Z. Wahrscheinlichkeitstheorie verw. Gebiete
57, 235–251 (1981)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Iterated Logarithm Laws with Random Subsequences

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0. Introduction

It is the purpose of this article to prove iterated logarithm laws for random subsequences $S_{T(n)}$ of partial sums S_n of independent random variables and concomittantly to show how these may be exploited to obtain simple proofs of iterated logarithm laws for stopping rules, last times, etc., obtained by Chow and Hsiung (1976), and Chow, Hsiung and Yu (1980). The latter contains many other results not dealt with here. A similar juxtaposition is quite familiar in the case of central limit theorems, where Anscombe's result in 1952 has been readily available.

If, under norming with $k(n)$, $T(n)$ converges almost surely, then random subsequences $S_{T(n)}$ are related to fixed subsequences $S_{k(n)}$, and these, in turn, when the underlying random variables are independent and identically distributed, may be handled via the work of Qualls (1977) on subsequences of Brownian motion. On the other hand, when the basic random variables are merely independent such aid is unavailable and an alternative approach is devised.

The i.i.d. case is treated in Sect. 1, where applications to stopping rules are given. Section 2 is devoted to the independent case where the iterated logarithm law of Teicher (1974) is extended to random subsequences. In Sect. 3, similar results are proved for tail sums, thereby generalizing results of Chow and Teicher (1973).

1. The i.i.d. Case

As has been pointed out by Qualls (1977), in dealing with Brownian motion, if k_n is a nondecreasing positive integer-valued sequence with $k_n \to \infty$, it may be necessary to define a thinner subsequence k'_n via

$$k'_{n+1} = \inf\{k_m : k_m \geqq k'_n \exp(c/\log n)\}, \qquad n \geqq 2, \tag{1.1}$$

* Research supported by NSF Grant MCS-78-09179

0044-3719/81/0057/0235/$03.40

where $k'_1 = k_1$; $k'_2 = k_2$, and c is any positive number. It is not difficult to verify that the convergence or divergence of the series (1.3) below is independent of c, and for this reason c may be chosen at one's convenience.

Theorem 1.1. *Let $\{X, X_n, n \geqq 1\}$ be independent and identically distributed random variables with $EX=0$, $EX^2=1$ and $\{T_n, n \geqq 1\}$ be positive integer-valued random variables with $T_n/k_n \to 1$ a.s. for some nondecreasing sequence k_n of positive integers with $k_n \to \infty$. Then*

$$\limsup_{n\to\infty} \frac{X_1+\ldots+X_{T_n}}{(2T_n \log\log T_n)^{\frac{1}{2}}} = 1 \qquad \text{a.s.,} \tag{1.2}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{(\log k'_n)^{1-\varepsilon}} = \infty, \qquad \textit{for all } \varepsilon > 0, \tag{1.3}$$

where $\{k'_n, n \geqq 1\}$ is the thinned subsequence of (1.1).

Proof. Set $S_n = X_1 + \ldots + X_n$, and let β be an arbitrary number in $(0,1)$. With probability one, for all large n, the random variables T_n will be in $[(1-\beta)k_n, (1+\beta)k_n] = I_n$, say. Hence with probability one

$$\limsup_{n\to\infty} \frac{|S_{T_n} - S_{k_n}|}{(2k_n \log\log k_n)^{\frac{1}{2}}} \leqq \limsup_{n\to\infty} \max_{j\in I_n} \frac{|S_j - S_{k_n}|}{(2k_n \log\log k_n)^{\frac{1}{2}}} \leqq 2\beta^{\frac{1}{2}}, \qquad \text{a.s.}$$

according to a theorem of Lai (1974). Thus, the hypotheses of the theorem entail

$$\lim_{n\to\infty} \frac{|S_{T_n} - S_{k_n}|}{(2k_n \log\log k_n)^{\frac{1}{2}}} = 0 \qquad \text{a.s.} \tag{1.4}$$

By Strassen's strong invariance principle (1964), if $\eta^*(t)$ is the function obtained by linearly interpolating S_n at n, that is,

$$\eta^*(t) = ([t]+1-t)\,S_{[t]} + (t-[t])\,S_{[t]+1},$$

there are a Brownian motion process $\xi(t)$ and a process $\eta(t)$ on a probability space with $\eta(t) \stackrel{\mathscr{L}}{=} \eta^*(t)$, and

$$\lim_{t\to\infty} (2t \log\log t)^{-\frac{1}{2}} \sup_{\tau \leqq t} |\xi(\tau) - \eta(\tau)| = 0 \qquad \text{a.s.}$$

According to Qualls (1977), for any ε in $(0,1)$

$$\begin{aligned} P[S_{k_n} &> (2(1-\varepsilon)\,k_n \log\log k_n)^{\frac{1}{2}}, \text{ i.o.}] \\ &= P[\xi(k_n) > (2(1-\varepsilon)\,k_n \log\log k_n)^{\frac{1}{2}}, \text{ i.o.}] = 1, \end{aligned} \tag{1.5}$$

if and only if

$$\sum_{n=1}^{\infty} (\log\log k'_n)^{-\frac{1}{2}} (\log k'_n)^{\varepsilon-1} = \infty,$$

hence if and only if (1.3) holds. Thus under (1.3), it follows from (1.4), (1.5), the Hartman-Wintner theorem and

$$\frac{S_{T_n}}{(2T_n \log\log T_n)^{\frac{1}{2}}} = \left(\frac{S_{k_n}}{(2k_n \log\log k_n)^{\frac{1}{2}}} + \frac{S_{T_n} - S_{k_n}}{(2k_n \log\log k_n)^{\frac{1}{2}}}\right)\left(\frac{k_n \log\log k_n}{T_n \log\log T_n}\right)^{\frac{1}{2}}, \tag{1.6}$$

that

$$\limsup_{n\to\infty} \frac{S_{T_n}}{(2T_n \log\log T_n)^{\frac{1}{2}}} = 1 \quad \text{a.s.} \tag{1.7}$$

Conversely, convergence of the series in (1.3) for some $\varepsilon > 0$ elicits via (1.5) and (1.6)

$$\limsup_{n\to\infty} \frac{S_{T_n}}{(2T_n \log\log T_n)^{\frac{1}{2}}} \leqq (1-\varepsilon)^{\frac{1}{2}}, \quad \text{a.s.,}$$

and hence the negation of (1.2).

The following corollary, due to Hartman (1941) in the special case where $T_n \equiv k_n$, provides a criterion directly in terms of k_n.

Corollary 1.1. *Let $\{X, X_n, n \geqq 1\}$ be independent and identically distributed random variables with $EX = 0$, $EX^2 = 1$ and $\{T_n,\ n \geqq 1\}$ be positive integer-valued random variables with $T_n/k_n \to 1$ a.s. for some nondecreasing sequence k_n of positive integers. If $k_n \to \infty$ and*

$$k_{n+1} = O(k_n), \tag{1.8}$$

then (1.2) holds.

Proof. Define $n_1 = n_1' = 1$, $n_2 = n_2' = 2$, and for $m \geqq 2$

$$n_{m+1} = \inf\{j > n_m \colon k_j \geqq 2k_{n_m}\},$$
$$n_{m+1}' = \inf\{j > n_m' \colon k_j \geqq k_{n_m'} \exp(\log 2/\log m)\},$$

and note that $k_m' = k_{n_m'}$ is the thinned subsequence of (1.1), and that $k_m' \leqq h_m \equiv k_{n_m}$. Moreover, in view of (1.8), for some M in (h_1, ∞)

$$\frac{h_{m+1}}{h_m} \leqq \frac{M}{2h_m} k_{n_{m+1}-1} \leqq M$$

implying $k_m' \leqq h_m \leqq M^m$ and hence

$$\sum_{m=1}^{\infty} \frac{1}{(\log k_m')^{1-\varepsilon}} = \infty, \quad \text{for all } \varepsilon > 0.$$

Since $T_m'/k_m' \to 1$, a.s. where $T_m' = T_{n_m}$, it follows via (1.6) as in the proof of Theorem 1.1 that

$$\limsup_{n\to\infty} \frac{S_{T_n}}{(2T_n \log\log T_n)^{\frac{1}{2}}} \geqq \limsup_{n\to\infty} \frac{S_{T_m'}}{(2T_m' \log\log T_m')^{\frac{1}{2}}} = 1 \quad \text{a.s.,}$$

and the reverse inequality obtains via the Hartman-Winter theorem.

Next, the prior results will be applied to a positively drifting process U_n and its corresponding first passage times N_λ, T_λ, etc., defined in the paragraphs below. The statistical motivation underlying the consideration of these entities is spelled out in Chow, Hsiung and Yu (1980), and the interested reader is referred to this article.

Throughout the rest of this section $\{X, X_n,\ n \geqq 1\}$ are independent and identically distributed random variables with $EX=1$, $E(X-1)^2=\sigma^2 \in (0,\infty)$, and $\{R_n,\ n\geqq 1\}$ are random variables with $R_n = o(1)$, a.s. as $n\to\infty$. Set

$$S_n=\sum_{i=1}^{n} X_i, \qquad U_n=n^{-1}S_n+R_n, \qquad S'_n=\sum_{i=1}^{n}(X_i-1), \tag{1.9}$$

and let $\{a_n,\ n\geqq 1\}$ be a positive regularly varying sequence with index $\rho>0$. The properties of such sequences are expounded in Bojanic and Seneta (1973). A statement $\overline{\lim}_{n\to\infty} Q_n=\pm\sigma$ is to be interpreted as $\overline{\lim}_{n\to\infty} Q_n=\sigma$ and $\underline{\lim}_{n\to\infty} Q_n=-\sigma$.

Theorem 1.2. *If a_n is eventually nondecreasing and*

$$n^{\frac{1}{2}}(2\log\log n)^{-\frac{1}{2}}R_n=c+o(1), \qquad \text{a.s.,} \tag{1.10}$$

then

$$\overline{\lim_{n\to\infty}}\, n^{\frac{1}{2}}(2\log\log n)^{-\frac{1}{2}}(1-\max_{j\leqq n} a_j U_j/a_n)+c=\pm\sigma \qquad \text{a.s.,} \tag{1.11}$$

$$\overline{\lim_{n\to\infty}}\, n^{\frac{1}{2}}(2\log\log n)^{-\frac{1}{2}}(1-\inf_{j\geqq n} a_j U_j/a_n)+c=\pm\sigma \qquad \text{a.s.,} \tag{1.12}$$

$$\overline{\lim_{n\to\infty}}\, n^{\frac{1}{2}}(2\log\log n)^{-\frac{1}{2}}(1-\sup_{j\geqq n} a_n U_j/a_j)+c=\pm\sigma \qquad \text{a.s.,} \tag{1.13}$$

$$\overline{\lim_{n\to\infty}}\, n^{\frac{1}{2}}(2\log\log n)^{-\frac{1}{2}}(1-\min_{n_0\leqq j\leqq n} a_n U_j/a_j)+c=\pm\sigma \qquad \text{a.s.,} \tag{1.14}$$

where $n_0=\inf\{k\geqq 1:\ U_n>0,$ for all $n\geqq k\}$.

Only (1.11) will be proved since the proofs of the remaining portions of the theorem are similar. It is convenient to first verify

Lemma 1.1. *If for each $n\geqq 1$,*

$$T_n=\inf\{k\geqq 1:\ a_k U_k=\max_{j\leqq n} a_j U_j\},$$

and if (1.10) holds, then as $n\to\infty$

$$T_n/n\to 1 \qquad \text{a.s.,} \tag{1.15}$$

and

$$n^{\frac{1}{2}}(\log\log n)^{-\frac{1}{2}}(1-a(T_n)/a(n))\to 0, \qquad \text{a.s.,}$$

where for notational convenience, a_n is sometimes designated $a(n)$.

Proof. Let $0<\alpha<1$. Since $a(\alpha n)/a(n)\to\alpha^\rho$ and $U_n\to 1$ a.s. and a_n does not decrease eventually, necessarily $\max_{j\leqq \alpha n} a_j U_j/a_n\to\alpha^\rho$ a.s.. If δ is in $(1, \alpha^{-\rho})$, then with probability one, $\max_{1\leqq j\leqq \alpha n} a_j U_j \leqq \delta\alpha^\rho a_n$ for all large n. But with probability one,

$U_n \geqq \delta \alpha^\rho T$ for all large n. Hence with probability one, for all large n

$$\alpha n \leqq T_n \leqq n.$$

As $\alpha \nearrow 1$, it follows that $T_n/n \to 1$, a.s. Next, by the definition of T_n, $a(T_n) U_{T_n} \geqq a(n) U_n$. Hence eventually

$$a(T_n)\frac{S'_{T_n}}{T_n} - a(n)\frac{S'_n}{n} + a(T_n) R_{T_n} - a(n) R_n \geqq a(n) - a(T_n) \geqq 0. \tag{1.16}$$

However, with probability one

$$\begin{aligned}\frac{a(T_n)}{a(n)}\frac{S'_{T_n}}{T_n} - \frac{S'_n}{n} &= \left(1 + +o(1)\right)\frac{S'_{T_n}}{n} - \frac{S'_n}{n} = o(1)\frac{S'_{T_n}}{n} + \frac{S'_{T_n} - S'_n}{n}\\ &= o(n^{-\frac{1}{2}} (\log\log n)^{\frac{1}{2}})\end{aligned}$$

by (1.2) and (1.4). Moreover, by (1.10)

$$\begin{aligned}&n^{\frac{1}{2}}(2\log\log n)^{-\frac{1}{2}} \left(\frac{a(T_n)}{a(n)} R_{T_n} - R_n\right)\\ &\quad = (1+o(1))(c+o(1)) - (c+o(1)) = o(1) \qquad \text{a.s.,}\end{aligned}$$

and so (1.15) follows from (1.16).

Proof of Theorem 1.2. Clearly

$$1 - \max_{j \leqq n} a_j U_j / a_n = 1 - \frac{a(T_n)}{a(n)} \left(\frac{S'_{T_n}}{T_n} + 1 + R_{T_n}\right)$$

and

$$\frac{a(T_n)}{a(n)} R_{T_n} = (c+o(1))(n^{-\frac{1}{2}} (\log\log n)^{-\frac{1}{2}}) \qquad \text{a.s.,}$$

whence (1.11) follows via Corollary 1.1 and Lemma 1.1.

Suppose next that there is a positive, increasing function $a(\cdot)$ on $[o, \infty)$ such that $a(n) = a_n$, $n \geqq 1$, is regularly varying with index $\rho > 0$. Then $a(\lambda x)/a(x) \to \lambda^\rho$ as $x \to \infty$, and one may define, for $\lambda > 0$,

$$a^{-1}(\lambda) = \inf\{x > 0 \colon a(x) = \lambda\}.$$

Furthermore, for $0 < n_\lambda = o(a^{-1}(\lambda))$, define

$$N_\lambda = \inf\{n \geqq n_\lambda \colon a_n U_n \geqq \lambda\}, \quad T_\lambda = \inf\{n \geqq n_\lambda \colon 0 < U_n \leqq \lambda^{-1} a_n\}, \tag{1.17}$$

$$N'_\lambda = \sup\{n \geqq n_\lambda \colon a_n U_n < \lambda\}, T'_\lambda = \sup\{n \geqq n_\lambda \colon U_n > \lambda^{-1} a_n\}, \tag{1.18}$$

$$N''_\lambda = \sum_{n=n_\lambda}^{\infty} I_{[a_n U_n < \lambda]}, \qquad T''_\lambda = \sum_{n=n_\lambda}^{\infty} I_{[U_n > \lambda^{-1} a_n]}. \tag{1.19}$$

Theorem 1.3. (i) *If for each $n \geqq 1$, U_n and $a(\cdot)$ are as specified, then as $\lambda \to \infty$,*

$$N_\lambda / a^{-1}(\lambda) \to 1, \qquad T_\lambda / a^{-1}(\lambda) \to 1, \qquad \text{a.s.,} \tag{1.20}$$

$$\overline{\lim_{\lambda\to\infty}} \frac{S_{N_\lambda} - N_\lambda}{(2N_\lambda \log\log N_\lambda)^{\frac{1}{2}}} = \overline{\lim_{\lambda\to\infty}} \frac{S_T - T_\lambda}{(2T_\lambda \log\log T_\lambda)^{\frac{1}{2}}} = \pm\sigma \qquad \text{a.s.} \tag{1.21}$$

(ii) *Furthermore, if in addition* $a(x)$ *is differentiable with* $\lim_{x\to\infty} \frac{xa'(x)}{a(x)} = \rho$ *and* $n^{\frac{1}{2}}(2\log\log n)^{-\frac{1}{2}} R_n = c + o(1)$ a.s., *then*

$$\overline{\lim_{\lambda\to\infty}} \frac{\rho(N_\lambda - a^{-1}(\lambda))}{(2a^{-1}(\lambda)\log\log\lambda)^{\frac{1}{2}}} + c = \overline{\lim_{\lambda\to\infty}} \frac{\rho(T_\lambda - a^{-1}(\lambda))}{(2a^{-1}(\lambda)\log\log\lambda)^{\frac{1}{2}}} + c = \pm\sigma \quad \text{a.s.} \tag{1.22}$$

(iii) *Moreover* (1.20), (1.21) and (1.22) hold with N_λ, T_λ *replaced by the corresponding random variables of* (1.18) *and* (1.19).

Proof. We shall only prove the results for N_λ since the proofs for T_λ, N'_λ and T'_λ are similar. The results for N''_λ and T''_λ follow in view of the fact that $N_\lambda - 1 \leqq N''_\lambda \leqq N'_\lambda$ and $T_\lambda - 1 \leqq T''_\lambda \leqq T'_\lambda$. By definition,

$$U_{N_\lambda} \geqq \frac{\lambda}{a_{N_\lambda}} > \frac{a_{N_\lambda - 1}}{a_{N_\lambda}} U_{N_\lambda - 1},$$

and so $U_n \to 1$ a.s. and $N_\lambda \to \infty$ a.s. imply that

$$a_{N_\lambda}/\lambda \to 1 \quad \text{a.s.,}$$

which, in turn, ensures the first half of (1.20). Suppose to the contrary that the set $D = \{\overline{\lim_{\lambda\to\infty}} N_\lambda/a^{-1}(\lambda) > 1\}$ had positive probability. Then for $w \in D$, there exists $\delta = \delta(w)$ in $(0,1)$ and $\lambda_k = \lambda_k(w) \nearrow \infty$ with $\delta N_{\lambda_k} \geqq a^{-1}(\lambda_k)$, for all $k \geqq 1$ and hence $a(\delta N_k) \geqq \lambda_k$, $k \geqq 1$. Hence for $w \in D$

$$1 \geqq \overline{\lim_{k\to\infty}} \frac{\lambda_k}{a(\delta N_{\lambda_k})} = \lim_{k\to\infty} \frac{\lambda_k}{a(N_{\lambda_k})} \frac{a(N_{\lambda_k})}{a(\delta N_{\lambda_k})} = \delta^{-\rho} > 1,$$

which is a contradiction. Thus $P[D] = 0$. Likewise the probability of the set $\{\underline{\lim}_{\lambda\to\infty} N_\lambda/a^{-1}(\lambda) < 1\}$ is zero, and the first half of (1.20) follows. Since $a^{-1}(\lambda+1) \sim a^{-1}(\lambda)$, as it can be easily verified, (1.20) and Corollary 1.1 ensure (1.21). Next, we denote N_λ by N. For some $\bar{N}$ between N and $a^{-1}(\lambda)$,

$$\begin{aligned} \lambda \frac{N}{a(N)} - N &= N \frac{a(a^{-1}(\lambda)) - a(N)}{a(N)} = \frac{N}{a(N)} (a^{-1}(\lambda) - N) a'(\bar{N}) \\ &= (a^{-1}(\lambda) - N) \frac{N a(\bar{N})}{\bar{N} a(N)} \frac{\bar{N} a'(\bar{N})}{a(N)} = \rho(1 + o(1))(a^{-1}(\lambda) - N). \end{aligned}$$

Hence, since $S_N - N = (S_N - \lambda N/a(N)) + (\lambda N/a(N) - N)$, it suffices by (1.20) and (1.21) to show that

$$(S_N - \lambda N/a(N))(2N\log\log N)^{-\frac{1}{2}} = -c + o(1) \quad \text{a.s.}$$

To this end, note via the definition of N and the hypothesis of (ii) that

$$S_N - \lambda N/a(N) \geqq -NR_N = -(c + o(1))(2N\log\log N)^{\frac{1}{2}}, \quad \text{a.s.}$$

On the other hand, since

$$S_{N-1}+(N-1)R_{N-1}<\lambda(N-1)/a(N-1),$$
$$S_N-\lambda N/a(N)<(S_N-S_{N-1})-(N-1)R_{N-1}+\lambda(N-1)/a(N-1)-\lambda N/a(N)$$
$$=X_N-(c+o(1))(2N\log\log N)^{\frac{1}{2}}+\lambda N(1/a(N-1)-1/a(N))+o(1)$$
$$=o(N^{\frac{1}{2}})-(c+o(1))(2N\log\log N)^{\frac{1}{2}}+\lambda Na'(\bar{N})/(a(N)\,a(N-1))+o(1)$$
$$=o(N^{\frac{1}{2}})-(c+o(1))(2N\log\log N)^{\frac{1}{2}}+O(\rho)+o(1)$$
$$=-(c+o(1))(2N\log\log N)^{\frac{1}{2}}),\quad \text{a.s.,}$$

completing the proof of the theorem.

Theorems 1.2 and 1.3 have been proved in Chow, Hsiung and Yu (1980) for specific a_n. The argument presented here is somewhat simpler.

Central limit theorems for $\max_{j\leq n} a_j U_j/a_n$, etc., can be proved by the same method as that of Theorem 1.2. It suffices to substitute Anscombe's result (1952) for Theorem 1.1 and to replace the condition on R_n by $n^{\frac{1}{2}}R_n=c+o(1)$ a.s. as $n\to\infty$.

2. The Independent Case

In this section, we shall assume that $\{X_n,\ n\geqq 1\}$ is a sequence of independent random variables with $EX_n=0$, $EX_n^2=\sigma_n^2<\infty$, $S_n=\sum_1^n X_j$, $s_n^2=\sum_1^n\sigma_j^2\to\infty$, $v_n^2=2\log\log s_n^2$, and that $\{k_n,\ n\geqq 1\}$ is a subsequence of the positive integers with $k_n\to\infty$. The law of the iterated logarithm will be proved first for fixed subsequences S_{k_n} of bounded random variables X_n, and then extended to random subsequences. A result of Teicher (1974) will be generalized.

Lemma 2.1. *Assume, for all large n, that there exist constants $c_n=o(v_n^{-1})$ such that for $0<|t|\,c_n\leqq 1$ and $1\leqq j\leqq n$*

$$\exp\left\{(1-|t|\,c_n)\frac{t^2\sigma_j^2}{2s_n^2}\right\}<E\exp\left\{\frac{tX_j}{s_n}\right\}<\exp\left\{\left(1+\frac{|t|\,c_n}{2}\right)\frac{t^2\sigma_j^2}{2s_n^2}\right\},\tag{2.1}$$

and that as $n\to\infty$,

$$s_{k_{n+1}}\sim s_{k_n}.\tag{2.2}$$

Then

$$\limsup_{n\to\infty}\frac{S_{k_n}}{s_{k_n}v_{k_n}}=1\quad \text{a.s.}\tag{2.3}$$

Proof. If $k_n=n$, Lemma 2.1 is due to Tomkins (1972). In general, put $X'_n=X_{k_{n-1}+1}+\ldots+X_{k_n}$ and $(\sigma'_n)^2=\sigma^2_{k_{n-1}+1}+\ldots+\sigma^2_{k_n}$ and $(s'_n)^2=(\sigma'_1)^2+\ldots+(\sigma'_n)^2$ for each $n\geqq 1$, where $k_0=0$. Then (2.1) is satisfied by X'_n and hence Lemma 2.1 follows from Tomkins' result (1972).

Theorem 2.1. *Assume that $|X_n|\leqq M_n$ a.s., where M_n are constants satisfying*

$$M_n=o(s_n/(\log\log s_n)^{\frac{1}{2}}).\tag{2.4}$$

If (2.2) *holds, then*

$$\limsup_{n\to\infty} \frac{S_{k_n}}{s_{k_n} v_{k_n}} = 1 \qquad \text{a.s.}$$

Proof. By Lemma 2.1, it suffices to verify (2.1), which is a consequence of Kolmogorov's proof of the law of the iterated logarithm. See, for example, Chow and Teicher (1978).

Lemma 2.2. *Assume that for all large n, there exist costants* $c_n = o(v_n^{-1})$ *such that for* $0 < |t|\, c_n \leqq 1,\ 1 \leqq j \leqq n$,

$$E \exp\left\{\frac{t X_j}{s_n}\right\} < \exp\left\{\left(1 + \frac{|t|\, c_n}{2}\right) \frac{t^2 \sigma_j^2}{2 s_n^2}\right\}, \tag{2.5}$$

and that for any $\beta > 0$, *there exists an* $\varepsilon > 0$ *such that for all large n*

$$\sum_{p_n+1}^{r_n} \sigma_j^2 \leqq \beta s_{k_n}^2, \qquad \text{where } p_n = [(1-\varepsilon) k_n] \text{ and } r_n = [(1+\varepsilon) k_n]. \tag{2.6}$$

If (2.2) *holds, then there is a subsequence* m_j *of* k_n *such that*

$$\lim_{j\to\infty} \max_{(1-\varepsilon) m_j \leqq i \leqq (1+\varepsilon) m_j} \frac{|S_i - S_{m_j}|}{s_{m_j} v_{m_j}} = 0 \qquad \text{a.s.} \tag{2.7}$$

Proof. For any $c > 1$ and any $j \geqq 1$, define

$$n_j = \inf\{n \geqq 1 : s_{k_n} > c^j\}.$$

Then putting $m_j = k_{n_j}$, we have $s_{m_{j-1}} \leqq c^j < s_{m_j}$, whence $s_{m_j} \sim c^j$. Let $q_j = r_{n_j}$. For $\delta > 0$, by the martingale inequality

$$\begin{aligned} P[&\max_{m_j < i \leqq q_j} (X_{m_j+1} + \ldots + X_i) > \delta s_{m_j} v_{m_j}] \\ &\leqq \exp\{-t \delta v_{m_j} s_{m_j}/s_{q_j}\}\, E \exp\{t(X_{m_j+1} + \ldots + X_{q_j})/s_{q_j}\} \\ &\leqq \exp\left\{-t \delta v_{m_j} s_{m_j}/s_{q_j} + \left(1 + \frac{t c_{q_j}}{2}\right) \frac{t^2}{2 s_{q_j}^2} \sum_{m_j+1}^{q_j} \sigma_i^2\right\}. \end{aligned} \tag{2.8}$$

Put $t = \gamma \delta^{-1} v_{m_j} s_{q_j}/s_{m_j}$. Since $v_{m_j} \sim v_{q_j}$ and $c_{q_j} = o(v_{m_j}^{-1})$ for all large j, then by (2.6), the righthand side of (2.8) is

$$\begin{aligned} &\leqq \exp\left\{-\gamma v_{m_j}^2 + (1 + o(1)) \frac{\gamma^2 v_{m_j}^2}{2\delta^2 s_{m_j}^2} \sum_{m_j+1}^{q_j} \sigma_i^2\right\} \\ &\leqq \exp\left\{-\frac{\gamma}{3} \log\log s_{m_j}^2\right\}, \end{aligned}$$

by choosing $\beta = \delta^2/\gamma$ in (2.6). Hence, taking $\gamma > 3$, we have

$$\sum_{j=1}^{\infty} P[\max_{m_j < i \leqq q_j} (X_{m_j+1} + \ldots + X_i) > \delta s_{m_j} v_{m_j}] < \infty$$

and the Borel-Cantelli Lemma ensures

$$\limsup_{j\to\infty} \max_{m_j<i\leqq q_j} (X_{m_j+1}+\ldots+X_i)/s_{m_j}v_{m_j}=0 \quad \text{a.s.}$$

Replacing X_j by $-X_j$ in the above proof, we can conclude

$$\lim_{j\to\infty} \max_{m_j<i\leqq q_j} |X_{m_j+1}+\ldots+X_i|/s_{m_j}v_{m_j}=0 \quad \text{a.s.} \tag{2.9}$$

Similarly

$$\lim_{j\to\infty} \max_{(1-\varepsilon)m_j<i\leqq m_j} |X_i+\ldots+X_{m_j}|/s_{m_j}v_{m_j}=0 \quad \text{a.s.} \tag{2.10}$$

(2.9) and (2.10) together imply the desired result (2.7).

Theorem 2.2. *Assume that* (2.1), (2.2) *and* (2.6) *hold. If* $\{T_n, n\geqq 1\}$ *is a sequence of positive integer-valued random variables with*

$$\frac{T_n}{k_n}\to 1 \text{ a.s.} \quad \text{as } n\to\infty, \tag{2.11}$$

then

$$\limsup_{n\to\infty}\frac{S_{T_n}}{s_{T_n}v_{T_n}}=\limsup_{n\to\infty}\frac{S_{T_n}}{s_{k_n}v_{k_n}}=1 \quad \text{a.s.} \tag{2.12}$$

Proof. For almost all ω, for all $\varepsilon>0$ and for sufficiently large n

$$(1-\varepsilon)k_n<T_n(\omega)<(1+\varepsilon)k_n. \tag{2.13}$$

Therefore $s_{T_n}^2/s_{k_n}^2\to 1$ a.s. as $n\to\infty$. Hence

$$\limsup_{n\to\infty}\frac{S_{T_n}}{s_{k_n}v_{k_n}}=\limsup_{n\to\infty}\frac{S_{T_n}}{s_{T_n}v_{T_n}}\left(\frac{s_{T_n}v_{T_n}}{s_{k_n}v_{k_n}}\right)$$
$$=\limsup_{n\to\infty}\frac{S_{T_n}}{s_{T_n}v_{T_n}} \quad \text{a.s.}$$

Since $s_{k_{n+1}}\sim s_{k_n}$ implies $s_{n+1}\sim s_n$, by Tomkins' result (1972)

$$\limsup_{n\to\infty}\frac{S_{T_n}}{s_{k_n}v_{k_n}}=\limsup_{n\to\infty}\frac{S_{T_n}}{s_{T_n}v_{T_n}}\leqq 1 \quad \text{a.s.,}$$

and so it suffices to show that

$$\limsup_{n\to\infty}\frac{S_{T_n}}{s_{k_n}v_{k_n}}\geqq 1 \quad \text{a.s.}$$

Let n_j and m_j be defined as in Lemma 2.2, then

$$\frac{S_{T_{n_j}}}{s_{m_j}v_{m_j}}=\frac{S_{m_j}}{s_{m_j}v_{m_j}}+\frac{S_{T_{n_j}}-S_{m_j}}{s_{m_j}v_{m_j}}. \tag{2.14}$$

By Lemma 2.2 and (2.13),

$$\lim_{j\to\infty} \frac{|S_{T_{n_j}} - S_{m_j}|}{s_{m_j} v_{m_j}} = 0 \quad \text{a.s.}$$

For the first term on the right hand side of (2.14), put

$$y_j^2 = s_{m_j}^2 - s_{m_{j-1}}^2, \qquad w_j^2 = 2\log\log y_j^2,$$

and note that $y_j^2 \sim s_{m_j}^2(1-c^{-2})$ and $w_j^2 \sim v_{m_j}^2$. Let $c_j' = c_{m_j} s_{m_j}/y_j$. Then $c_j' = o(1)$ and $0 < t c_j' \leqq 1$ for all large j. Let $u = t s_{m_j}/y_j$, and replace t by u in (2.1), obtaining

$$\exp\left\{(1-u c_{m_j}) \frac{u^2 \sigma_i^2}{2 s_{m_j}^2}\right\} < E \exp\{u X_i / s_{m_j}\}$$

and hence

$$\exp\left\{(1-t c_j') \frac{t^2}{2}\right\} < E \exp\{t(S_{m_j} - S_{m_{j-1}})/y_j\}. \tag{2.15}$$

As in the derivation of Kolmogorov's exponential bound, for $\gamma > 0$, if $b = b(\gamma)$ is sufficiently large, and then $c = c(\gamma)$ is sufficiently small in (2.15), then

$$P[S_{m_j} - S_{m_{j-1}} > b y_j] > \exp\{-(1+\gamma) b^2/2\}. \tag{2.16}$$

Let $0 < \delta < 1$; choose $\gamma = (1-\delta)^{-2} - 1$, $b_j = (1-\delta) w_j$ and note that $c_j' = O(c_{m_j})$, $b_j c_j' = O(c_{m_j} w_j) = O(c_{m_j} v_{m_j}) = o(1)$. Therefore for all large j, and by (2.16)

$$\begin{aligned} P[S_{m_j} - S_{m_{j-1}} > (1-\delta) w_j y_j] &> \exp\{-(1-\delta)^2 w_j^2 (1+\gamma)/2\} \\ &= \exp\{-\log\log y_j^2\} = (\log y_j^2)^{-1} \\ &\simeq (\log s_{m_j}^2 (1-c^{-2}))^{-1} \simeq (2j \log c - \log(1-c^{-2}))^{-1} \end{aligned}$$

implying that

$$\sum_{j=1}^{\infty} P[S_{m_j} - S_{m_{j-1}} > (1-\delta) w_j y_j] = \infty.$$

By the Borel-Cantelli Lemma,

$$P[S_{m_j} - S_{m_{j-1}} > (1-\delta) w_j y_j, \text{ i.o.}] = 1.$$

Hence

$$\limsup_{j\to\infty} \frac{S_{m_j} - S_{m_{j-1}}}{s_{m_j} v_{m_j}} \geqq (1-\delta)(1-c^{-2})^{\frac{1}{2}} \quad \text{a.s.} \tag{2.17}$$

Replacing X_j by $-X_j$ in (2.3), we have

$$\liminf_{n\to\infty} \frac{S_{k_n}}{s_{k_n} v_{k_n}} \geqq -1 \quad \text{a.s.,}$$

A fortiori

$$\liminf_{j\to\infty} \frac{S_{m_{j-1}}}{s_{m_{j-1}} v_{m_{j-1}}} \geqq -1 \quad \text{a.s.,}$$

whence

$$\liminf_{j\to\infty} \frac{S_{m_{j-1}}}{s_{m_j} v_{m_j}} \geqq -\frac{1}{c} \quad \text{a.s.} \tag{2.18}$$

By (2.17) and (2.18), with probability one

$$\limsup_{j\to\infty} \frac{S_{m_j}}{s_{m_j} v_{m_j}} \geqq \limsup_{j\to\infty} \frac{S_{m_j} - S_{m_{j-1}}}{s_{m_j} v_{m_j}} + \liminf_{j\to\infty} \frac{S_{m_{j-1}}}{s_{m_j} v_{m_j}}$$
$$\geqq (1-\delta)(1-c^{-2})^{\frac{1}{2}} - c^{-1}.$$

Choose c large and δ small; then it follows the first term on the righthand side of (2.14) has limsup equal to one; and the result follows immediately.

Corollary 2.1. *Assume* $|X_n| \leqq M_n$ a.s., *where* M_n *are constants satisfying* (2.4) *and assume that* (2.2) *and* (2.6) *hold. If* $\{T_n, n \geqq 1\}$ *is a sequence of positive integer-valued random variables satisfying* (2.11), *then*

$$\limsup_{n\to\infty} \frac{S_{T_n}}{s_{T_n} v_{T_n}} = \limsup_{n\to\infty} \frac{S_{T_n}}{s_{k_n} v_{k_n}} = 1 \quad \text{a.s.}$$

Proof. The result follows directly from Theorem 2.2 and the proof of Theorem 2.1.

Theorem 2.3. *Assume* (2.2), (2.6) *and that* $\{T_n, n \geqq 1\}$ *is a sequence of positive integer-valued random variables satisfying* (2.11). *Assume that for some* $\delta > 0$ *and for all* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[|X_n| > \delta s_n (\log\log s_n^2)^{\frac{1}{2}}] < \infty \tag{2.19}$$

$$\sum_{j=1}^{n} E X_j^2 I_{[X_j^2 > \varepsilon s_j^2/\log\log s_j^2]} = o(s_n^2), \tag{2.20}$$

$$\sum_{n=1}^{\infty} \frac{1}{s_n^2 \log\log s_n^2} E X_n^2 I_{[\varepsilon s_n^2/\log\log s_n^2 \leqq X_n^2 \leqq \delta^2 s_n^2 \log\log s_n^2]} < \infty. \tag{2.21}$$

Then

$$\limsup_{n\to\infty} \frac{S_{T_n}}{s_{T_n} v_{T_n}} = \limsup_{n\to\infty} \frac{S_{T_n}}{s_{k_n} v_{k_n}} = 1 \quad \text{a.s.}$$

Proof. The argument follows the general pattern, but not all the strands of Teicher's Theorem (1974). Define truncation constants $b_n = o(s_n/(\log\log s_n^2)^{\frac{1}{2}})$ exactly as in that theorem and set

$$X_n' = X_n I_{[|X_n| \leqq b_n]},$$
$$X_n''' = X_n I_{[|X_n| > \delta s_n (\log\log s_n^2)^{\frac{1}{2}}]},$$
$$X_n'' = X_n - X_n' - X_n''',$$

and let S_n', S_n'' and S_n''' be the corresponding partial sums. As has been demonstrated in the course of the proof of that theorem

$$S_n'' - ES_n'' = o(s_n \log\log s_n^{2})^{\frac{1}{2}}). \quad \text{a.s.};$$
$$S_n''' - ES_n''' = o(s_n(\log\log s_n^2)^{\frac{1}{2}}), \quad \text{a.s.} \tag{2.22}$$

whence (2.22) likewise holds with n replaced by T_n. On the other hand, Corollary 2.1 ensures that the conclusion holds with S_{T_n} replaced by $S'_{T_n} - ES'_{T_n}$. The conclusion follows directly.

3. Tail Sums of Independent Random Variables

This section treats the random tail sums of independent random variables. Throughout this section we shall assume that $\{X_n,\ n \geqq 1\}$ is a sequence of independent random variables with $EX_n = 0$, $EX_n^2 = \sigma_n^2 > 0$, and $\sum_1^\infty \sigma_n^2 < \infty$. For each $n \geqq 1$, put

$$U_n = \sum_{j=n}^\infty X_j, \quad u_n^2 = \sum_{j=n}^\infty \sigma_j^2 \quad \text{and} \quad v_n = 2\log\log u_n^{-2}.$$

We also assume that $\{k_n,\ n \geqq 1\}$ is a subsequence of the positive integers with $k_n \to \infty$.

Lemma 3.1. *Assume, for all large n and $j \geqq n$, that there exist constants $c_n = o(v_n^{-1})$ such that for $o < |t|\, c_n \leqq 1$*

$$\exp\left\{(1 - |t|\, c_n)\frac{t^2\sigma_j^2}{2u_n^2}\right\} < E\exp\{tX_j/u_n\} < \exp\left\{\left(1 + \frac{|t|\, c_n}{2}\right)\frac{t^2\sigma_j^2}{2u_n^2}\right\}, \tag{3.1}$$

and assume that as $n \to \infty$

$$\sum_{k_n+1}^{k_{n+1}} \sigma_j^2 = o(u_{k_n}^2). \tag{3.2}$$

Then

$$\limsup_{n\to\infty} \frac{U_{k_n}}{u_{k_n} v_{k_n}} = 1 \quad \text{a.s.} \tag{3.3}$$

Proof. If $k_n = n$, then Lemma 3.1 is proved in Chow and Teicher (1973). In general, put $X_n' = X_{k_{n-1}+1} + \ldots + X_{k_n}$ and $(\sigma_n')^2 = \sigma_{k_{n-1}+1}^2 + \ldots + \sigma_{k_n}^2$ and $(u_n')^2 = \sum_{j=n}^\infty (\sigma_j')^2$ for each $n \geqq 1$, where $k_0 = 0$. Then (3.1) is satisfied by X_n' and hence Lemma 3.1 follows from Chow and Teicher (1973).

Theorem 3.1. *Assume* $|X_n| \leqq M_n$ a.s., *where M_n are constants satisfying*

$$c_n \equiv \frac{1}{u_n}\max_{j\geqq n} M_j = o((\log\log u_n^{-}2)^{-\frac{1}{2}}). \tag{3.4}$$

Assume (3.2) *holds. Then*

$$\limsup_{n\to\infty} \frac{U_{k_n}}{u_{k_n} v_{k_n}} = 1 \quad \text{a.s.}$$

Proof. As shown in Chow and Teicher (1973), condition (3.1) is satisfied. Hence the result follows from Lemma 3.1.

Corollary 3.1. *Let $\{Y, Y_n, n \geqq 1\}$ be independent and identically distributed random variables with $EY=0$ and $EY^2=1$ and $\{a_n, n \geqq 1\}$ be constants satisfying*

$$\frac{a_n^2}{\sum_n^\infty a_j^2} \leqq \frac{C}{n}, \quad \sum_1^\infty a_n^2 < \infty \quad \text{for some constant } C, \tag{3.5}$$

and

$$\sum_{k_n+1}^{k_{n+1}} a_j^2 = o\left(\sum_{k_n}^\infty a_j^2\right). \tag{3.6}$$

Then

$$\limsup_{n\to\infty} \frac{\sum_{k_n}^\infty a_j Y_j}{\left(2\sum_{k_n}^\infty a_j^2 \log\log\left(\sum_{k_n}^\infty a_j^2\right)^{-1}\right)^{\frac{1}{2}}} = 1 \quad \text{a.s.} \tag{3.7}$$

Proof. The argument follows that in Chow and Teicher (1973) by applying Theorem 3.1.

Remark. Admissible sequences for a_n are $a_n = \pm n^\beta$, $\beta < -\frac{1}{2}$; $a_n = \pm n^{\beta_1}(\log^n)^{\beta_2}$, $\beta_1 < -\frac{1}{2}$ or $\beta_2 < -\frac{1}{2} = \beta_1$. When $k_n = n$, the above results have been obtained in Chow and Teicher (1973).

Lemma 3.2. *Assume that for all large n and $j \geqq n$, there exist constants $c_n = o(v_n^{-1})$ such that for $0 < |t| c_n \leqq 1$,*

$$E\exp\{tX_j/u_n\} < \exp\left\{\left(1+\frac{|t|c_n}{2}\right)\frac{t^2\sigma_j^2}{2u_n^2}\right\}. \tag{3.8}$$

Assume (3.2) *holds and that for any $\beta > 0$, there exists an $\varepsilon > 0$ such that for all large n*

$$\sum_{p_n+1}^{r_n} \sigma_j^2 \leqq \beta u_{k_n}^2, \quad \text{where } p_n = [(1-\varepsilon)k_n] \text{ and } r_n = [(1+\varepsilon)k_n]. \tag{3.9}$$

Then there is a subsequence m_j of k_n such that

$$\lim_{j\to\infty} \max_{(1-\varepsilon)m_j \leqq i \leqq (1+\varepsilon)m_j} \frac{|U_i - U_{m_j}|}{u_{m_j} v_{m_j}} = 0 \quad \text{a.s.} \tag{3.10}$$

Proof. For any $c > 1$ and any $j \geqq 1$, define

$$n_j = \inf\{n \geqq 1 : u_{k_n} < c^{-j}\}. \tag{3.11}$$

Then putting $m_j = k_{n_j}$, we have $u_{m_j} < c^{-j} \leqq u_{m_{j-1}}$, whence $u_{n_j} \sim c^{-j}$. Letting $r_n = [(1+\varepsilon)k_n]$ and $q_j = r_{n_j}$. For $\delta > 0$, by the martingale inequality and (3.8),

$$\begin{aligned}
&P[\max_{m_j<i\leqq q_j}(X_{m_j+1}+\ldots+X_i)>\delta u_{m_j}v_{m_j}] \\
&\quad\leqq \exp\{-t\delta v_{m_j}\}\,E\exp\{t(X_{m_j+1}+\ldots+X_{q_j})/u_{m_j}\} \\
&\quad\leqq \exp\left\{-t\delta v_{m_j}+\left(1+\frac{tc_{m_j}}{2}\right)\frac{t^2}{2u_{m_j}^2}\sum_{m_j+1}^{q_j}\sigma_i^2\right\}. \qquad (3.12)
\end{aligned}$$

Let $t=\gamma\delta^{-1}v_{m_j}$. Then since $c_{m_j}=o(v_{m_j}^{-1})$, (3.12) becomes

$$\begin{aligned}
&\leqq \exp\left\{-\gamma v_{m_j}^2+(1+o(1))\frac{\gamma^2 v_{m_j}^2}{2\delta^2 u_{m_j}^2}\sum_{m_j+1}^{q_j}\sigma_i^2\right\} \\
&\leqq \exp\left\{-\gamma v_{m_j}^2+\gamma v_{m_j}^2\frac{\gamma\beta}{\delta^2}\right\} \\
&=\exp\left\{-2\gamma\left(1-\frac{\gamma\beta}{\delta^2}\right)\log\log u_{m_j}^{-2}\right\} \\
&=\frac{1}{(\log u_{m_j}^{-2})^{2\gamma(1-\eta)}}\sim\frac{1}{2j^{2\gamma(1-\eta)}(\log c)^{2\gamma(1-\eta)}},
\end{aligned}$$

where $\eta=\gamma\beta/\delta^2$. If γ,η are such that $2\gamma(1-\eta)>1$, then

$$\sum_{j=1}^{\infty}P[\max_{m_j<i\leqq q_j}(X_{m_j+1}+\ldots+X_i)>\delta u_{m_j}v_{m_j}]<\infty.$$

By the Borel-Cantelli Lemma,

$$P[\max_{m_j<i\leqq q_j}(X_{m_j+1}+\ldots+X_i)/u_{m_j}v_{m_j}>\delta,\ \text{i.o.}]=0$$

implying

$$\limsup_{j\to\infty}\max_{m_j<i\leqq q_j}(X_{m_j+1}+\ldots+X_i)/u_{m_j}v_{m_j}=0 \quad \text{a.s.}$$

Replacing X_j by $-X_j$ in the above argument, we get

$$\lim_{j\to\infty}\max_{mj<i\leqq q_j}|X_{m_j+1}+\ldots+X_i|/u_{m_j}v_{m_j}=0 \quad \text{a.s.}$$

Similarly

$$\lim_{j\to\infty}\max_{(1-\varepsilon)m_j\leqq i\leqq m_j}|X_i+\ldots+X_{m_j}|/u_{m_j}v_{m_j}=0 \quad \text{a.s.}$$

Hence the result (3.10) follows immediately.

Theorem 3.2. *Assume that* (3.1), (3.2) *and* (3.9) *hold. If* $\{T_n,\ n\geqq 1\}$ *is a sequence of positive integer-valued random variables such that*

$$\frac{T_n}{k_n}\to 1 \quad \text{a.s.} \quad \text{as} \quad n\to\infty, \qquad (3.13)$$

then

$$\limsup_{n\to\infty}\frac{U_{T_n}}{u_{T_n}v_{T_n}}=\limsup_{n\to\infty}\frac{U_{T_n}}{u_{k_n}v_{k_n}}=1 \quad \text{a.s.} \qquad (3.14)$$

Proof. For almost all ω, for $\varepsilon>0$ and for all sufficiently large n,

$$(1-\varepsilon)k_n < T_n(\omega) < (1+\varepsilon)k_n.$$

Therefore $u_{T_n}^2/u_{k_n}^2 \to 1$ a.s. as $n\to\infty$. Hence

$$\limsup_{n\to\infty} \frac{U_{T_n}}{u_{k_n}v_{k_n}} = \limsup_{n\to\infty} \frac{U_{T_n}}{u_{T_n}v_{T_n}}\left(\frac{u_{T_n}v_{T_n}}{u_{k_n}v_{k_n}}\right) = \limsup_{n\to\infty} \frac{U_{T_n}}{u_{T_n}v_{T_n}} \quad \text{a.s.}$$

Since (3.2) implies $\sigma_n = o(u_n)$ and by Chow and Teicher (1973)

$$\limsup_{n\to\infty} \frac{U_{T_n}}{u_{k_n}v_{kn}} = \limsup_{n\to\infty} \frac{U_{T_n}}{u_{T_n}v_{T_n}} \leqq 1 \quad \text{a.s.,}$$

it suffices to show that

$$\limsup_{n\to\infty} \frac{U_{Tn}}{u_{k_n}v_{k_n}} \geqq 1 \quad \text{a.s.}$$

Define n_j as in Lemma 3.2. Then again setting $m_j = k_{n_j}$,

$$\frac{U_{T_{n_j}}}{u_{m_j}v_{m_j}} = \frac{U_{m_j}}{u_{m_j}v_{m_j}} + \frac{U_{T_{n_j}} - U_{m_j}}{u_{m_j} - v_{m_j}}. \tag{3.15}$$

By Lemma 3.2, the second term on the righthand side of (3.15) will go to zero a.s., and for the first term, put

$$y_j^2 = u_{m_j}^2 - u_{m_{j+1}}^2, \qquad w_j^2 = 2\log\log y_j^{-2},$$

and note that $y_j^2 \sim u_{m_j}^2(1-c^{-2})$ and $w_j^2 \sim v_{m_j}^2$. If $c_j' = \frac{c_{m_j}v_{m_j}}{y_j}$, then $c_j' = o(1)$ and $0 < tc_j' \leqq 1$ for all large j. Set $z = \frac{t u_{m_j}}{y_j}$, and replace t by z in (3.1) obtaining

$$\exp\left\{(1-tc_j')\frac{t^2}{2}\right\} < E\exp\{t(U_{m_j} - U_{m_{j+1}})/y_j\}. \tag{3.16}$$

As in the derivation of Kolmogorov's exponential bound, for $\gamma>0$, if $b=b(\gamma)$ us sufficiently large and then $c=c(\gamma)$ is sufficiently small in (3.16), then

$$P[U_{m_j} - U_{m_{j+1}} > b y_j] > \exp\{-b^2(1+\gamma)/2\}. \tag{3.17}$$

Let $0<\delta<1$; choose $\gamma = (1-\delta)^{-2} - 1$, $b_j = (1-\delta)w_j$ and note that $c_j' = O(c_{m_j})$, $b_j c_j' = O(c_{m_j} w_j) = O(c_{m_j} v_{m_j}) = o(1)$; therefore for large j and by (3.17)

$$\begin{aligned} P[U_{m_j} - U_{m_{j+1}} > b_j y_j] &> \exp\{-(1-\delta)^2 w_j^2(1+\gamma)/2\} \\ &= \exp\{-\log\log y_j^{-2}\} \sim (\log v_{m_j}^{-2}(1-c^{-2})^{-1})^{-1} \\ &\sim (\log c^{2j} - \log(1-c^{-2}))^{-1} - (2j\log c - \log(1-c^{-2}))^{-1} \end{aligned}$$

implying

$$\sum_{j=1}^{\infty} P[U_{m_j} - U_{m_{j+1}} > b_j y_j] = \infty,$$

and by the Borel-Cantelli Lemma,

$$P[U_{m_j} - U_{m_{j+1}} > (1-\delta) w_j y_j, \text{ i.o.}] = 1.$$

Hence

$$\limsup_{j\to\infty} \frac{U_{m_j} - U_{m_{j+1}}}{u_{m_j} v_{m_j}} \geqq (1-\delta)(1-c^{-2})^{\frac{1}{2}} \quad \text{a.s.} \tag{3.18}$$

Replacing X_j by $-X_j$ in (3.1) and by Chow and Teicher (1973)

$$\liminf_{n\to\infty} \frac{U_{k_n}}{u_{k_n} v_{k_n}} \geqq -1 \quad \text{a.s.}$$

A fortiori

$$\liminf_{j\to\infty} \frac{U_{m_{j+1}}}{c^{-1} u_{m_j} v_{m_j}} = \liminf_{j\to\infty} \frac{U_{m_{j+1}}}{u_{m_{j+1}} v_{m_{j+1}}} \geqq -1 \quad \text{a.s.,}$$

whence

$$\liminf_{j\to\infty} \frac{U_{m_{j+1}}}{u_{m_j} v_{m_j}} \geqq -\frac{1}{c} \quad \text{a.s.} \tag{3.19}$$

By (3.18) and (3.19), with probability one

$$\limsup_{j\to\infty} \frac{U_{m_j}}{u_{m_j} v_{m_j}} \geqq \limsup_{j\to\infty} \frac{U_{m_j} - U_{m_{j+1}}}{u_{m_j} v_{m_j}} + \liminf_{j\to\infty} \frac{U_{m_{j+1}}}{u_{m_j} v_{m_j}}$$
$$\geqq (1-\delta)(1-c^{-2})^{\frac{1}{2}} - c^{-1}.$$

Choose c large and δ small; then

$$\limsup_{j\to\infty} \frac{U_{m_j}}{u_{m_j} v_{m_j}} \geqq 1 \quad \text{a.s.,}$$

and the result follows.

Corollary 3.2. *Assume that* $|X_n| \leqq M_n$ *a.s. where* M_n *are constant satisfying* (3.4) *and that* (3.2) *and* (3.9) *hold. If* $\{T_n, n \geqq 1\}$ *is a sequence of positive integer-valued random variables satisfying* (3.13), *then*

$$\limsup_{n\to\infty} \frac{U_{T_n}}{u_{T_n} v_{T_n}} = \limsup_{n\to\infty} \frac{U_{T_n}}{u_{k_n} v_{k_n}} = 1 \quad \text{a.s.}$$

Proof. The result follows directly from Theorem 3.2 and the proof of Theorem 3.1.

Corollary 3.3. *Let* $\{Y, Y_n, n \geqq 1\}$ *be independent and identically distributed random variables* $EY=0$ *and* $EY^2=1$ *and* $\{a_n, n \geqq 1\}$ *be constants satisfying* (3.5) *and* (3.6). *Assume that for any* $\beta > 0$, *there exists an* $\varepsilon > 0$ *such that for all large* n

$$\sum_{p_n+1}^{r_n} a_j^2 \leqq \beta \sum_{k_n}^{\infty} a_j^2, \quad \text{where } p_n = [(1-\varepsilon)k_n],\ r_n = [(1+\varepsilon)k_n]. \tag{3.20}$$

If $\{T_n, n \geqq 1\}$ is a sequence of positive integer-valued random variables satisfying (3.13), *then*

$$\limsup_{n\to\infty} \frac{\sum_{T_n}^{\infty} a_j Y_j}{\left(2\sum_{T_n}^{\infty} a_j^2 \log\log\left(\sum_{T_n}^{\infty} a_i^2\right)^{-1}\right)^{\frac{1}{2}}}$$

$$= \limsup_{n\to\infty} \frac{\sum_{T_n}^{\infty} a_j Y_j}{\left(2\sum_{k_n}^{\infty} a_j^2 \log\log\left(\sum_{k_n}^{\infty} a_i^2\right)^{-1}\right)^{\frac{1}{2}}} = 1 \quad \text{a.s.}$$

Proof. The argument follows that in Chow and Teicher (1973). By Corollary 3.1, Theorem 3.2 and Lemma 3.2, the result follows.

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Received March 20, 1980

Note Added in Proof.

Research of H. Teicher was supported by NSF Grant MCS-80-05481.

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