{x}_n;\, n\in\mathbb{N}^q\}$ denote generically a collection of independent copies of $\hat{x}$. Set $\hat{S}_n=\sum_{m\leqq n}\hat{x}_m$.

The point of Lemma 4.3 is that we need only obtain the conclusion of Proposition 4.1 when $B=\mathbb{R}^p$ for some $p\geqq 1$. This is accomplished over the course of the next two sections.

5. Weak Law of the Iterated Logarithm

Let

$$c_m=(c_m(1),\ldots,c_m(q))$$

with

$$(5.1)\qquad c_m=\begin{cases}0, & e<m\\ [c^{m_i}/(c-1)], & e<m;\ i=1,\ldots,q\end{cases}$$

for $m=(m_1,\ldots,m_q)\in\mathbb{N}^q$ and some $1<c<2$. For each subset $J\subset\{1,\ldots,q\}$ put

$$(5.2)\qquad H_m(J)=\left\{n\in\mathbb{N}^q\,\middle|\,\begin{matrix}c_m(i)<n_i\leqq c_{m+e} & \text{for } i\notin J\\ 1\leqq n_i\leqq c_m(i) & \text{for } i\in J\end{matrix}\right\}, \qquad m\in\mathbb{N}^q.$$

The sets $H_m(\phi)$ so defined partition $\mathbb{N}^q$ as m runs through $\mathbb{N}^q$. Set

(5.3) $$h_m(J)=|H_m(J)|;\ m\in\mathbb{N}^q,\ J\subset\{1,\dots,q\}.$$

Then observe that there is an absolute constant C_0 so that

(5.4) $$1/C_0<h_n(J)(c-1)^{|J|}c^{-\sum_{i=1}^{q}n_i}<C_0;$$
$$n\in\mathbb{N}^q,\quad J\subset\{1,\dots,q\},\quad 1<c<2.$$

Recalling (2.2)(iv), (4.7) and (5.1)–(5.3) we now define for all $n\in\mathbb{N}^q$ and $J\subset\{1,\dots,q\}$,

$$X_m(J)=h_m^{-1/2}(J)\sum_{n\in H_m(J)}\hat{x}_n$$

(5.5) and

$$X'_m(J)=h_m^{-1/2}(J)\sum_{n\in H_m(J)}(\hat{x}_n 1_{\{|\hat{x}_n|\leqq\alpha([c_m])\}}-E(\hat{x}_n 1_{\{|\hat{x}_n|\leqq\alpha([c_m])\}})).$$

Write $F_m^{(J)}$ (resp. $F_m^{(J)'}$) for the distribution of $X_m(J)$ (resp. $X'_m(J)$) and G for the Gaussian distribution on $\mathbb{R}^p$ with covariance matrix I_p.

For a set $A\subset\mathbb{R}^p$ and $r\geqq 0$ we denote $A^r=\bigcup_{\xi\in A}\{\eta:|\eta-\xi|<r\}$.

(5.6) **Proposition.** *Let $J\subset\{1,\dots,q\}$. Let $\rho>0$ and put*

(5.7) $$\rho_m=\rho(\log\log^+[c_m])^{1/2}.$$

Then there exist non-negative numbers $(\sigma_m,\ m\in\mathbb{N}^q)$ such that

$$\sum_{m\in\mathbb{N}^q}\sigma_m<\infty$$

and

$$F_m^{(J)}(A)\leqq G(A^{\rho_m})+\sigma_m\qquad \textit{for each Borel set } A\subset\mathbb{R}^p.$$

Throughout the remainder of this section we fix J and drop the dependence on J from our notation. Let G'_m denote the Gaussian distribution on $\mathbb{R}^p$ with covariance matrix

$$\Gamma'_m=E\,X'_m X'^{\mathrm{T}}_m.$$

Define the Prohorov distance $d(F,G)$ between distributions F and G on $\mathbb{R}^p$ as $d(F,G)=\inf\{\varepsilon>0: F(A)\leqq G(A^\varepsilon)+\varepsilon$ for all Borel sets $A\subset\mathbb{R}^p\}$. We shall obtain:

(5.9) (i) $$\sum_{m\in\mathbb{N}^q}P(|X_m-X'_m|>\tfrac{1}{3}\rho_m)<\infty$$

(ii) $$\sum_{m\in\mathbb{N}^q}d(F'_m,G'_m)<\infty$$

and (iii) $$\sum_{m\in\mathbb{N}^q}P(|Y-Y'_m|>\tfrac{1}{3}\rho_m)<\infty$$

where $Y'_m \sim G'_m$ and $Y \sim G$. From (5.9)(i)–(iii) one deduces that

$$\begin{aligned}F_m(A) &\leqq P(x'_m \in A^{\rho_m}) + \tfrac{1}{3}\sigma_m \\ &\leqq P(Y'_m \in A^{(\rho_m+\sigma_m)/3}) + \tfrac{2}{3}\sigma_m \\ &\leqq P(Y \in A^{(2\rho_m+\sigma_m)/3}) + \sigma_m \leqq G(A^{\rho_m}) + \sigma_m\end{aligned}$$

for some non-negative numbers $(\sigma_m, m \in \mathbb{N}^q)$ with $\sum_m \sigma_m < \infty$ and any Borel set $A \subset \mathbb{R}^p$. Thus, to prove Proposition 5.6 it suffices to verify (5.9)(i)–(iii).

(5.10) **Lemma.** *Take* Γ''_m *as given by* (5.8) *and denote by* $\langle \cdot, \cdot \rangle$ *the standard inner product for* $\mathbb{R}^p$. *Then*

$$\lim_m \sup_{\xi \in \mathbb{R}^p, \langle \xi, \xi \rangle \leqq 1} \langle \xi, (\Gamma'_m - I_p)\xi \rangle = 0.$$

Proof. Since all norms on a finite dimensional space are equivalent, by (4.7), (5.5) and (5.8) it suffices to show that

$$\lim_m E|X_m - X'_m|^2 = 0.$$

But, by (5.5),

$$E|X_m - X'_m|^2 \leqq 4E(|\hat{x}|^2 1_{\{|\hat{x}| > \alpha([c_m])\}})$$

since by (4.7) $\hat{x} \in L^2$ and $E\hat{x} = 0$. Thus, by (2.2)(iv) and (5.1) the proof is complete. □

Proof of (5.9)(i). Let $\hat{\sigma}$ denote the distribution of $\hat{x}$. Define $\hat{w}_n$ by the R.H.S. of (2.3) with σ replaced by $\hat{\sigma}$. Then by Markov's inequality, (5.1)–(5.4) and (5.7),

$$\begin{aligned}P(|X_m - X'_m| > \tfrac{1}{3}\rho_m) &\leqq \text{const}\, h_m^{-1/2} \rho_m^{-1} \sum_{n \in H_m(\phi)} E|\hat{x}_n 1_{\{\hat{x}_n > \alpha([c_m])\}}| \\ &\leqq \text{const}\, a_n^{-1} \sum_{n \in H_m(\phi)} \hat{w}_n, \quad \text{uniformly in } m.\end{aligned}$$

An application of Lemma 2.4 now finishes the proof. □

Proof of (5.9)(ii). By the main theorem of Yurinskii [17] we calculate that

$$\begin{aligned}d(F'_m, G'_m) &\leqq \text{const} \frac{\sum_{i \leqq p} \sum_{n \in H_m} E|\hat{x}_{n,i} 1_{\{|\hat{x}_n| \leqq \alpha([c_m])\}} - E(\hat{x}_{n,i} 1_{\{|\hat{x}_n| \leqq \alpha([c_m])\}})|^3}{(E\{(h_m^{1/2} X'_m)^2\})^{3/2}} \\ &\leqq \text{const}\, h_m^{-1/2} E(|\hat{x}|^3 1_{\{|\hat{x}| \leqq \alpha([c_m])\}}) \quad \text{for large } [m].\end{aligned}$$

Thus, setting $s(m) = \sum_{i=1}^{q} m_i$, we have

$$\begin{aligned}
&\sum_m d(F'_m, G'_m)\\
&\leqq \text{const}\left(1+\sum_m c^{-\frac{1}{2}s(m)} \sum_{\mu=1}^{\frac{1}{2}(s(m)-\log\log^+ s(m))} c^{\frac{3}{2}\mu} P(c^{\mu/2}<|\hat{x}|\leqq c^{(\mu+1)/2})\right)\\
&\leqq \text{const}\left(1+\sum_{\nu=1}^{\infty} \nu^{q-1} c^{-\frac{1}{2}(\nu+\log\log^+\nu)} \sum_{\mu\leqq\nu} c^{\frac{3}{2}\mu} P(c^{\mu/2}<|\hat{x}|\leqq c^{(\mu+1)/2})\right)\\
&=\text{const}\left(1+\sum_{\mu=1}^{\infty} c^{\frac{3}{2}\mu} P(c^{\mu/2}<|\hat{x}|\leqq c^{(\mu+1)/2}) \sum_{\nu=\mu}^{\infty} \nu^{q-1} c^{-\frac{1}{2}(\nu+\log\log^+\nu)}\right)\\
&\leqq \text{const}\left(1+\sum_{\mu=1}^{\infty} (\log^+(c^\mu))^{q-1} c^\mu P(c^{\mu/2}<|\hat{x}|\leqq c^{(\mu+1)/2})/\log\log^+\mu\right)\Bigg)\\
&\leqq \text{const}\,(1+E(|\hat{x}|^2(\log^+|\hat{x}|)^{q-1}/\log\log^+|\hat{x}|))<\infty. \quad \square
\end{aligned}$$

Proof of (5.9)(iii). By (4.7), (5.4) and (5.5) it suffices to show that for large $[m]$,

$$\int_A \exp(-\tfrac{1}{2}\langle\xi,\xi\rangle\, d\xi \leqq (\det\Gamma'_m)^{-\frac{1}{2}} \int_{A^{\rho_m/3}} \exp(-\tfrac{1}{2}\langle\xi,\Gamma'^{-1}_m\xi\rangle)\,d\xi + \text{const}\,[m]^{-2}, \quad \text{for all Borel sets } A\subset\mathbb{R}^p. \tag{5.11}$$

The change of variable $\eta=\Gamma'^{-\frac{1}{2}}_m\xi$ takes condition (5.11) into the form

$$\int_A \exp(-\tfrac{1}{2}\langle\xi,\xi\rangle)\,d\xi \leqq \int_{\Gamma'^{-\frac{1}{2}}_m} (A^{\rho_m/3}) \exp(-\tfrac{1}{2}\langle\eta,\eta\rangle)\,d\eta + \text{const}\,[m]^{-2}, \quad \text{for all Borel sets } A\subset\mathbb{R}^p.$$

But (5.12) holds if we can show it holds with

$$A\subset\{\xi\in\mathbb{R}^p\colon |\xi|\leqq\text{const}(\log\log^+[c_m]^{\frac{1}{2}})\}.$$

Now, by Lemma 5.10 and (5.7), if $\eta\in A$ and A satisfies (5.13) then

$$|\eta-\Gamma'^{\frac{1}{2}}_m\eta|\leqq\rho_m/3 \quad \text{and} \quad \Gamma'_m \text{ is non-singular}$$

since $[m]$ is large. But then $\Gamma'^{\frac{1}{2}}_m A\subset A^{\rho_m/3}$ or $A\subset\Gamma'^{-\frac{1}{2}}_m(A^{\rho_m/3})$. Thus (5.11) holds as does (5.12). $\square$

The proof of Proposition 5.6 is now complete as the statements (5.9)(i)–(iii) have all been verified. $\square$

6. Proof of Proposition 4.1

The following Theorem is due to Philipp [13]. It generalizes Theorem 2 of Berkes and Philipp [1].

Theorem. *Let $\{B_k, m_k, k\geqq 1\}$ be a sequence of complete separable metric spaces. Let $\{X_k; k\geqq 1\}$ be a sequence of random variables with values in B_k and let $\{L_k; k\geqq 1\}$ be a sequence of σ-fields such that X_k is L_k-measurable. Suppose that* *for some sequence* $\{\Phi_k, k \geqq 1\}$ *of non-negative numbers*

$$|P(AB) - P(A)P(B)| \leqq \Phi_k P(A)$$

for all $k \geqq 1$ *and all* $A \in \bigvee_{j<k} L_j$ *and* $B \in L_k$. *Denote by* F_k *the distribution of* X_k *and let* $\{G_k, k \geqq 1\}$ *be a sequence of distributions* (G_k *a distribution on* B_k) *such that for some non-negative numbers* ρ_k *and* σ_k

$$F_k(A) \leqq G_k\left(\bigcup_{\xi \in A} \{\eta: m_k(\xi, \eta) < \rho\}\right) + \sigma_k$$

for all Borel sets $A \subset B_k$. *Then without changing its distribution we can redefine the sequence* $\{X_k; k \geqq 1\}$ *on a richer probability space on which there exists a sequence* $\{Y_k; k \geqq 1\}$ *of independent random variables* Y_k *with distribution* G_k *such that for all* $k \geqq 1$

$$P(m_k(X_k, Y_k) \geqq 2(\Phi_k + \rho_k) \leqq 2(\rho_k + \sigma_k).$$

By the conclusion of Proposition 5.6 we can apply the above theorem directly (with $\Phi_i \equiv 0$) because we are working with independent random vectors. Thus, for independent vectors Y_m, we have:

$$\sum_m P(|X_m(\phi) - Y_m| \geqq 2\rho_m) < \infty \tag{6.1}$$

where $Y_m \sim \mathcal{N}(0, I_p)$ and $X_m(\phi)$ is given by (5.5). Now if $\{W_\theta(t); t \in [0, \infty)^q\}$ is a Brownian sheet in B with covariance function $T(\cdot,\cdot)$ then the action of the canonical maps Π_N defined by (4.2) render $\Pi_N(W(t)) \stackrel{\text{def}}{=} \hat{W}(t) \sim \mathcal{N}(0, [t] I_p)$. Hence by (6.1), the assumption that x is pregaussian and Kolmogorov's existence theorem,

$$\sum_m P(h_m^{-\frac{1}{2}}(\phi) | \sum_{n \in H_m(\phi)} \hat{x}_n - \hat{W}_\theta(H_m(\phi))| \geqq 2\rho_m) < \infty. \tag{6.2}$$

Here we have put

$$\hat{W}_\theta(\{k \in \mathbb{N}^q: m \leqq k \leqq n\}) = \sum_{k: k_i = m_i - 1 \text{ or } n_i} (\pm)\, \hat{W}_\theta(k)$$

for $m, n \in \mathbb{N}^q$ with $m \leqq n$.

Next, by (5.1)–(5.4) and (5.7),

$$\begin{aligned}\sum_{m \leqq n} h_m^{\frac{1}{2}}(\phi)\, \rho_m &\leqq \text{const}\, \rho \sum_{m \leqq n} c^{\frac{1}{2}\sum_{i=1}^{q} m_i} (\text{loglog}^+ [c_m])^{\frac{1}{2}} \\ &\leqq \text{const}\, \rho (c^{\frac{1}{2}} - 1)^{-q} a_{t_n} \quad \text{for} \quad 1 < c < 2.\end{aligned}$$

This, together with (6.2), yields

$$\overline{\lim_m}\, a_{c_m}^{-1} \|\hat{S}_{c_{m+e}} - W_\theta(c_{m+e})\| \leqq \text{const}\, \rho (c^{\frac{1}{2}} - 1)^{-q} \quad \text{a.s.} \tag{6.3}$$

To obtain Proposition 4.1 from (6.3) and Lemma 4.3 we need the following lemma which gives us a bound on the fluctuation of $\hat{S}_n$ over the set $H_m(\phi)$.

(6.4) **Lemma.** *Set* $c=\theta^3+1$ *in definitions* (5.1)–(5.3). *Then for any proper subset* $J\subset\{1,\ldots,q\}$ *and small positive number* θ,

$$\frac{1}{\theta}\overline{\lim_m}\, a_{c_m}^{-1} \max_{n\in H_m(\phi)} \Big\|\sum_{k\in H_m(J),\,k\leqq n} \hat{x}_k\Big\| \leqq \frac{2^{-q}}{12} \quad a.s.$$

Proof. We apply Lemma 2.9 to obtain

$$\begin{aligned} &P(\max_{n\in H_m(\phi)} \Big\|\sum_{k\in H_m(J),\,k\leqq n} \hat{x}_k\Big\| > 2^{-q}\,\theta a_{c_m}/12) \\ &\leqq \mathrm{const}(\theta)\cdot P(\Big\|\sum_{n\in H_m(J)} \hat{x}_n\Big\| > 2^{-2q}\,\theta a_{c_m}/12) \end{aligned} \tag{6.5}$$

for $[m]$ sufficiently large (depending on θ). This is valid because the mean zero random vectors $\{\hat{x}_n;\, n\in\mathbb{N}^q\}$ are finite dimensional and independent. Hence the conditions of Lemma 2.9 are easily seen to be satisfied by applying Čebyšev's inequality.

Now set $\rho=\theta^{2q}$ in (5.7). By Proposition 5.6,

$$\begin{aligned} &P(\Big\|\sum_{n\in H_m(J)} \hat{x}_n\Big\| > 2^{-q}\,\theta a_{c_m}/12) \\ &\leqq F_m^{(J)}(\{\xi\in\mathbb{R}^p\colon |\xi| > 2^{-2q}\,\theta a_{c_m} h_m^{-\frac{1}{2}}(J)/12E^{\frac{1}{2}}\|x\|^2) \\ &\leqq \sigma_m + G(\{\xi\in\mathbb{R}^p\colon |\xi| > (C_0'\,\theta(c-1)^{-\frac{1}{2}}-\theta^{2q})(\log\log^+[c_m])^{\frac{1}{2}}) \end{aligned}$$

for some absolute constant C_0'. Thus, if θ is sufficiently small,

$$\sum_m \text{L.H.S. of (6.5)} \leqq \mathrm{const}(\theta)\left(1+\sum_m\left\{\sigma_m(\theta)+P\left(\chi^2_{p(\theta)}>3q\sum_{i=1}^q m_i\right)\right\}\right).$$

An application of the Borel-Cantelli lemma completes the proof. □

We are now ready to finish the proof of Proposition 4.1. We take $c=\theta^3+1$ and $\rho=\theta^{2q}$. Let $n\in\mathbb{N}^q$. This determines $m=m_n\in\mathbb{N}^q$ by

$$c_m < n \leqq c_{m+e}$$

We then write

$$\begin{aligned} a_n^{-1}\|\hat{S}_n - W_\theta(v)\| &\leqq a_{c_m}^{-1}\|\hat{S}_{c_{m+e}} - \hat{W}_\theta(c_{m+e})\| \\ &+ a_{c_m}^{-1}\sum_{J\subsetneqq\{1,\ldots,q\}} \max_{k\in H_m(J)} (\|\hat{W}_\theta(\{j\in H_m(J), j\leqq k\})\| + \Big\|\sum_{j\in H_m(J),\, j\leqq k} \hat{x}_j\Big\|). \end{aligned}$$

Therefore by (6.3), Lemma 6.4 and our choices of ρ and c,

$$\overline{\lim_n}\, a_n^{-1}\|\hat{S}_n - \hat{W}_\theta(n)\| \leqq \frac{\theta}{3} \quad \text{a.s.}$$

when θ is small. This, together with Lemma 4.3, yields the conclusion of Proposition 4.1. □

7. Proof of Theorem 1

By utilizing Proposition 4.1 we can patch together various independent Brownian sheets to obtain a single Brownian sheet satisfying (1.4). Our method is similar to that employed by Major [11].

For each $i \geqq 0$ we construct pairs of processes $\{x_n^{(i)};\, n \in \mathbb{N}^q\}$, $\{W^{(i)}(t);\, t \in [0, \infty)^q\}$ which are independent for different i. Further, for each i we take

$$\{x_n^{(i)};\, n \in \mathbb{N}^q\} = \text{a collection of independent copies of } x$$

and

$$\{W^{(i)}(t);\, t \in [0, \infty)^q\} = \text{a Brownian sheet in } B$$

with covariance function $T(\cdot,\cdot)$ given by (1.3). Put $S_n^{(i)} = \sum_{m \leqq n} x_m^{(i)};\, n \in \mathbb{N}^q,\, i \geqq 0$. By Proposition 4.1 we construct these processes so that

$$\overline{\lim_n}\, a_n^{-1} \|S_n^{(i)} - W^{(i)}(n)\| \leqq 2^{-i} \quad \text{a.s., } i \geqq 0. \tag{7.1}$$

Using (7.1) and Lebesgue's bounded convergence theorem we choose an increasing sequence $\{v_i, i \geqq 1\}$ of q^{th} powers of positive integers such that

$$P(\sup_{[n]} \geqq v_i\, a_n^{-1} \|S_n^{(i)} - W^{(i)}(n)\| \geqq 2^{-i+1}) \leqq 2^{-i}. \tag{7.2}$$

We pick a subsequence $\{v_i'\}$ of $\{v_i\}$ so that with $n(i) = (v_i')^{1/q} \cdot e \in \mathbb{N}^q$ we get

$$\sum_{i \geqq 1} P\{a_{n(i)}^{-1}(\|S_{n(i)}^{(0)}\| + \|W^{(0)}(n(i))\|) > 2^{-i}\} < \infty$$

(7.3) and

$$\sum_{i \geqq 1} P\{a_{n(i)}^{-1}(\|S_{n(i)}^{(i)}\| + \|W^{(i)}(n(i))\|) > 2^{-i}\} < \infty.$$

This is possible because $\overline{\lim_n}\, P\{a_n^{-1}(\|S_n^{(i)}\| + \|W^{(i)}(n)\|) > 2^{-i}\} = 0$ for each $i \geqq 0$.

We now define inductively $\{W(t);\, t \in [0, \infty)^q\}$ and $\{S_n;\, n \in \mathbb{N}^q\}$ by putting $W(0) = 0$, $S_0 = 0$, $n(0) = 0$, $v_0' = 0$, $t = (t_1, \ldots, t_q)$ and

$$W(t) = W^{(i)}(t) - W^{(i)}(n(i)) + W(n(i)) \quad \text{for } v_i' \leqq t_1 \ldots t_q < v_{i+1}'$$

(7.4) and

$$S_n = S_n^{(i)} - S_{n(i)}^{(i)} + S_{n(i)} \quad \text{for } v_i' \leqq [n] < v_{i+1}';\, i \geqq 0.$$

In this way

$$\{S_n;\, n \in \mathbb{N}^q\} \stackrel{D}{=} \{S_n^{(0)};\, n \in \mathbb{N}^q\}$$

(7.5) and

$$\{W(t);\, t \in [0, \infty)^q\} \stackrel{D}{=} \{W^{(0)}(t);\, t \in [0, \infty)^q\}.$$

Thus, from (7.3), (7.5) and the Borel Cantelli lemma,

$$\lim_{i\to\infty} a_{n(i)}^{-1}(\|W^{(i)}(n(i))\| + \|W(n(i))\| + \|S_{n(i)}^{(i)}\| + \|S_{n(i)}\|) = 0 \quad \text{a.s.} \tag{7.6}$$

Moreover, from (7.2) and the Borel Cantelli lemma,

$$\lim_{i\to\infty} \sup_{[n]\geqq v_i'} a_n^{-1} \|S_n^{(i)} - W^{(i)}(n)\| = 0 \quad \text{a.s.} \tag{7.7}$$

Finally, let $n\in\mathbb{N}^q$. Then for some $i=i(n)$ we have $v_i'\leqq[n]<v_{i+1}'$, and, from (7.4),

$$\begin{aligned} a_n^{-1}\|W(n)-S(n)\| \leqq & \sup_{v_i'\leqq[n]<v_{i+1}'} a_n^{-1}\|S_n^{(i)} - W^{(i)}(n)\| \\ & + a_n^{-1}(\|W^{(i)}(n(i))\| + \|W(n(i))\| + \|S_{n(i)}^{(i)}\| + \|S_{n(i)}\|). \end{aligned} \tag{7.8}$$

Hence, (1.4) follows from (7.6)-(7.8). □

8. Functional Law of the Iterated Logarithm

We first note that a Brownian sheet $\{W(t);\, t\in[0,\infty)^q\}$ in B has continuous sample paths. To see this we use the argument in [2], p. 258-259. Let $D_N = \{2^{-N}n: 1\leqq n_i\leqq 2^N, i=1,\dots,q; n\in\mathbb{N}^q\}$ and put $D=\bigcup_{N\geqq 1} D_N$. Define

$$U_v = \sup_{\substack{s,t\in D \\ |s_i-t_i|\leqq 2^{-v}; i=1,\dots,q}} \|W(t)-W(s)\|.$$

We must show that $\lim_v U_v = 0$ a.s. and for this it is enough to show that $\lim_v P(U_v>\delta)=0$ for each $\delta>0$ since U_v is non-increasing in v.

Put $Y_n = Y_{n,v} = \sup_{\substack{t\in D \\ (n_i-1)2^{-v}\leqq t_i\leqq n_i 2^{-v}; i=1,\dots,q}} \|W(t)-W(2^{-v}(n-e))\|$. Then

$$U_v \leqq 3 \max_{\{n\in\mathbb{N}^q:\, 1\leqq n_i\leqq 2^v; i=1,\dots,q\}} Y_n,$$

so

$$\begin{aligned} P(U_v>\delta) &\leqq \sum_{\substack{1\leqq n_i\leqq 2^v \\ i=1,\dots,q}} P(Y_n\geqq\delta/3) \\ &= 2^{vq} P(Y_e\geqq\delta/3). \end{aligned}$$

Now observe that $P(Y_e\geqq\delta/3) = \lim_{N\to\infty} P(\sup_{t\in D_N, t\leqq 2^{-v}e} \|W(t)\|\geqq\delta/3)$. Moreover, by a result of Fernique [3] there is some $\alpha>0$ for which $E\exp(\alpha\|W(e)\|^2)<\infty$. Thus by Lemma 2.9,

$$\begin{aligned} & P(\sup_{t\in D_N, t\leqq 2^{-v}e} \|W(t)\|\geqq\delta/3) \\ & \leqq C\exp(-\alpha\delta^2 2^{vq}/9) \end{aligned}$$

for some constant C depending only on α, δ and q. Therefore $P(U_\nu > \delta) \leqq C 2^{\nu q} \exp(-ad^2 2^{\nu q}/9)$, and this last expression clearly tends to zero as $\nu \to \infty$.

We now show that Theorem 2 holds when we replace B by $\Pi_N(B)$ for any $N \geqq 1$. Here, Π_N is the canonical map on B as given in (4.2). To do this we need only modify slightly Pyke's proof of a functional law of the iterated logarithm for a Brownian sheet $\{W^0(t); t \in [0,\infty)^q\}$ in $\mathbb{R}^p$ with covariance matrix I_p ([14], Theorem 4).

First, if $p = \min(N, \dim H_T)$ the limit set $\mathscr{K}_T$ defined in (1.6) with B replaced by $\Pi_N(B)$ is just the set

$$(8.1)\qquad \mathscr{K}^{(p)} = \left\{ f \in C_{\mathbb{R}^p}([0,1]^q) \;\middle|\; \begin{array}{l} f \text{ is absolutely continuous w.r.t. Lebesgue} \\ \text{measure } l \text{ on } [0,1]^q,\ f(t)=0 \text{ if} \\ t_i = 0 \text{ for some } i=1,\dots,q \text{ and} \\ \int_{[0,1]^q} \langle df/dl, df/dl \rangle \, dl \leqq 1 \end{array} \right\}$$

Let us write $|\xi|_\infty = \max_{i=1,\dots p} |\xi_i|$ for $\xi = (\xi_1, \dots \xi_p) \in \mathbb{R}^p$. We put

$$\mathscr{K}_\varepsilon^{(p)} = \{ g \in C_{\mathbb{R}^p}([0,1]^q) : |f-g|_\infty \leqq \varepsilon \quad \text{for some } f \in K^{(p)} \}$$

and

$$W_n^0(t) = a_n^{-1} W^0(n_1 t_1, \dots, n_q t_q) \qquad t \in [0,1]^q,\ n \in \mathbb{N}^q.$$

Then, just as in [14],

$$P(W_n^0 \notin \mathscr{K}_\varepsilon^{(p)}) \leqq (\log [n])^{-(1+\delta)q}$$

for some $\delta = \delta_\varepsilon > 0$. Thus, with c_m given by (5.1) one deduces that $P(W_{c_m}^0 \in \mathscr{K}_\varepsilon^{(p)}$ for large $[m]) = 1$ since $\sum_m (m_1 + \dots + m_q)^{-(1+\delta)q} < \infty$.

Let now $n \in \mathbb{N}^q$. Suppose that $c_{m-e} \leqq n \leqq c_m$. For $t = (t_1, \dots, t_q) \in [0,1]^q$ we define $t_n = (t_n(1), \dots, t_n(q)) \in [0,1]^q$ by setting $t_n(i) = n_i t_i / c_m(i)$ $i = 1, \dots, q$. For any $g \in \mathscr{K}^{(p)}$ and $t \in [0,1]^q$ we write

$$W_n^0(t) - g(t) = (W_{c_m}^0(t_n) - g(t_n)) \frac{a_{c_m}}{a_n} + \left(\frac{a_{c_m}}{a_n} - 1 \right) g(t_n) + (g(t_n) - g(t)) \frac{a_{c_m}}{a_n}.$$

From (8.1) and the Cauchy-Schwarz inequality, observe that for any rectangle $R \subset [0,1]^q$ the increment $g(R)$ satisfies

$$(8.2)\qquad |g(R)|_\infty \leqq l(R).$$

Therefore

$$\begin{aligned} |W_n^0 - g|_\infty &\leqq |W_{c_m}^0 - g|_\infty \, a_{c_m} a_n^{-1} \\ &+ (a_{c_m} a_n^{-1} - 1) |g|_\infty + 2^q \max_{i=1,\dots,q} (1 - n_i / c_m(i)) \, a_{c_m} a_n^{-1} \end{aligned}$$

Thus, by taking c close to 1,

$$P(W_n^0 \in \mathscr{K}_{2\varepsilon}^{(p)} \text{ whenever } [n] \text{ is large}) = 1. \tag{8.3}$$

Therefore, since $\mathscr{K}^{(p)}$ is $|\cdot|_\infty$-compact, with probability 1 the collection $\{W_n^0;\, n \in \mathbb{N}^q\}$ is relatively compact and has its limit points in $\mathscr{K}^{(p)}$ (with respect to the norm $|\cdot|_\infty$). We prove that the limit set of $\{W_n^0;\, n \in \mathbb{N}^q\}$ is almost surely equal to $\mathscr{K}^{(p)}$.

Let μ be a positive integer. We divide the unit cube $[0,1]^q$ into μ^q cubes ι each having sides of length $1/\mu$. Denote by $\mathscr{P}_\mu$ the collection of all such cubes which do not have some face in one of the planes $\{t=(t_1, \ldots, t_q)\colon t_i = 0\}$, $i = 1, \ldots, q$. Let $g \in \mathscr{K}^{(p)}$ with $\int_{[0,1]^q} \langle dg/dl, dg/dl \rangle \, dl = 1-\delta$ for some $\delta > 0$. By (8.2), (8.3) and the separability of $\mathscr{K}^{(p)}$ it suffices to show that infinitely many of the events

$$A_n = \{|W_n^0(\iota) - g(\iota)|_{\mathbb{R}^p} < \varepsilon,\, \iota \in \mathscr{P}_\mu\},\ n \in \mathbb{N}^q$$

occur with probability 1 for each $\mu \geqq 1$ and $\varepsilon > 0$ (cf. the argument of Strassen [15]). As in [15] we estimate that

$$P(A_n) \geqq \text{const}\,(\log [n])^{-(1-\delta)q} (\mu \log\log [n])^{-\mu^q p/2}$$

for $[n]$ sufficiently large. Then, putting $n^{(k)} = (\mu^{k_1}, \ldots, \mu^{k_q})$ for each $k = (k_1, \ldots, k_q) \in \mathbb{N}^q$, we notice that the events $A_{n^{(k)}}$ are independent by our choice of $\mathscr{P}_\mu$. Moreover,

$$\sum_k P(A_{n^{(k)}}) = \infty$$

and

$$\sum_k (k_1 + \ldots + k_q)^{(-1+\delta)q} = \infty.$$

By the Borel-Cantelli lemma we have accomplished what we set out to do.

We have thus proved Theorem 2 with f_n, B and $\mathscr{K}_T$ replaced by W_n^0, $\Pi_N(B)$ and $\mathscr{K}^{(p)}$ respectively. But, if f_n^0 is defined by the R.H.S. of (1.5) with $W^0(m)$ instead of $W(m)$ we get $\lim_n |f_n^0 - W_n^0|_\infty = 0$ a.s. Therefore

$$P(\lim_n \inf_{f \in \mathscr{K}^{(p)}} \|f_n^0 - f\|_{B,\infty} = 0) = 1$$

(8.4) and

$$P(\{f \in C_{\mathbb{R}^p}([0,1]^q)\colon f \text{ is a } |\cdot|_\infty\text{-limit point of } \{f_n^0;\, n \in \mathbb{N}^q\}\} = \mathscr{K}^{(p)}) = 1.$$

We now pass to the general case. Let $\{W(t);\, t \in [0,\infty)^q\}$ be a Brownian sheet in B with covariance function $T(\cdot,\cdot)$. Let $\theta > 0$. From (1.5) and Lemma 4.3 there is $N = N_\theta$ such that

$$P(\overline{\lim_n}\, \|(I - \Pi_N)(f_n)\|_{B,\infty} \leqq \theta) = 1. \tag{8.5}$$

From (1.6), (4.2) and (8.1),

$$\Pi_N(\mathscr{K}_T)=\mathscr{K}^{(p)} \quad \text{with } p=\min(N, \dim H_T). \tag{8.6}$$

Moreover, by (1.6), (4.2) and (4.6),

$$\lim_N \|\Pi_N f - f\|_{B,\infty}=0 \qquad f\in\mathscr{K}_T. \tag{8.7}$$

Therefore, because $\{\Pi_N(W(t));\, t\in[0,\infty)^q\}$ is a Brownian sheet in $\mathbb{R}^p$ with covariance matrix I_p, (8.4)-(8.7) yield Theorem 2. □

To prove the Note following Theorem 2 we calculate by Lemma 2.9, the aforementioned result of Fernique [3] and the Borel-Cantelli lemma that

$$\lim_n \|f_n - W_n\|_{B,\infty}=0 \quad \text{a.s.} \quad \square$$

9. Proof of Theorem 3

Proof of (1.8). Let g_n denote the R.H.S. of (1.5) with S_m in place of $W(m)$. Let $T=T(\cdot,\cdot)$, $\mathscr{K}_T$ and Π_N be as defined in (1.3), (1.6) and (4.2) respectively. We mention that the definitions of H_T and $\mathscr{K}_T$ depend only on T. Observe that for any $N\geqq 1$, $\Pi_N(x)$ satisfies the hypotheses of Theorem 1. Hence, by Theorems 1 and 2 it follows that

$$P(\lim_n \inf_{f\in\mathscr{K}_T} \|\Pi_N(g_n-f)\|_{B,\infty}=0)=1 \tag{9.1}$$

and

$$P(\{f\in C_B([0,1]^q)\colon \Pi_N(f) \text{ is a } \|\cdot\|_{B,\infty}\text{-limit point of } \{\Pi_N(g_n);\, n\in\mathbb{N}^q\}\}=\Pi_N(\mathscr{K}_T))=1. \tag{9.2}$$

Furthermore by Lemma 4.3, (8.5) holds with f_n replaced by g_n. This together with (8.7), (9.1) and (9.2) yields (1.8).

Proof of (1.9). For any $\varepsilon>0$

$$\begin{aligned}\overline{\lim_n}\, P(a_n^{-1}\|S_n\|>\varepsilon) &\leqq \overline{\lim_n}\, P(a_n^{-1}\|\Pi_N(S_n)\|>\varepsilon/2)\\ &\quad+\overline{\lim_n}\, P(a_n^{-1}\|Q_N(S_n)\|>\varepsilon/2)\end{aligned} \tag{9.3}$$

where $Q_N=I-\Pi_N$. But, because $\Pi_N(B)$ is a finite-dimensional space, by Čebyšev's inequality it is clear that

$$\lim_n P(a_n^{-1}\|\Pi_N(S_n)\|>\varepsilon/2)=0,\ N\geqq 1. \tag{9.4}$$

Further by (4.5) and (1.7) (*i*)

$$\lim_{N\to\infty} \overline{\lim_n}\, P(a_n^{-1}\|Q_N(S_n)\|>\varepsilon/2)=0. \tag{9.5}$$

As ε is arbitrary, (9.3)–(9.5) give

$$(9.6)\qquad P\lim_n a_n^{-1} S_n=0.$$

To finish the proof of (1.9) we define x'_n by (3.5) and put $S'_n=\sum_{m\leq n} x'_m$. Then by (2.3) and Lemma 2.4,

$$(9.7)\qquad \overline{\lim_n}\, a_n^{-1} E\,\|S_n-S'_n\|\leqq\overline{\lim_n}\, a_n^{-1}\sum_{m\leqq n} w_m=0.$$

The last equality follows by the argument just preceding (3.6). From (9.6) and (9.7) we get $P\lim_n a_n^{-1} S'_n=0$. We now apply Lemma 2.15 together with the Remark included in its proof to obtain $\lim_n a_n^{-1} E\,\|S'_n\|=0$. This together with (9.7) yields (1.9). □

10. Proof of Theorem 4

Let $\alpha(\cdot)$ be as defined in (2.2) (iv). For each $\delta\geqq 0$, put

$$(10.1)\qquad u_n^{(\delta)}=\int_{|\xi|\geqq(\operatorname{loglog}^+[n])^\delta\alpha([n])} |\xi|\, P\{x\in d\xi\}.$$

The proof of Lemma 2.4 can easily be adjusted to yield the following.

(10.2) **Lemma.** *Let $x\in L^2(\log^{q-1} L)/\operatorname{loglog} L$ for some $q\geqq 2$. Then for each $\delta\geqq 0$ we have $\sum_n(\operatorname{loglog}^+[n])^{-1+\delta} a_n^{-1} u_n^{(\delta)}<\infty$.*

Our next result is a direct analogue of Proposition 3.1.

(10.3) **Proposition.** *Let x be a mean zero random variable taking values in a separable Hilbert space $(H,|\cdot|)$ with $x\in L^2(\log^{q-1} L)/\operatorname{loglog} L$ for some $q\geqq 1$. Then $x\in$BLIL.*

Proof. By Proposition 3.1 this result holds for $q=1$ since, in fact, $\lim_n a_n^{-1} E\,|S_n|=0$ by Čebyšev's inequality. So, we take $q\geqq 2$. Also, as in the proof of Proposition of 3.1 it is enough to assume that x is symmetric, so we do.

The argument below follows along the lines of Wichura [16]. We set:

$$\beta(r)=(\operatorname{loglog}^+ r)^{3/8}\, r^{1/2},\ \gamma(r)=(\operatorname{loglog}^+ r)\,\alpha(r)$$
$$x'_n=x_n\,1_{\{|x_n|\leqq\alpha([n])\}},\ x''_n=x_n\,1_{\{|x_n|\leqq\gamma([n])\}}$$
$$x^*_n=x_n\,1_{\{\alpha([n])<|x_n|\leqq\beta([n])\}},\ x^{**}_n=x''_n-x'_n-x^*_n.$$

$$S'_n=\sum_{m\leqq n} x'_m,\ S''_n=\sum_{m\leqq n} x''_m,\ S^*_n=\sum_{m\leqq n} x^*_m,\ S^{**}_n=S''_n-S'_n-S^*_n.$$

By Lemma 10.2 we have (as in (3.6)) that

$$\lim_n a_n^{-1}\,\|S_n-S''_n\|=0\quad\text{a.s.}$$

Thus, by the argument of Sect. 3 it suffices to have

$$\lim_n a_n^{-1} \|S_n'' - S_n'\| = 0 \quad \text{a.s.} \tag{10.4}$$

To obtain (10.4) we shall in turn show that

$$\lim_n a_n^{-1} \|S_n^*\| = 0 \quad \text{a.s.} \tag{10.5}$$

and

$$\lim_n a_n^{-1} \|S_n^{**}\| = 0 \quad \text{a.s.} \tag{10.6}$$

Write:

$$d_m = (2^{m_1} - 2, \ldots, 2^{m_q} - 2), m = (m_1, \ldots, m_q) \in \mathbb{N}^q$$

(10.7) and

$$\Delta_m = \{n \in \mathbb{N}^q : d_m < n \leqq d_{m+e}\}.$$

Let $\varepsilon > 0$. By the 4th moment form of Čebyšev's inequality,

$$\begin{aligned} & P(|\sum_{n\in\Delta_m} x_n^*| > \varepsilon a_{d_m}) \\ & \leqq (\varepsilon a_{d_m})^{-4} E \sum_{j,k\in\Delta_m} \langle x_j^*, x_k^* \rangle^2 \\ & \leqq (\varepsilon a_{d_m})^{-4} (1 + E|x|^2)(\beta^2([d_{m+e}] + [d_{m+e}]) \sum_{n\in\Delta_m} E|x_n^*|^4) \\ & \leqq \text{const}(\varepsilon)(\text{loglog}^+ [d_m])^{-5/4} [d_m]^{-1} \sum_{n\in\Delta_m} E|x_n^*|^2 \end{aligned} \tag{10.8}$$

Since this bound tends to zero as $[m] \to \infty$ we can apply Lemma 2.9 to the events

$$\{\max_{n\in\Delta_m} |\sum_{k\in\Delta_m, k\leqq n} x_k^*| > \varepsilon a_{d_m}\}, m \in \mathbb{N}^q. \tag{10.9}$$

Thus, by (10.7)–(10.9) and the Borel-Cantelli lemma, to establish (10.5) it is enough that

$$\sum_m \text{loglog}^+ [d_m]^{-5/4} [d_m]^{-1} \sum_{n\in\Delta_m} E|x_n^*|^2 < \infty \tag{10.10}$$

We employ the inequality:

$$E|x|^2 1_{\{a\leqq|x|\leqq b\}} \leqq a^2 P\{|x| > a\} + 2\int_a^b r P(|x| > r)\, dr.$$

For $n \in \Delta_m$ we estimate that

$$\begin{aligned} E|x_n^*|^2 \leqq & \alpha^2([d_{m+e}]) P\{|x| > \alpha([d_m])\} \\ & + 2 \int_{\alpha([d_m])}^{(\text{loglog}^+ [d_m])^{5/8} \alpha([d_{m+e}])} r P(|x| > r)\, dr + \beta([d_{m+e}])\, u_{d_m}^{(5/8)}. \end{aligned}$$

Then, breaking up the above integral and making trivial estimates, by (10.1) and (10.7),

$$E|x_n^*|^2 \leqq \sum_{i=1}^{10} [d_{m+e}] (\log\log [d_{m+e}])^{-1+i/8} P\{|x| > (\log\log^+ [d_m])^{\frac{i-1}{16}}\}$$
$$+ [d_{m+e}] (\log\log^+ [d_m])^{3/8} u_{d_m}^{(5/8)}$$

Hence

$$\sum_m (\log\log^+ [d_m])^{-5/4} [d_m]^{-1} \sum_{n \in \Delta_m} E|x_n^*|^2$$
$$= O\left(\sum_{i=1}^{10} \sum_n a_n^{-1} (\log\log^+ [n])\right) (-1 + \tfrac{i-1}{16}) + (\tfrac{i+1}{16} - \tfrac{3}{4}) u_n(\tfrac{i-1}{16})$$
$$+ O(\sum_n a_n^{-1} (\log\log^+ [n])^{-3/8} u_n^{(5/8)}).$$

Thus, by Lemma 10.2, (10.10) holds and as already stated this yields (10.5).

To establish (10.6), let

$$E_m = \{x_n^{**} \neq 0 \quad \text{for at least two } n\text{'s in } \Delta_m\}.$$

Since $\lim_n a_n^{-1} \gamma([n]) = 0$, by (10.7) and the Borel-Cantelli lemma it is enough to show that $\sum_m P(E_m) < \infty$. Now, by independence and Čebyšev's inequality,

$$P(E_m) \leqq \sum_{n \in \Delta_m} P(x_n^{**} \neq 0) P(x_k^{**} \neq 0 \quad \text{for some } k \neq n, k \in \Delta_m)$$
$$\leqq \sum_{n \in \Delta_m} \beta^{-4}([n]) E|x_n^{**}|^4 \sum_{k \in \Delta_m} \beta^{-2}([k]) E|x|^2$$
$$= O([d_m]^{-1} (\log\log^+ [d_m])^{-5/4} \sum_{n \in \Delta_m} E|x_n^{**}|^2$$

But, we have the bound: $E|x_n^{**}| \leqq \gamma([n]) u_n^{(7/8)}$. Therefore,

$$\sum_m P(E_m) = O(\sum_n a_n^{-1} (\log\log^+ [n])^{-1/4} u_n^{(7/8)}).$$

This last expression is finite by Lemma 10.2. Whence (10.6) holds. □

To finish the proof of Theorem 4 we proceed exactly as in the proof of Theorem 1. Since x takes values in a separable Hilbert space it is well known that x is pregaussian. The analysis of Sects. 4, 6, and 7 therefore goes through with Proposition 10.3 taking the place of Proposition 3.1 if we can show Proposition 5.6 holds under the hypothesis of Theorem 4.

By checking the proof of Proposition 5.6 one sees that it is only necessary to demonstrate the validity of (5.9) (i). Fix a subset $J \subset \{1, \ldots, q\}$. In what follows we drop the dependence on J from our notation. Let X_m and X'_m be as defined in (5.5). Put

$$X_m^* = h_m^{-\frac{1}{2}} \sum_{n \in H_m} (\hat{x}_n 1_{\{\alpha([n]) \leqq |\hat{x}| \leqq \beta([n])\}} - E(\hat{x}_n 1_{\{\alpha([n]) < |\hat{x}_n| \leqq \beta([n])\}})),$$
$$X_m^{**} = h_m^{-\frac{1}{2}} \sum_{n \in H_m} (\hat{x}_n 1_{\{\beta([n]) < |\hat{x}_n| \leqq \gamma([n])\}} - E(\hat{x}_n 1_{\{\beta([n]) < |\hat{x}_n| \leqq \gamma([n])\}}))$$

(10.11) and $$X''_m = X'_m + X_m^* + X_m^{**}$$

Then, using Lemma 10.2,

$$\sum_m P(|X'_m - X''_m| > \tfrac{1}{9}\rho_m) < \infty. \tag{10.12}$$

Further, by the same argument used to establish (10.5),

$$\sum_m P(|X^*_m| > \tfrac{1}{9}\rho_m) < \infty. \tag{10.13}$$

Finally,

$$P(X^{**}_m > \tfrac{1}{9}\rho_m) \leqq P(|\sum_{n\in H_m} \hat{x}_n 1_{\{\beta([n]) < |\hat{x}| \leqq \gamma([n])\}}| > \tfrac{1}{9}\rho_m h_m - \sum_{n\in H_m} E(|\hat{x}_n| 1_{\{\beta([n]) < |\hat{x}_n| \leqq \gamma([n])\}}),$$

while

$$\overline{\lim_m}\, a_{c_m}^{-1} \sum_{n\in H_m} E(|\hat{x}_n| 1_{\{\beta([n]) < |\hat{x}_n| \leqq \gamma([n])\}}) \leqq \lim_m a_{c_m}^{-1}(\beta([c_m]))^{-1} h_m E|\hat{x}|^2 = 0.$$

Hence, in the same way that we verified (10.6),

$$\sum_m P(|X^{**}_m| > \tfrac{1}{9}\rho_m) < \infty. \tag{10.14}$$

Combining (10.11)–(10.14) and Lemma 10.2 we obtain (5.9) (i). □

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Received April 11, 1980; Revised form September 7, 1980.