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On the Construction Problem for Markov Chains. II

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This note is the result of a tidying-up operation on the author's paper [*4*], the notation and terminology of which will be used without more ado. We shall deal here only with the construction of strictly stochastic transition matrices though the substochastic case may be dealt with by a trivial modification.

The argument in [*4*] consisted of three parts. First it was shown that a stochastic transition function $P = \{P(t) : t \geqq 0\}$, whose Q-matrix $\underline{Q}$ satisfied Assumptions A and B, defined, by formulae stated explicitly, matrix functions $\eta = \eta(P)$ and $T = T(P)$ satisfying the conditions set out in Theorem 3 of [*4*]. Next, it was shown that *any* matrix functions η and $\underline{T}$ satisfying these conditions defined, again by formulae stated explicitly, a strictly stochastic transition function $P = P(\eta, T)$ with Q-matrix $\underline{Q}$. Lastly, probabilistic interpretations were given for the matrices η and $\underline{T}$ and for other matrices associated with them.

The 'analytic' section of the paper therefore presented a 'necessary and sufficient' construction. What was unsatisfactory was the following situation.

Starting with $\underline{Q}$ satisfying Assumptions A and B, one could choose η and $\underline{T}$ suitably and then construct from them the transition function $P = P(\eta, T)$. If one then calculated $\bar{\eta} = \eta(P)$ and $\bar{T} = T(P)$, one found that, in general, $\bar{\eta} \neq \eta$ and $\bar{T} \neq T$. (The example following Theorem 18.5 of CHUNG [*2*] illustrates this.) There is no contradiction here. All that was asserted in [*4*] was the true result:

$$P = P(\eta, T) = P(\bar{\eta}, \bar{T}).$$

Let $\{X(t)\}$ be a Markov chain on $\underline{E}$ with transition matrix $\{P(t)\}$ where $P'(0) = Q$ and $\underline{Q}$ satisfies Assumptions A and B. Then (see CHUNG [*1*]) $X(t)$ will at certain times reach points of the exit boundary. An exit boundary point $\underline{b}$ may be '*stable*' in that definite time intervals elapse between successive visits to $\underline{b}$. On the other hand, $\underline{b}$ may be '*instantaneous*' in that infinitely many visits to $\underline{b}$ occur in a finite time interval.

It is at first sight surprising that the law of succession of the exit boundary states qua limit points is in general different from the law of succession of states in B under the process $\{Z(t)\}$ of § 4 of [*4*]. However, there is no great mystery here, only a 'Principle of Superposition of States'. Let us examine this in a trivial case.

Suppose that one started with the substochastic transition function on two states $\underline{i}$ and $\underline{j}$ with Q-matrix

$$\begin{array}{cc} & \begin{array}{cc} i & \;\; j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \end{array}.$$

One could build this up to a strictly stochastic transition function as follows: Adjoin two 'boundary' states $\underset{\sim}{c}$ and $\underset{\sim}{d}$ and set up the Q-matrix

$$\begin{array}{c} \\ i \\ j \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} i & j & c & d \end{array} \\ \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{pmatrix} \end{array}. \tag{1}$$

Let $\{Z(t)\}$ be a Markov chain with Q-matrix as at (1) and let $\{X(t)\}$ be the process on $\underset{\sim}{i}$ and $\underset{\sim}{j}$ obtained by deleting the states $\underset{\sim}{c}$ and $\underset{\sim}{d}$ from $\{Z(t)\}$. Then $\{X(t)\}$ has Q-matrix

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

We could however obtain a process with the same Q-matrix as $\{X(t)\}$ by adjoining 'boundary' states $\underset{\sim}{a}$ and $\underset{\sim}{b}$, forming instead of the matrix at (1), the matrix

$$\begin{array}{c} \\ i \\ j \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cccc} i & j & a & b \end{array} \\ \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{pmatrix} \end{array}$$

and then deleting $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

What is important is that, in an obvious sense, $c = a$, $d = \frac{1}{2}(a + b)$. It is this type of stochastic lumping together of *stable* boundary states which occurs when $\bar{\eta} \neq \eta$ and $\bar{T} \neq T$. The point is that making a boundary state real involves coupling together an exit law and an entrance law. An *instantaneous* boundary state is the natural fusion of an exit law with a definite matching entrance law. (See the author's paper [4].) A *stable* exit law may be coupled with more or less arbitrary combinations of entrance laws.

It *is* true that there is a '*canonical*' process extended to the boundary. See the discussion in § 16 of Chung [2]. We shall approach the problem differently.

Suppose that $\underset{\sim}{Q}$ satisfies Assumptions A and B. If $\underset{\sim}{P}$ is a strictly stochastic transition function with $P'(0) = Q$, let us call $\underset{\sim}{P}$ *non-redundant* if the vectors $y^a(\lambda)$ are all different and extremal in the sense of § 3.4 of [4]. This implies that $G = I$ and that $U(\lambda) = V(\lambda)$.

For non-redundant transition matrices, Theorem 3 of [4] may be replaced by the following more precise result.

Theorem. *Let E be a countable set and let Q be an $E \times E$ matrix satisfying Assumptions* A *and* B. *Define the minimal resolvent $\Phi(\lambda)$ as in Lemma* 5 *of* [4] *and the vector x^0 as in Lemma* 6 *of* [4].

Let x^b $(b \in B)$ be the extremal (sojourn) non-negative solutions of $Qx = 0$ such that

$$\sum_{b \in B} x^b = 1 - x^0 .$$

For $\lambda > 0$ define $x^b(\lambda) = x^b - \lambda \Phi(\lambda) x^b \geqq 0$ $(b \in B)$. For each b in B choose an

l valued function $\eta^b(\cdot)$ on $(0, \infty)$ satisfying $\langle \eta^b(1), 1\rangle = 1$,

$$\eta^b(\mu) = \eta^b(\lambda) A(\lambda, \mu) \qquad (\lambda, \mu > 0)$$

(see Lemma 7 of [4]) and also

$$\lim_{\lambda\to\infty} \langle \lambda \eta^b(\lambda), 1 - x^b\rangle < \infty \qquad (b \in B)\,.$$

Define

$$U^{ab}(\lambda) = \langle \lambda \eta^a(\lambda), x^b\rangle \qquad (a, b \in B;\, \lambda > 0)\,.$$

Next choose a matrix T on $B \times B$ such that

$$\begin{array}{llllll} -T^{ab} = U^{ab}(\infty) & \text{when} & a \neq b & \text{and} & U^{bb}(\infty) & \text{is finite,} \\ -T^{ab} \geqq U^{ab}(\infty) & \text{when} & a \neq b & \text{and} & U^{bb}(\infty) & \text{is infinite,} \end{array}$$

$$\sum_{c\in B} T^{ac} = \tau^a \qquad (a \in B)\,.$$

Then the matrix $M(\lambda) = [U(\lambda) + T]^{-1}$ exists and is non-negative. Now set

$$r_{ij}(\lambda) = \Phi_{ij}(\lambda) + \sum_{a\in B} \sum_{b\in B} x_i^a(\lambda) M^{ab}(\lambda) \eta_j^b(\lambda)\,.$$

Then $R(\lambda) = \{r_{ij}(\lambda)\}$ is the resolvent of a strictly stochastic transition function $\{P(t)\}$ with $P'(0) = Q$. Every non-redundant such $\{P(t)\}$ may be constructed in the above manner.

With the notation used above, we now have

$$\overleftarrow{\eta} = \eta,\ \overleftarrow{T} = T\,.$$

Assuming T chosen as just described, let $\{Z(t)\}$ be the Markov chain described in § 4.2 of [4]. Then the law of succession of states in B under $\{Z(t)\}$ is identical to the law of succession of boundary points qua limit points of paths traversed by the Markov chain $\{Z^E(t)\}$. We recall that $\{Z^E(t)\}$ has transition matrix $\{P(t)\}$ and that the law of succession of states in B under $\{Z(t)\}$ is determined by the ratios T^{ab}/T^{aa}, $(a \neq b)$.

Proof of the Theorem. Suppose $\underline{P}$ given with $P'(0) = Q$, $\underline{Q}$ satisfying Assumptions A and B. The matrix $T(P)$ is the (subsequential) limit of the matrix $T(\mu)$ where

$$T^{ac}(\mu) = \frac{\delta^{ac} - \sum_b M^{ab}(\mu)\, U^{bc}(\mu)}{\sum_t M^{at}(\mu)}\,.$$

(See equation (48) of [4].)

If $U^{cc}(\infty)$ is finite, in which case $U^{bc}(\infty)$ is finite for all $\underline{b}$ then, in a suitable subsequence,

$$-T^{ac}(\mu) = \sum H^{ab}(\mu)\, U^{bc}(\mu) \to \sum H^{ab}\, U^{bc}(\infty)$$

with $H^{ab}(\mu)$ as in equation (39) of [4] and $H^{ab} = \lim H^{ab}(\mu)$. From equation (43) of [4], $H\,U(\mu) = U(\mu)$ for all μ and hence if $U^{cc}(\infty)$ is finite we have

$$-T^{ac} = \sum H^{ab}\, U^{bc}(\infty) = U^{ac}(\infty) \qquad (a \neq c)$$

as required.

Conversely, suppose that $\underline{T}$ has been chosen so that $-T^{ac} = U^{ac}(\infty)$ when $U^{cc}(\infty)$ is finite and that the transition function $\{P(t)\}$ has been constructed as explained in the theorem. Then since

$$M(\mu)[T + U(\mu)] = I,$$

we have for $a \neq c$,

$$\sum_b M^{ab}(\mu)[T + U(\mu)]^{bc} = 0.$$

If $U^{cc}(\infty)$ is finite, this implies that

$$0 = \sum H^{ab}[T + U(\infty)]^{bc} = H^{ac}[T^{cc} + U^{cc}(\infty)].$$

Hence $H^{ac} = 0$.

If $U^{cc}(\infty) = \infty$, then if $a \neq c$,

$$-\sum_{b \neq c} H^{ab}(\mu)[T + U(\mu)]^{bc} = H^{ac}(\mu)[T^{cc} + U^{cc}(\mu)].$$

Let μ tend to infinity to yield

$$-\sum_{b \neq c} H^{ab}[T + U(\mu)]^{bc} = H^{ac}[T^{cc} + U^{cc}(\infty)]$$

so that, again, $H^{ac} = 0$. Hence $H = I$.

We now show that $\bar{\eta}^a(\lambda) = \eta^a(\lambda)$ for $\underline{a}$ in B. From equation (33) of [4],

$$\bar{\eta}^a(\lambda) = \lim \frac{y^a(\mu)\, A(\mu, \lambda)}{\| y^a(\mu)\, A(\mu, 1)\|} = \lim \frac{\sum_b M^{ab}(\mu)\, \eta^b(\lambda)}{\sum_t M^{at}(\mu)} =$$

$$= \lim \sum_b H^{ab}(\mu)\eta^b(\lambda) = \sum_b H^{ab}\eta^b(\lambda) = \eta^a(\lambda).$$

Since $M(\mu)\, T = I - M(\mu)\, U(\mu)$,

$$\bar{T}^{ac}(\mu) = \frac{\delta^{ac} - [M(\mu)\, U(\mu)]^{ac}}{\sum M^{at}(\mu)} = \frac{\sum_b M^{ab}(\mu)\, T^{bc}}{\sum M^{at}(\mu)} =$$

$$= [H(\mu)\, T]^{ac} \to T^{ac}.$$

We have shown that, under the new conditions,

$$M^{ab}(\infty) = 0, \qquad (a \neq b).$$

Consulting Theorems 4 and 5 of [4], we see that if $\{Z(t)\}$ starts in $\underline{a}$, the time it spends in $b\,(\neq a)$ before entering the set $\underline{E}$ is zero with probability one. This shows that the law of succession of states in $\underline{B}$ is the correct one.

Our restriction to non-redundant $\{P(t)\}$ is not important. In the general situation one needs to define the 'boundary' in terms of the non-extremal vectors z^b of Theorem 3 of [4] and to replace x's by z's throughout the whole theory.

Bibliography

[1] Chung, K. L.: On the boundary theory for Markov chains. Acta math. **110**, 19–77 (1963).
[2] — On the boundary theory for Markov chains, II. Acta math. **115**, 111–163 (1966).
[3] Williams, D.: The process extended to the boundary. Z. Wahrscheinlichkeitstheorie verw. Gebiete **2**, 332–339 (1962).
[4] — On the construction problem for Markov chains. Z. Wahrscheinlichkeitstheorie verw. Gebiete **3**, 227–246 (1964).

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