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The Survival of Branching Annihilating Random Walk

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Summary. Branching annihilating random walk is an interacting particle system on $\mathbb{Z}$. As time evolves, particles execute random walks and branch, and disappear when they meet other particles. It is shown here that starting from a finite number of particles, the system will survive with positive probability if the random walk rate is low enough relative to the branching rate, but will die out with probability one if the random walk rate is high. Since the branching annihilating random walk is non-attractive, standard techniques usually employed for interacting particle systems are not applicable. Instead, a modification of a contour argument by Gray and Griffeath is used.

1. Introduction

In this paper we will study an interacting particle system, called the *branching annihilating random walk.* This model is a Markov process with transition rates which are determined by a parameter. We will show that, depending on the choice of parameter value, two different types of long-term behavior are possible, which we call *extinction* and *survival.* Finding, classifying and understanding models with these kinds of behavior is one of the major problems in interacting particle systems (see Griffeath [3], Liggett [4], or Stroock [5]). Our model is one of the first such examples which fails to have a certain monotonicity property known as *attractiveness* (see below).

The model is easy to describe. The state space is the set of finite subsets of $\mathbb{Z}$. We will use the notation ξ_t for the state at time t. This state is thought of as the set of sites in $\mathbb{Z}$ which are *occupied* by particles at time t. The set ξ_t^c is the set of *vacant* sites at time t. Each particle can do two different things. After an exponential holding time with mean 1, a particle will give birth to a new particle at one of the two neighboring sites. This is called *branching* and the two neighboring sites are equally likely to receive the offspring of a particle. Also, after an exponential time with mean $1/\rho$, where $\rho > 0$ is a parameter, a particle moves to one of the two neighboring sites. This is called *jumping* (or random walking), and the particle is equally likely to jump to the left or the right. Thus, the term "branching random walk." The branching and jumping actions of each particle are independent of one another, and all particles act independently except when two particles attempt to occupy the same site. If a particle lands on a site which is already occupied, either by jumping there or as the result of branching, then both particles disappear, i.e., are *annihilated.* Thus, if two particles are at neighboring sites, it is possible for both sites to be vacated simultaneously (if one particle jumps on the other), or for only one site to be vacated (if one particle has an offspring which jumps to the site of the other particle). Fig. 1 in the next section shows one possible realization of this process and the different types of movement which can occur[1].

The main question that we wish to study is: are there values of the parameter ρ for which the process has a positive probability of survival, that is, such that

$$P(\xi_t \neq \emptyset \text{ for } all\ t \geqq 0) > 0$$

for some non-empty (finite) initial state? We will show that there is survival for all sufficiently small $\rho > 0$, and we will also show that the probability of survival is 0 for large ρ.

The annihilating feature of the process (ξ_t) makes it different in an important way from nearly all models for which the survival question has been answered. (The centered long-range contact process in Bramson and Gray [1] is another example.) For many models, the presence of more particles increases the lifetime of the process in a strong sense. Such models are called *attractive.* More precisely, an attractive model is one in which the addition of an extra occupied site to the state of the process does not decrease the exponential rate at which any of the vacant sites become occupied, nor does it increase the rate at which the other occupied sites are vacated. The process (ξ_t) studied here is not attractive – the addition of a new particle at a vacant site may in time reduce rather than increase the total population.

Previous methods for proving survival have relied heavily on the monotonicity properties possessed by attractive systems. However, by considerably modifying the technique used in Gray and Griffeath [2], we can show survival for small ρ for the branching annihilating random walk. While it would have been preferable to invent an entirely new technique (desperately needed for non-attractive systems), we feel that this line of reasoning provides some hope for future progress. Moreover, it can also be applied to the problem in [2] to provide a much shorter, more accessible solution.

2. Comparison to Sums of i.i.d. Random Variables

We will show here that for small values of ρ, there is a tendency toward growth at the ends of the set ξ_t. As a result, the width of ξ_t will tend to

[1] Our interest in branching annihilating random walk was kindled by an address by Ulam [6], in which much more general multitype branching random walks involving annihilation and coalescence of different types of particles were mentioned in conjunction with elementary particle physics

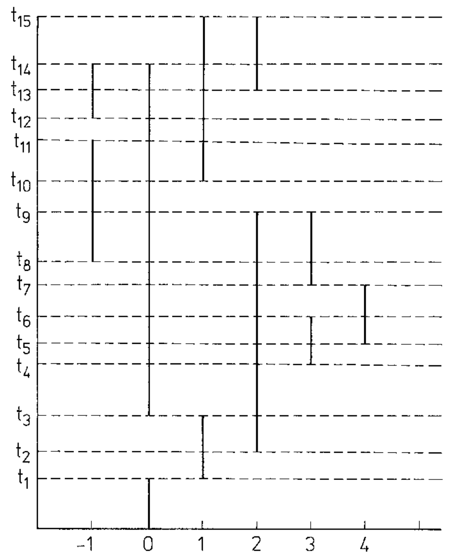

Fig. 1

increase, which favors survival of (ξ_t). This tendency holds up as long as the position of an end does not change too much at one time. We deal with the problem of large changes in the next two sections.

To illustrate various ideas, we provide a picture of a realization of the process (ξ_t). The initial state is the singleton $\{0\}$, and on this particular sample path, the process dies out after 15 transitions. The figure is drawn on a space-time graph with the time exis running upward. The vertical lines represent occupied sites in space-time. The times $t_1, t_2, \ldots, t_{15}$ are transition times.

In this figure, jumps occur at times t_1, t_3, t_7, t_9, t_{14} and t_{15}. The last three of these are with annihilation. Branchings occur at the other times, two of which (at times t_6 and t_{11}) occur with annihilation.

We will concentrate on the *width* of ξ_t, that is, the distance from the leftmost particle to the rightmost particle plus 1. We first note that when this width is 1 (when ξ_t is a singleton), the process cannot die out before the width increases to width 2. Furthermore, when ξ_t is a single particle, it may jump around some (as it does at time t_1 in the picture), but it must eventually become a pair (as at time t_2). Next, we see that the width can increase by at most 1 unit at a time, and that this can happen in two different ways, through a jump or through branching, illustrated by the transitions at times t_3 and t_{13}, among others. The width can decrease by 1 unit, also in two different ways – see times t_7 and t_{11}. Finally, the width can decrease by 2 or more units, and this always happens in the same way, via a jumping annihilation, as at times t_9, t_{14} and t_{15}.

To further analyze the behavior of the ends, we define a process which is essentially the width of ξ_t: let

$$
\begin{aligned}
W_t &= \textit{the width of } \xi_t \text{ if } \xi_t \textit{ is not a singleton or the empty set}\\
&= 2 \qquad \text{if } \xi_t \textit{ is a singleton}\\
&= 0 \qquad \text{if } \xi_t = \emptyset.
\end{aligned}
$$

We have defined W_t to be 2 when the width of ξ_t is 1 to simplify the statements of results below. Also, recall the comments above concerning ξ_t when it is a singleton.

If we ignore increments in the width that are larger than 2 units, there is a sense in which W_t is dominated from below by a random walk with positive mean. To make this precise, let $\tau_1, \tau_2, \ldots$ be the transition times of the process (W_t). (If there are only n times, let $\tau_{n+1} = \tau_{n+2} = \ldots = \infty$.) Let

$$N = \min\{n \in \mathbb{Z} \colon W_{\tau_n} = 0 \quad or \quad W_{\tau_n} - W_{\tau_{n-1}} < -2\}.$$

Define i.i.d. random variables $Y_1, Y_2, Y_3, \ldots$ such that

$$
\begin{aligned}
P(Y_n = -2) &= 32/1600\\
P(Y_n = -1) &= 751/1600\\
P(Y_n = 1) &= 817/1600.
\end{aligned}
$$

Also assume that the Y_n's are independent of the process (ξ_t), and let

$$
\begin{aligned}
X_n &= W_{\tau_n} - W_{\tau_{n-1}} \quad \text{if } n < N.\\
&= Y_n \qquad\qquad\quad \text{if } n \geqq N.
\end{aligned}
$$

We may now state the main result of the section.

Proposition. *If* $\rho < 1/100$, *then for all* $m \in \mathbb{Z}$,

$$P(X_1 + X_2 + \ldots + X_n \leqq m) \leqq P(Y_1 + Y_2 + \ldots + Y_n \leqq m).$$

Proof. Since the X_n's can only take on the values 1, -1, -2, it is enough to prove that

$$P(X_n = i \mid X_1, X_2, \ldots, X_{n-1}) \leqq P(Y_n = i) \tag{1}$$

for $i = -1, -2$ and $\rho < 1/100$. We will instead prove that for all such ρ

$$P(X_n = -2 \mid \xi_{\tau_{n-1}}, n < N) \leqq 2\rho \tag{2a}$$

and

$$P(X_n = -1 \mid \xi_{\tau_{n-1}}, n < N) \leqq 15/32 + \rho/16. \tag{2b}$$

Since $X_n = Y_n$ for $n \geqq N$, the strong Markov property and the definition of Y_n imply that (1) follows from (2).

Demonstration of (2a). First note that the conditional probabilities in (2) are zero if $W_{\tau_{n-1}} = 2$, since $n < N$ and since the process (W_t) never takes the value 1. So we can assume that $W_{\tau_{n-1}} > 2$. To prove (2a), we determine the maximum rate at which W_t can decrease by 2 and the minimum rate at which it can increase by 1. A decrease by 2 can only be due to a jumping annihilation at either end of ξ_t, so the maximum rate is 2ρ. The minimum rate of increase is $1+\rho$, due to branching and random walk at either end. It follows that

$$P(X_n=-2\mid\xi_{\tau_{n-1}},\, n<N)\leqq 2\rho/(1+3\rho)<2\rho.$$

Demonstration of (2b). *Transition rates which vary with the state.* We will prove (2b) in somewhat the same fashion that we proved (2a), although a little more care is now needed. As before, we will consider the rates at which various changes in W_t occur. However, we no longer can ignore the way in which these rates depend on the state of the process. These rates depend in a crucial way on whether the sites next to the ends are occupied.

If the site next to an end is occupied, then W_t will change by -1 if the particle next to the end branches onto the end particle. (See time t_{11} in Fig. 1.) Otherwise, W_t can change by -1 due to a jump toward the middle of the occupied set by the end particle. (See time t_7 in Fig. 1.) In the first case, the rate is 1/2 for each end to which the case applies, in the second it is $\rho/2$. As always, the rate at which W_t increases is $1/2+\rho/2$ at each end.

We will find it useful to do some further conditioning. Let

ℓ = *the position of the left end of* $\xi_{\tau_{n-1}}$
r = *the position of the right end of* $\xi_{\tau_{n-1}}$
E = *the event that the change that occurs at time* τ_n *occurs at the left end.*

Applying the strong Markov property and the rates given above, we see that

$$P(X_n=-1\mid\xi_{\tau_{n-1}},\, n<N, E, \textit{ and the site } \ell+1 \textit{ is occupied at time } \tau_n^-)\leqq\frac{1}{2+\rho}<\frac{1}{2}$$

and

$$P(X_n=-1\mid\xi_{\tau_{n-1}},\, n<N, E, \textit{ and the site } \ell+1 \textit{ is vacant at time } \tau_n^-)\leqq\frac{\rho}{1+2\rho}<\rho.$$

It therefore follows that

$$\begin{aligned}P(X_n=-1\mid\xi_{\tau_{n-1}},\, n<N, E)&\\ <\tfrac{1}{2}P(\ell+1 \textit{ is occupied at time } \tau_n^-\mid\xi_{\tau_{n-1}},\, n<N, E)&\\ +\rho P(\ell+1 \textit{ is vacant at time } \tau_n^-\mid\xi_{\tau_{n-1}},\, n<N, E).&\end{aligned}\tag{3}$$

Conclusion. We need to bound the first conditional probability on the right side of (3) away from 1. We will show

$$P(\ell+1 \textit{ is vacant at time } \tau_n^-\mid\xi_{\tau_{n-1}},\, n<N, E)>1/16.\tag{4}$$

Application of (4) to (3) then shows that

$$P(X_n=-1\mid\xi_{\tau_{n-1}},\, n<N, E)<\frac{15}{32}+\frac{\rho}{16}.$$

A similar computation works for the right end of $\xi_{\tau_{n-1}}$ and so (2b) follows. To prove (4), we start with the case in which $\xi_{\tau_{n-1}}$ contains the site $\ell+1$. We compute the probability that the particle at $\ell+1$ is first annihilated (before time τ_n) and then the particle at ℓ branches to the left or jumps before the site $\ell+1$ is reoccupied. The rate at which the particle at $\ell+1$ is annihilated is at least 1/2 (due to branching of the left end particle). The total rate at which the left end position changes is no more than $3\rho/2+1$, so the probability for the first step in the above procedure is at least $1/(3\rho+3)>4/15$ when $\rho<1/4$. For the second step, the rate at which the left end particle branches to the left or jumps is $1/2+\rho$, while the site $\ell+1$ becomes reoccupied (by branching from either side or a jump from the site $\ell+2$) at a rate no larger than $1+\rho/2$. So the conditional probability for the second step (given the first step) is at least $(1+2\rho)/(3+3\rho)>4/15$ when $\rho<1/4$. Multiplying the two probabilities one obtains the lower bound $(4/15)^2>1/16$. The computation for the case where $\xi_{\tau_{n-1}}$ does not contain the site $\ell+1$ is the same as in the second step just computed above, and so one obtains the lower bound $4/15>1/16$ that $\ell+1$ is vacant at time τ_{n-1}^- in this case as well. Together, these two bounds imply (4), which completes the proof of the proposition. $\square$

If it were not possible for the increments of (W_t) to be less than -2, Proposition 1 would imply survival of (ξ_t) for $\rho<1/100$ since the width would then grow faster than a random walk with positive mean. In the next two sections, we will develop the techniques needed for dealing with the greater changes in (W_t).

3. Tree Sets

We start by defining a "thickened up" version of the occupation set in space-time: let

$$T=\{(x,t)\in\mathbb{R}\times[0,\infty): |x-y|\leqq 1/2 \textit{ for some } y\in\xi_t\}.$$

We call this set the *tree set* of the process (ξ_t) because it shares certain features with trees from graph theory, as we shall see. In Fig. 2, we have drawn the set T for the same realization used for Fig. 1. The shaded region is T.

Certain crucial times related to the tree set are labeled on the right side of the figure. The time labeled "1" (*time* t_3) is called a *separation time*. The times labeled 2 and 3 (*times* t_9 and t_{15}) are called *terminal times*. The terminal times are those times at which either the width of ξ_t decreases by more than 2 or the process dies out. The separation times can be described rigorously in terms of the shape of the outside boundary of T. They occur wherever the outside boundary of T contains a U-shaped piece formed by one horizontal and two vertical line segments (the horizontal segment is labelled H in Fig. 2).

In Fig. 2, T is connected and bounded. The event that T is bounded is almost surely equivalent to the event that the process (ξ_t) dies out. The connectedness of T is dependent partly on the initial state. If the initial state is a set containing no vacant sites between the two ends, then T is always connected. Since any non-empty state can be reached from any other non-empty state, we have the following useful criterion:

$$P(\xi_t=\emptyset \text{ for } \textit{some } t\geqq 0)<1 \tag{5a}$$

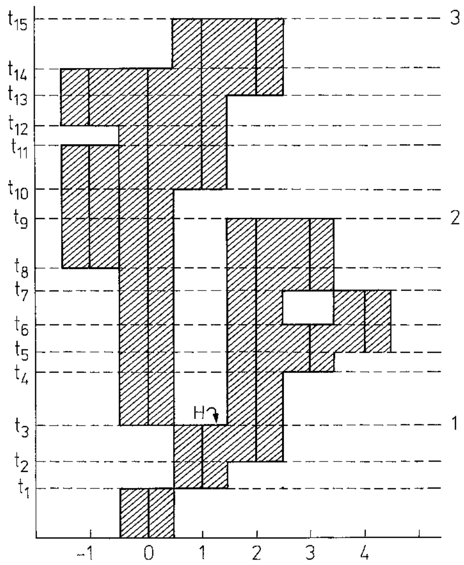

Fig. 2

for all non-empty initial states iff

$$P(T \textit{ is bounded} \text{ and } \textit{connected}) < 1 \tag{5b}$$

4. Survival

In this section, we prove that branching annihilating random walk survives with positive probability for a sufficiently low random walk rate.

Theorem 1. *For sufficiently small* $\rho > 0$ *and all non-empty initial states,*

$$P(\xi_t = \emptyset \text{ for } \textit{some } t \geqq 0) < 1.$$

Proof. We will make use of the equivalence in (5). It is convenient to abbreviate (5b) by letting

$$A = \{T \textit{ is bounded} \text{ and } \textit{connected}\}.$$

An important quantity in our proof will be

$$K = \textit{the number of separation times.}$$

If T is bounded, K is almost surely finite. We will prove by induction on $k = 0, 1, 2, \ldots$ that for sufficiently small ρ there exist positive constants C and R, with $R < 1$, such that for all initial states of width ℓ (i.e., $\ell = \max \xi_0 - \min \xi_0 + 1$),

$$P(A \text{ and } K \leqq k) \leqq \frac{C \rho R^{\ell}}{\ell^2}. \tag{6}$$

Since the right side of (6) is independent of k, we see upon letting $k \to \infty$ that (5b) follows from (6).

The case $K=0$.

We first recall that

$$W_{\tau_n} = \ell + X_1 + X_2 + \dots + X_n \quad \text{for } n < N, \tag{7}$$

where τ_n and N are defined in Sect. 2. For $K=0$ there are no separation times. If the tree is connected, it is easy to check that its width therefore never decreases by more than 2 units. Consequently, on A, the number of times that the process (W_t) hits the value 2 is bounded by the random variable

$$M = \textit{the number of integers } n \textit{ such that}$$
$$X_1 + X_2 + \dots + X_n = -\ell + 2$$

if $K=0$.

The event that the process (ξ_t) dies out implies that there is some time at which the width becomes 2 units and then the two remaining particles disappear due to a jumping annihilation. Each time the process reaches a state with width 2, the probability that such an annihilation occurs before the width changes to 3 is $\rho/(2\rho+1)$, as can be computed directly from the rates. One concludes from the strong Markov property that

$$P(A \text{ and } K=0) \leqq \rho EM/(2\rho+1) < \rho EM. \tag{8}$$

Now, recall that $EY_n > 0$ for Y_n defined as in Proposition 1. It therefore follows from Proposition 1 and a simple large deviation estimate on $P(Y_1 + \dots + Y_n \leqq -\ell + 2)$ that there exist positive constants $\bar{C}$ and $\bar{R}$, with $\bar{R} < 1$, such that

$$EM \leqq \bar{C}\bar{R}^{\ell-2}$$

whenever $0 < \rho < 1/100$. Set $R = \sqrt{\bar{R}}$ and

$$C = (2\bar{C}/R^4)(\sup_{\ell \geqq 1} R^{\ell}\ell^2).$$

Because of (8),

$$P(A \text{ and } K=0) \leqq C\rho R^{\ell}/2\ell^2 \tag{9}$$

which implies (6) for $k=0$. (We will need the extra factor of 2 in the denominator.)

The Inductive Step

The Basic Idea. We will let C and R be as in the previous paragraph, and assume that (6) holds for fixed $\rho > 0$ and for $0 \leqq k < m$, where $m > 0$. We will show that under this assumption, (6) holds for $k=m$ for ρ chosen small enough (where the bound on ρ does not depend on m). By (9), it is sufficient to prove

$$P(A \text{ and } 1 \leqq K \leqq m) < C\rho R^{\ell}/2\ell^2. \tag{10}$$

Our approach will be basically the following. If $1 \leqq K \leqq m$, there is a smallest separation time, which we will call σ. At time σ, a gap occurs in the set of occupied sites that splits the population of particles into two subpopulations which remain separated after time σ by at least one vacant site. In Fig. 2, this gap has length 1 and appears at site 1 at time t_3. In general, the gap may have length 1 or 2, and it may appear at any site or pair of adjacent sites between the two end particles.

Thus, after time σ, the process (ξ_t) can be conveniently viewed as two separate processes which start on either side of the gap, with σ playing the role of the initial time for these processes. With this point of view, we can define tree sets T_1 and T_2 for these two "sub-processes" in the same fashion as before. It is not hard to see that if T is bounded and connected, then so are T_1 and T_2; in fact, they are the closures of the two connected components of the set $T \cap \{t > \sigma\}$. These new tree sets have a total of $K-1$ separation times, so if $K \leqq m$, they each have strictly less than m branch times. We will be able to apply the inductive hypothesis to the two subprocesses. In doing so, we must avoid the temptation to use the strong Markov property to restart the process at time σ, since σ is not Markovian.

Terminology. We will say that ξ_t has an i-gap if

$$\min \xi_t + i \notin \xi_t$$

$$\min \xi_t + i - 1 \in \xi_t$$

and either

$$\min \xi_t + i + 1 \in \xi_t$$

or

$$\min \xi_t + i + 2 \in \xi_t .$$

Thus an i-gap is a gap of 1 or 2 vacant sites that starts i sites from the left end of ξ_t. Any transition that produces an i-gap will be called an *i-transition*. Note that an i-transition occurs at time σ for some unique i strictly between 0 and W_σ. We will say that the process is *permanently separated* by an i-gap at time s if ξ_s has an i-gap and if the part of the process which starts to the left of the i-gap at time s is separated from the part of the process which starts to the right of the i-gap at time s by at least one vacant site for all times $t \geqq s$. If T is bounded and connected and $K \geqq 1$, then the process is permanently separated by an i-gap at time σ, where i is the (unique) integer such that σ is the time of an i-transition. This permanant separation is the separation into two subprocesses described in the previous paragraph. Note that the shifted process $(\tilde{\xi}_t) = (\xi_{\sigma+t})$, $t \geqq 0$, is permanently separated by an i-gap at time 0 for the same value of i, and that the tree set of $(\tilde{\xi}_t)$ has exactly two bounded components and strictly less than m separation times when the tree set of the original process is connected and has at most m separation times. We define

$A_i = \{$the process is permanently separated by an i-gap at time 0 and T has exactly two bounded components$\}$.

Executing the Basic Idea. Let $\sigma_1^i, \sigma_2^i, \sigma_3^i, \ldots$ be the successive times at which i-transitions occur. Then by the strong Markov property and the fact that $\sigma < \tau_N$

when $K \geqq 1$ (see Sect. 2 for the definition of τ_N),

$$P(A \text{ and } 1 \leqq K \leqq m) = \sum_{j=3}^{\infty} \sum_{i=1}^{j-1} \sum_{n=1}^{\infty} P(A, 1 \leqq K \leqq m, \sigma = \sigma_n^i \text{ and } W_\sigma = j)$$

$$\leqq \sum_{j=3}^{\infty} \sum_{i=1}^{j-1} \sum_{n=1}^{\infty} \sum_{\xi} P_{\xi_0}(\sigma_n^i < \tau_N, W_{\sigma_n^i} = j, \xi_{\sigma_n^i} = \xi) P_\xi(A_i \text{ and } 0 \leqq K < m). \tag{11}$$

We have been careful to indicate initial states by subscripts in the last term. The innermost summation is taken over states ξ that have an i-gap and that satisfy $\max \xi - \min \xi + 1 = j$. For such ξ, we will now estimate $P_\xi(A_i$ and $0 \leqq K < m)$.

Let

ξ_t^1 = *particles in* ξ_t *descended from particles on the left of the i-gap in the initial state* ξ.
ξ_t^2 = *particles in* ξ_t *descended from the remaining particles in* ξ.
T_1 = *the tree set of* (ξ_t^1).
T_2 = *the tree set of* (ξ_t^2).

If the process is permanently separated by an i-gap at time 0, then $T_1 \cap T_2 = \emptyset$. Under the restriction that $T_1 \cap T_2 = \emptyset$, the two processes (ξ_t^1) and (ξ_t^2) remain separated by at least one vacant site at all times, so they have the same probability law on this restricted part of the probability space as two independent branching annihilating random walks under the corresponding restriction (i.e., that they remain separated, or equivalently, that their tree sets are disjoint). It follows from the inductive hypothesis that

$$P_\xi(A_i \text{ and } 0 \leqq K < m) \leqq \left(\frac{C\rho R^i}{i^2}\right)\left(\frac{C\rho R^{j-i-2}}{((j-i-2) \vee 1)^2}\right) \leqq \frac{9C^2\rho^2 R^{j-2}}{i^2(j-i)^2}. \tag{12}$$

We have used the facts that T_1 and T_2 each have strictly less than m separation times if T does, and that the initial states ξ_0^1 and ξ_0^2 have "widths" (in the sense of the process (W_t) defined earlier) of at least i and $(j-i-2) \vee 1$ respectively. If we plug (12) into (11) and sum over ξ, we obtain

$$P(A \text{ and } 1 \leqq K \leqq m) \leqq \sum_{j=3}^{\infty} \sum_{i=1}^{j-1} \frac{(9C^2\rho^2 R^{j-2})}{i^2(j-i)^2} \sum_{n=1}^{\infty} P(\sigma_n^i < \tau_N, W_{\sigma_n^i} = j). \tag{13}$$

Estimating (13). We first estimate the innermost sum in (13). Since a site can be vacated at a maximum rate $1+2\rho$, this is bounded above by $1+2\rho$ times the expected Lebesque measure of the set of times $t < \tau_N$ such that $W_t = j$. The rate at which W_t changes is always more than 1, so this measure is bounded by the expected number of times the process (W_t) visits j before time τ_N. As in the derivation of (9), we can reason that for $t < \tau_N$, (W_t) can be compared with a process of sums of i.i.d. random variables with bounded increments and positive mean, so that this expected value is bounded above by

$$\begin{array}{ll} \bar{C}\bar{R}^{i-j} & \text{if } j<\ell \\ \bar{C} & \text{if } j\geqq\ell, \end{array}$$

where $\ell=W_0$ and $\bar{C}$ and $\bar{R}$ are chosen as before. If we substitute this bound in for the bracketed quantity in (13) and recall the relationships between $\bar{C}$, $\bar{R}$, C and R, we obtain the following:

$$P(A \text{ and } 1\leqq K\leqq m)\leqq\frac{9C^3\rho^2(1+2\rho)}{R^2} \cdot\left(\sum_{j=3}^{\ell-1}\sum_{i=1}^{j-1}\frac{R^\ell}{i^2(j-i)^2(\ell-j)^2}+\sum_{j=\ell}^{\infty}\sum_{i=1}^{j-1}\frac{R^j}{i^2(j-i)^2}\right). \tag{14}$$

Since

$$\sum_{i=1}^{j-1}\frac{1}{i^2(j-i)^2}<\frac{16}{j^2},$$

the quantity in (14) is less than

$$\frac{144\,C^3\rho^2(1+2\rho)}{R^2}\left(\sum_{j=1}^{\ell-1}\frac{R^\ell}{j^2(\ell-j)^2}+\sum_{j=\ell}^{\infty}\frac{R^j}{j^2}\right)$$

which is less than

$$M\rho^2R^\ell/\ell^2$$

for some constant M which does not depend on $\rho\leqq 1$, ℓ or m. If we choose $\rho<C/2M\wedge 1/100$, then (10) follows, which completes the proof of the inductive step, and hence of the theorem. □

5. Extinction

In this section we show that branching annihilating random walk dies out with probability one for a high enough random walk rate.

Theorem 2. *For sufficiently large ρ and any finite initial state ξ_0,*

$$P(\xi_t=\emptyset \text{ for some } t\geqq 0)=1. \tag{15}$$

We will find it convenient to introduce the processes (ξ_t^x) and $({}^c\xi_t^x)$ associated with (ξ_t) so that for $x\in\xi_0$,

$\xi_t^x=$ those particles in ξ_t which are descended from the particle at x at time 0,
${}^c\xi_t^x=$ those particles in ξ_t which are descended from particles not at x at time 0.

Of course, $\xi_t^x\cup{}^c\xi_t^x=\xi_t$. To demonstrate Theorem 2, it is enough for us to demonstrate the following proposition.

Proposition 2. *For sufficiently large ρ and for $x\in\xi_0$,*

$$E|\xi_4^x|<1/2. \tag{16}$$

Theorem 2 follows from Proposition 2, since summing (16) over $x \in \xi_0$ implies that

$$E|\xi_4| < \tfrac{1}{2}|\xi_0|,$$

and therefore by induction, that

$$E|\xi_{4n}| < \frac{1}{2^n}|\xi_0|.$$

This quantity $\to 0$ as $n \to \infty$, which implies (15).

Proof of Proposition 2. We first introduce the following notation. Let

σ = time at which (ξ_t^x) first branches or becomes extinct,

τ = time at which (ξ_t^x) branches again ($= \infty$ if (ξ_t^x) becomes extinct first),

A_1 = event that $\sigma \leqq 4 - \varepsilon$,

A_2 = event that $\tau > \sigma + \varepsilon$,

A_3 = event that no particle from (ξ_t^x) meets any particle from $({}^c\xi_t^x)$ in $(\sigma, \sigma + \varepsilon]$.

Here $\varepsilon \leqq 1$ and will be further restricted later on.

We may partition the space of realizations of ξ_4^x so as to write:

$$\begin{aligned} E|\xi_4^x| = {} & E[|\xi_4^x|; A_1^c] + E[|\xi_4^x|; A_1 \cap A_2^c] + E[|\xi_4^x|; A_1 \cap A_2 \cap A_3^c] \\ & + E[|\xi_4^x|; A_1 \cap A_2 \cap A_3]. \end{aligned} \tag{17}$$

We proceed to show that each of the four quantities on the right is small: the first three because the associated events have low probability, and the fourth because $\xi_4^x = \emptyset$ is usually the case here.

$E[|\xi_4^x|; A_1^c]$. It is not difficult to show that $(|\xi_t^x|)$, $t \geqq 4 - \varepsilon$, is pathwise dominated by an appropriate version of the binary branching process $(Z_{t-4+\varepsilon})$ having initial state $|\xi_{4-\varepsilon}^x|$. Therefore,

$$E[|\xi_4^x|; A_1^c] \leqq E[Z_\varepsilon; A_1^c] = e^\varepsilon E[Z_0; A_1^c] = e^\varepsilon E[|\xi_{4-\varepsilon}^x|; A_1^c].$$

Each particle in (ξ_t^x) branches at exponential rate 1, and so

$$P(A_1^c) \leqq e^{-(4-\varepsilon)}.$$

Since $|\xi_{4-\varepsilon}^x| = 1$ under A_1^c, it follows that

$$E[|\xi_4^x|; A_1^c] \leqq e^{-(4-2\varepsilon)} < 1/8$$

for $\varepsilon \leqq 1/2$.

$E[|\xi_4^x|;\ A_1 \cap A_2^c]$. One can employ the strong Markov property to show that $(|\xi_{\tau+t}^x|)$ is pathwise dominated by an appropriate version of (Z_t) with initial state $|\xi_\tau^x|$. Therefore,

$$E[|\xi_4^x|; A_1 \cap A_2^c] \leqq E[Z_{4-\tau}; A_1 \cap A_2^c] \tag{18}$$

which, since (Z_t) is increasing and $\tau \leqq 4$, is at most

$$E[Z_4; A_1 \cap A_2^c] = e^4 E[Z_0; A_1 \cap A_2^c] = e^4 E[|\xi_\tau^x|; A_1 \cap A_2^c]. \tag{19}$$

Since $|\xi_t^x| \leqq 2$ for $t < \tau$ and each particle branches at rate 1,

$$P(A_2^c) \leqq 2(1 - e^{-\varepsilon}).$$

Because $|\xi_\tau^x| \leqq 3$, it therefore follows from (18) and (19) that

$$E[|\xi_4^x|; A_1 \cap A_2^c] \leqq 6(1 - e^{-\varepsilon})\, e^4.$$

Choosing $\varepsilon > 0$ small enough, one obtains

$$E[|\xi_4^x|; A_1 \cap A_2^c] < 1/8.$$

$E[|\xi_4^x|; A_1 \cap A_2 \cap A_3^c]$. Under A_2, (ξ_t^x) has either two distinct particles in $(\sigma, \sigma + \varepsilon]$ (if (ξ_t^x) branches at σ) or none at all. Therefore,

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3^c) \leqq\; & P((^c\xi_t^x) \text{ hits a simple random walk starting} \\ & \text{at } x \text{ with jump rate } \rho \text{ in } (\sigma, \sigma+\varepsilon];\ \xi_\sigma^x \neq \emptyset) \\ & + P((^c\xi_t^x) \text{ hits a simple random walk starting} \\ & \text{at } x, \text{ with jump rate } \rho \text{ and which jumps} \\ & \text{at } \sigma, \text{ in } (\sigma, \sigma+\varepsilon];\ \xi_\sigma^x \neq \emptyset,\ \sigma \leqq 4 - \varepsilon). \end{aligned} \tag{20}$$

Let $G(s) =$ probability $(^c\xi_t^x)$ and the first random walk hit by time s. Then the first quantity on the right is at most

$$\int_0^\infty P(\sigma \in [s - \varepsilon, s);\ \xi_\sigma^x \neq \emptyset)\, dG(s) \leqq \varepsilon \int_0^\infty dG(s) \leqq \varepsilon, \tag{21}$$

since particles branch at rate 1. On the other hand, one can check that the probability that a random walk with jump rate $\rho + 1$ has simultaneously no more than $4\varepsilon(\rho+1)$ jumps in any interval of length ε in $[0, 4]$ is more than $1 - \varepsilon$ for $\rho > 4/\varepsilon^2$. This is true in particular for $[\sigma' - \varepsilon, \sigma)$, where σ' is the hitting time in the second quantity on the right of (20). (Set $\sigma' = \infty$ if (ξ_t^x) and the random walk do not hit by $t = 4$.) Therefore, since σ (under $\xi_\sigma^x \neq \emptyset$) can be viewed as the first jump of a random walk with jump rate 1,

$$P(\sigma \in [\sigma' - \varepsilon, \sigma')) < 4\varepsilon + \varepsilon = 5\varepsilon. \tag{22}$$

Together, (21) and (22) show that

$$P(A_1 \cap A_2 \cap A_3^c) < 6\varepsilon.$$

Under A_2, $|\xi_{\sigma+\varepsilon}^x| \leqq 2$, and so

$$E[|\xi_{\sigma+\varepsilon}^x|; A_1 \cap A_2 \cap A_3^c] < 12\varepsilon.$$

Reasoning as in (18) and (19), it follows that for $\varepsilon < e^{-4}/96$ and $\rho > 4/\varepsilon^3$,

$$\begin{aligned} E[|\xi_4^x|; A_1 \cap A_2 \cap A_3^c] &\leqq e^4 E[|\xi_{\sigma+\varepsilon}^x|; A_1 \cap A_2 \cap A_3^c] \\ &< 12\varepsilon\, e^4 < 1/8. \end{aligned}$$

$E[|\xi_4^x|; A_1 \cap A_2 \cap A_3]$. Either $\xi_\sigma^x = \emptyset$ or $|\xi_\sigma^x| = 2$; in the latter case the two neighboring particles execute simple random walks with jump rate ρ, and do not branch or meet any particles from $({}^c\xi_t^x)$ in $(\sigma, \sigma+\varepsilon]$. Therefore,

$$E[|\xi_{\sigma+\varepsilon}^x|; A_1 \cap A_2 \cap A_3] \leqq 2P(\text{simple random walks at neighboring sites with jump rate 1 do not meet by time } \varepsilon\rho).$$

By the recurrence of simple random walk, this quantity $\to 0$ as $\rho \to \infty$, and is therefore $< e^{-4}/8$ for ρ chosen large enough. Reasoning again as in (18) and (19), it follows that

$$E[|\xi_4^x|; A_1 \cap A_2 \cap A_3] \leqq e^4 E[|\xi_{\sigma+\varepsilon}^x|; A_1 \cap A_2 \cap A_3| < 1/8$$

for sufficiently large ρ. $\square$

We conclude with a few comments on Theorems 1 and 2. Theorem 1 states that in dimension 1, branching annihilating random walk survives with positive probability if $\rho > 0$ is small enough, whereas Theorem 2 states that the process dies out with probability 1 for large ρ. One can of course extend the definition of branching annihilating random walk to dimensions $d \geqq 2$ in the obvious manner, and ask about analogues of Theorems 1 and 2. It is possible to extend Theorem 1 to higher dimensions by projecting down onto the line $x_1 = x_2 = \ldots = x_d$. It can be shown that the one-dimensional projected process has the same kind of growth properties that were derived in Sect. 2, and the tree argument of Sect. 4 goes through with slight modifications. Theorem 2, however, is a consequence of the probability of eventual return to the origin of a random walk being greater than 1/2. Theorem 2 therefore also holds in $d = 2$, but its proof is inapplicable for $d \geqq 3$. (In three dimensions, the probability of return is ≈ 0.35.) It would be interesting to show that there is always a positive probability of survival for $d \geqq N$ for some N (which may be 3). Also, it would be interesting to have some statement on the survival of branching annihilating random walk at values ρ intermediate to those for which Theorems 1 and 2 hold. In particular, one might wish to show that there is a critical ρ_0 for which $\rho < \rho_0$ implies survival and $\rho > \rho_0$ implies extinction of the process.

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