Z. Wahrscheinlichkeitstheorie verw. Geb. 9, 16—19 (1967)

Note on Resolvents of Denumerable Submarkovian Processes

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Received April 17, 1967

1. It is well known that, if $\{R(\lambda), \lambda > 0\}$ is a family of matrices $R(\lambda) = (r_{ij}(\lambda);$ $i, j = 0, 1, 2, \ldots)$ such that

$$r_{ij}(\lambda) \geqq 0, \quad \sum_j \lambda r_{ij}(\lambda) \leqq 1, \tag{1}$$

$$R(\lambda) - R(\mu) + (\lambda - \mu) R(\lambda) R(\mu) = 0, \tag{2}$$

$$\lambda r_{ij}(\lambda) \to \delta_{ij} \quad \text{as} \quad \lambda \to \infty, \tag{3}$$

then $R(\lambda)$ is the resolvent for a submarkovian semigroup $\{P(t), t > 0\}$. That is, there exist functions p_{ij}, continuous on $(0, \infty)$, such that r_{ij} is the Laplace transform

$$r_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)\, dt$$

of p_{ij}, and such that $P(t) = (p_{ij}(t))$ satisfies

$$p_{ij}(t) \geqq 0, \quad \sum_j p_{ij}(t) \leqq 1, \tag{1*}$$

$$P(s+t) = P(s) P(t), \tag{2*}$$

$$p_{ij}(t) \to \delta_{ij} \quad \text{as} \quad t \to 0+. \tag{3*}$$

The proof (for which see e.g. [4]) comes by applying the Hille-Yosida theorem to $R(\lambda)$, acting as an operator on the Banach-space l by

$$(x R(\lambda))_j = \sum_i x_i r_{ij}(\lambda), \quad x \in l, \tag{4}$$

and condition (3) is needed to make the range of $R(\lambda)$ dense in l.

We shall show that $P(t)$ can still be obtained from $R(\lambda)$ if one drops (3) for $R(\lambda)$, of course at the expense of losing (3*) for $P(t)$. The proof will use direct real variable arguments because use of the Hille-Yosida theorem becomes inconvenient when the range of $R(\lambda)$ is not known to be dense, and because in any case such direct arguments may be interesting.

2. We now prove the following:

Theorem. (i) *Given $R(\lambda)$ satisfying* (1) *and* (2), *$r_{ij}(\lambda)$ is the Laplace transform of $p_{ij}(t)$, continuous on* $(0, \infty)$ *and such that $P(t) = (p_{ij}(t))$ satisfies* (1*) *and* (2*).

(ii) *The limits $\omega_{ij} = \lim_{t \to 0+} p_{ij}(t)$ exist, and $\lambda r_{ij}(\lambda) \to \omega_{ij}$ as $\lambda \to \infty$.*

Corollary. *If $R(\lambda)$ also satisfies* (3), *then $P(t)$ also satisfies* (3*).

Proof. We start from the familiar observation that (1) shows $R(\lambda)$, defined at (4), to be a positive linear operator on l with norm $\leqq \lambda^{-1}$, and that (2) then implies

$$\left(\frac{-d}{d\lambda}\right)^n R(\lambda) = n!\,(R(\lambda))^{n+1} \tag{5}$$

in the uniform operator topology. Hence

$$0 \leqq \left(-\frac{d}{d\lambda}\right)^n r_{ij}(\lambda) \leqq n!/\lambda^{n+1}, \tag{6}$$

and also

$$0 \leqq \left(-\frac{d}{d\lambda}\right)^n \sum_j r_{ij}(\lambda) \leqq n!/\lambda^{n+1}. \tag{7}$$

Using the theory of completely monotonic functions (for which see e.g. [3; 415—418]), we deduce from (6) that $r_{ij}(\lambda)$ is the Laplace transform of a measurable function $f_{ij}(t)$ such that

$$0 \leqq f_{ij}(t) \leqq 1, \quad t > 0, \tag{8}$$

and from (7) that

$$0 \leqq \sum_j f_{ij}(t) \leqq 1, \quad t > 0; \tag{9}$$

note that since r_{ij} determines f_{ij} only up to a null set, we can avoid exceptional null sets in (8) or (9).

Next, we show that

$$f_{ij}(s+t) = \sum_k f_{ik}(s)\,f_{kj}(t) \tag{10}$$

except perhaps on a measurable (s, t) — set of plane measure zero. This holds because the double Laplace transforms

$$\int_0^\infty \int_0^\infty \cdots e^{-\lambda s - \mu t}\,ds\,dt$$

of the two sides in equation (10) are

$$(\mu - \lambda)^{-1}(r_{ij}(\lambda) - r_{ij}(\mu)), \quad \sum_k r_{ik}(\lambda)\,r_{kj}(\mu)$$

for $\lambda \neq \mu$. These are equal by (2), and we can also get equality for $\lambda = \mu$ by letting $\lambda \to \mu$. The uniqueness theorem for double Laplace transforms shows that (10) holds a.e. (s, t).

Our problem now is simply how to alter the $f_{ij}(t)$ on a null set in such a way that (8) and (9) continue to hold, (10) holds for all $s > 0$, $t > 0$, and the new version $p_{ij}(t)$ of $f_{ij}(t)$ is continuous on $(0, \infty)$. Define, for $t > 0$,

$$p_{ij}(t) = t^{-1} \sum_k \int_0^t f_{ik}(u)\,f_{kj}(t-u)\,du \tag{11}$$

$$= t^{-1} \int_0^t \left(\sum_k f_{ik}(u)\,f_{kj}(t-u)\right) du. \tag{12}$$

Since (10) holds a.e. (s, t), it follows that for almost all $t > 0$ the integrand in (12) equals $f_{ij}(t)$ for almost all $t > 0$ the integrand in (12) equals $f_{ij}(t)$ for almost all $u \in (0, t)$, and therefore

$$p_{ij}(t) = f_{ij}(t), \quad \text{a.e.} \tag{13}$$

Next, (11) tells us that

$$p_{ij}(t) = t^{-1} \sum_k g_k(t), \tag{14}$$

where g_k is the convolution of the two bounded measurable functions f_{ik} and f_{kj} and is therefore continuous. Also the series $\sum_k g_k(t)$ is dominated by

$$\sum_k \int_0^t f_{ik}(u)\, du\,;$$

the terms of this series are continuous, and its sum

$$\int_0^t \Big(\sum_k f_{ik}(u)\Big)\, du$$

is continuous (see (9)), so that by Dini's theorem the series converges uniformly on any interval $[0, T]$. Hence $\sum_k g_k(t)$ is also uniformly convergent, and it follows that

$$p_{ij}(t) \text{ is continuous for } t > 0. \tag{15}$$

We can now deduce from (8), (13) and (15) that

$$0 \leqq p_{ij}(t) \leqq 1 \quad \text{for all} \quad t > 0, \tag{16}$$

and from (9) and (13) that

$$\sum_j p_{ij}(t) \leqq 1 \tag{17}$$

for almost all $t > 0$; but since p_{ij} is continuous the sum in (17) is lower semi-continuous, so that (17) holds for all $t > 0$. (Of course we shall see in the end that the sum in (17) is continuous).

We have now shown that p_{ij} is continuous and satisfies (1*), and continue the argument by deriving (2*) from (10). This amounts to removing the exceptional (s, t) — set in the statement

$$p_{ij}(s+t) = \sum_k p_{ik}(s)\, p_{kj}(t), \quad \text{a.e.} \quad (s, t), \tag{18}$$

obtained from (10) and (13). Since the left side is continuous for $s > 0$, $t > 0$, it will be enough to show that the right side is continuous in every set

$$\{a \leqq s \leqq b, \quad a \leqq t \leqq b\}, \quad 0 < a < b.$$

Because $0 \leqq p_{kj}(t) \leqq 1$ it will be enough to show that $\sum_k p_{ik}(s)$ is uniformly convergent on $[a, b]$, and this will follow from Dini's theorem if we can show $\sum_k p_{ik}(s)$ to be continuous. Now, from (12),

$$\begin{aligned}\sum_j p_{ij}(s) &= s^{-1} \int_0^s \Big(\sum_k \sum_j f_{ik}(u)\, f_{kj}(s-u)\Big)\, du \\ &= s^{-1} \sum_k \int_0^s f_{ik}(u) \Big(\sum_j f_{kj}(s-u)\Big)\, du.\end{aligned}$$

The series $\sum_k$ here is dominated, because of (9), by

$$\sum_k \int_0^s f_{ik}(u)\, du\,,$$

and we saw earlier that this series converges uniformly on any $[0, T]$. The uniform convergence of $\sum_j p_{ij}(s)$ on $[a, b]$ follows.

We have now proved (i) of the Theorem. As for (ii), the existence of

$$\omega_{ij} = \lim_{t \to 0+} p_{ij}(t)$$

is known ([2]; see also [1; II. 2.3]), and the fact that $\lambda r_{ij}(\lambda) \to \omega_{ij}$ as $\lambda \to \infty$ is a well-known abelian property for Laplace transforms.

3. The reader may well consider whether it is possible to deduce the Theorem from the Hille-Yosida theorem, even though the range of $R(\lambda)$ may not be dense. It seems to be possible to construct such a proof 'by hindsight' by converting known facts about the matrix $P(t)$, given in [1; Th. II. 2.3] into statements about $R(\lambda)$, which have then to be derived directly from (1)—(3). The argument, using the notation of [1], would begin by an examination of the range of $R(\lambda)$, leading to an identification of the classes $F, I, J, \ldots$ and the quantities u_j, followed by the construction of a resolvent with dense range on the state space $\{I, J, \ldots\}$. This would lead, via the Hille-Yosida theorem, to a transition matrix $(\Pi_{IJ}(t))$, and a further argument should lead to the functions $\Pi_{iJ}(t)$ $(i \in F)$. Although I have not gone through all details of the argument, I have little doubt that it would be quite as elaborate as the arguments in §2.

References

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