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On Optimal Stopping Rules

By

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1. Introduction

Let $y_1, y_2, \ldots$ be a sequence of random variables with a given joint distribution. Assume that we can observe the y's sequentially but that we must stop some time, and that if we stop with y_n we will receive a payoff $x_n = f_n(y_1, \ldots, y_n)$. What stopping rule will maximize the expected value of the payoff?

In this paper we attempt to give a reasonably general theory of the existence and computation of optimal stopping rules, previously discussed to some extent in [1] and [12]. We then apply the theory to two particular cases of interest in applications. One of these belongs to the general domain of dynamic programming; the other is the problem of showing the Bayesian character of the Wald sequential probability ratio test.

2. Existence of an optimal rule

Let $(\Omega, \mathscr{F}, P)$ be a probability space with points ω, let $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \ldots$ be a non-decreasing sequence of sub-σ-algebras of $\mathscr{F}$, and let $x_1, x_2, \ldots$ be a sequence of random variables defined on Ω with $E|x_n| < \infty$ and such that $x_n = x_n(\omega)$ is measurable $(\mathscr{F}_n)$. A sampling variable (s.v.) is a random variable (r.v.) $t = t(\omega)$ with values in the set of positive integers (not including $+\infty$) and such that $\{t(\omega) = n\} \in \mathscr{F}_n$ for each n, where by $\{\ldots\}$ we mean the set of all ω for which the indicated relation holds. For any s.v. t we may form the r.v. $x_t = x_{t(\omega)}(\omega)$. We shall be concerned with the problem of finding a s.v. t which maximizes the value of $E(x_t)$ in the class of all s.v.'s for which this expectation exists.

We shall use the notation $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, so that $x = x^+ - x^-$. To simplify matters we shall suppose that $E(\sup_n x_n^+) < \infty$; then for any s.v. t, $x_t \leqq \sup_n x_n^+$, and hence $-\infty \leqq E(x_t) \leqq E(\sup_n x_n^+) < \infty$. Denoting by C the class of all s.v.'s, it follows that $E(x_t)$ exists for all $t \in C$ but may have the value $-\infty$.

In what follows we shall occasionally refer to [1] for the details of certain proofs.

Definition. A s.v. t is *regular* if for all $n = 1, 2, \ldots$

(1) $$t > n \Rightarrow E(x_t | \mathscr{F}_n) > x_n .$$

Note that if t is any regular s.v. then

$$E(x_t) = \int_{\{t=1\}} x_t + \int_{\{t>1\}} x_t \geqq \int_{\{t=1\}} x_1 + \int_{\{t>1\}} x_1 = E(x_1) > -\infty .$$

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Lemma 1. *Given any s.v. t, define*

$$t' = \text{first integer } j \geqq 1 \text{ such that } E(x_t \mid \mathscr{F}_j) \leqq x_j .$$

Then t' is a s.v. and has the following properties:

(a) *t' is regular,*

(b) $t' \leqq t$,

(c) $E(x_{t'}) \geqq E(x_t)$.

Proof. If $t = n$ then $E(x_t \mid \mathscr{F}_n) = x_n$, so that $t' \leqq n$. Thus $t' \leqq t < \infty$, and hence (b) holds. For any $A \in \mathscr{F}_n$,

$$\int\limits_{A\{t' \geqq n\}} x_{t'} = \sum_{j=n}^{\infty} \int\limits_{A\{t'=j\}} x_j \geqq \sum_{j=n}^{\infty} \int\limits_{A\{t'=j\}} E(x_t \mid \mathscr{F}_j) = \int\limits_{A\{t' \geqq n\}} x_t . \tag{2}$$

Putting $n = 1$ and $A = \Omega$, (2) yields the inequality (c). Finally, from (2) and the definition of t'.

$$t' > n \Rightarrow E(x_{t'} \mid \mathscr{F}_n) \geqq E(x_t \mid \mathscr{F}_n) > x_n ,$$

which proves (a).

Lemma 2. *Let $t_1, t_2, \ldots$ be any sequence of regular s.v.'s and define*

$$\tau_i = \max(t_1, \ldots, t_i), \quad \tau = \sup_i t_i = \lim_{i \to \infty} \tau_i .$$

Then the τ_i are regular s.v.'s, $\tau_1 \leqq \tau_2 \leqq \cdots$, and

$$\max(E x_{t_1}, \ldots, E x_{t_i}) \leqq E(x_{\tau_i}) \leqq E(x_{\tau_{i+1}}) \leqq \cdots . \tag{3}$$

Moreover, if $P(\tau < \infty) = 1$ then τ is a regular s.v. and

$$E x_\tau \geqq \lim_{i \to \infty} E(x_{\tau_i}) \geqq \sup E x_{t_i} . \tag{4}$$

Proof. For any $i, n = 1, 2, \ldots$ and any $A \in \mathscr{F}_n$ we have

$$\int\limits_{A\{\tau_i \geqq n\}} x_{\tau_i} = \sum_{j=n}^{\infty} \left(\int\limits_{A\{\tau_i = j \geqq t_{i+1}\}} x_{\tau_{i+1}} + \int\limits_{A\{\tau_i = j < t_{i+1}\}} x_j \right) \leqq \sum_{j=n}^{\infty} \left(\int\limits_{A\{\tau_i = j \geqq t_{i+1}\}} x_{\tau_{i+1}} + \int\limits_{A\{\tau_i = j < t_{i+1}\}} x_{t_{i+1}} \right) = \int\limits_{A\{\tau_i \geqq n\}} x_{\tau_{i+1}} .$$

Hence, since $\tau_1 \leqq \tau_2 \leqq \cdots$, it follows that

$$\tau_i \geqq n \Rightarrow E(x_{\tau_i} \mid \mathscr{F}_n) \leqq E(x_{\tau_{i+1}} \mid \mathscr{F}_n) \leqq E(x_{\tau_{i+2}} \mid \mathscr{F}_n) \leqq \cdots . \tag{5}$$

Since t_1 is regular and $\tau_1 = t_1$, it follows that

$$t_1 > n \Rightarrow x_n < E(x_{t_1} \mid \mathscr{F}_n) = E(x_{\tau_1} \mid \mathscr{F}_n) \leqq E(x_{\tau_2} \mid \mathscr{F}_n) \leqq \cdots .$$

By symmetry,

$$t_j > n \Rightarrow E(x_{\tau_i} \mid \mathscr{F}_n) > x_n , \qquad j = 1, \ldots, i ,$$

and hence

$$\tau_i > n \Rightarrow E(x_{\tau_i} \mid \mathscr{F}_n) > x_n ,$$

so that each τ_i is regular. Setting $n = 1$ in (5) we obtain

$$E(x_{t_1} \mid \mathscr{F}_1) = E(x_{\tau_1} \mid \mathscr{F}_1) \leqq E(x_{\tau_2} \mid \mathscr{F}_1) \leqq \cdots , \tag{6}$$

so that

$$E(x_{t_1}) \leqq E(x_{\tau_1}) \leqq E(x_{\tau_2}) \leqq \cdots ,$$

and by symmetry

$$E(x_{t_j}) \leqq E(x_{\tau_i}) , \qquad j = 1, \ldots, i ,$$

which proves (3).

Turning our attention to τ we observe that since $x_\tau = \lim_{i\to\infty} x_{\tau_i}$, and since

$$E(\sup_i x_{\tau_i}) \leqq E(\sup_n x_n^+) < \infty\,,$$

we have by Fatou's lemma for conditional expectations [2, p. 348] that

$$E(x_\tau \mid \mathscr{F}_n) \geqq \limsup_{i\to\infty} E(x_{\tau_i} \mid \mathscr{F}_n)\,. \tag{7}$$

Hence by (5) and (7),

$$\begin{aligned} \tau > n \Rightarrow \tau_i > n \quad \text{for some} \quad i &\Rightarrow x_n < E(x_{\tau_i} \mid \mathscr{F}_n) \leqq E(x_{\tau_{i+1}} \mid \mathscr{F}_n) \leqq \cdots \\ &\Rightarrow \limsup_{i\to\infty} E(x_{\tau_i} \mid \mathscr{F}_n) > x_n \Rightarrow E(x_\tau \mid \mathscr{F}_n) > x_n\,, \end{aligned}$$

so that τ is regular. Finally, from (6) and (7) we have

$$E(x_\tau \mid \mathscr{F}_1) \geqq E(x_{\tau_i} \mid \mathscr{F}_1)\,,$$

so that (4) holds.

Corollary 1. *Let t_0 be any s.v., and let $C(t_0)$ denote the class of all s.v.'s t such that $t \leqq t_0$. Then there exists a s.v. $\tau \in C(t_0)$ such that*

$$E(x_\tau) = \sup_{t\in C(t_0)} E(x_t)\,. \tag{8}$$

Proof. Take any sequence $t_1, t_2, \ldots$ of s.v.'s in $C(t_0)$ such that

$$\sup_i E(x_{t_i}) = \sup_{t\in C(t_0)} E(x_t)\,.$$

By Lemma 1 we may assume that the t_i are regular. Set $\tau = \sup_i t_i$; then $\tau \in C(t_0)$ and the conclusion follows from Lemma 2.

Corollary 2. *Suppose there exists a s.v. τ_0 such that*

$$E(x_{\tau_0}) = \sup_{t\in C} E(x_t)\,. \tag{9}$$

Choose any sequence $t_1, t_2, \ldots$ of regular s.v.'s such that

$$\sup_i E(x_{t_i}) = \sup_{t\in C} E(x_t)\,, \tag{10}$$

and set $\tau = \sup_i t_i$. Then

$$\tau \leqq \tau_0\,, \tag{11}$$

so that τ is a s.v., and

$$E(x_\tau) = \sup_{t\in C} E(x_t)\,. \tag{12}$$

The s.v. τ thus defined does not depend on the particular choice of $\tau_0, t_1, t_2, \ldots$, since by (11) *and* (12) *it is the minimal s.v. τ such that* (12) *holds.*

Proof. By Lemma 1 of [1], $t_i \leqq \tau_0$ for each i, so that (11) holds, and (12) then follows from Lemma 2.

Lemma 3. *Assume that*

$$x_n = x_n' - x_n'' \tag{13}$$

where x_n', x_n'' are measurable ($\mathscr{F}_n$) for each n, and are such that

$$E[\sup_n (x_n')^+] = B < \infty, \tag{14}$$

$$x_n'' \geqq 0, \quad \lim_{n\to\infty} x_n'' = \infty. \tag{15}$$

Let $t_1, t_2, \ldots$ be any sequence of s.v.'s such that

$$E(x_{t_i}) \geqq K > -\infty, \tag{16}$$

and set $\tau = \liminf_{i\to\infty} t_i$. Then $P(\tau < \infty) = 1$.

Proof. For any integers i and m,

$$\int_{\{t_i \geqq m\}} x_{t_i} = \int_{\{t_i \geqq m\}} (x_{t_i}' - x_{t_i}'') \leqq \int_{\{t_i \geqq m\}} (\sup_n (x_n')^+ - \inf_{j \geqq m} x_j'') \leqq B - \int_{\{t_i \geqq m\}} w_m,$$

where we have set

$$w_m = \inf_{j \geqq m} x_j''.$$

Since

$$\int_{\{t_i < m\}} x_{t_i} \leqq B,$$

we have

$$K \leqq E(x_{t_i}) \leqq 2B - \int_{\{t_i \geqq m\}} w_m.$$

Let $A_i = \{\inf_{j \geqq i} t_j \geqq m\} \subset \{t_i \geqq m\}$; then since $w_m \geqq 0$,

$$K \leqq 2B - \int_{A_i} w_m,$$

and letting $i \to \infty$ we have

$$K \leqq 2B - \int_{\{\tau \geqq m\}} w_m \leqq 2B - \int_{\{\tau = \infty\}} w_m.$$

Let $m \to \infty$; then since

$$0 \leqq w_1 \leqq w_2 \leqq \cdots \to \liminf_{n\to\infty} x_n'' = \infty,$$

it follows that

$$\int_{\{\tau = \infty\}} \infty \leqq 2B - K < \infty,$$

so that $P(\tau = \infty) = 0$.

Lemma 4. *Under the assumptions* (13), (14), (15) *of Lemma 3, there exists a s.v. τ such that*

$$E(x_\tau) = \sup_{t \in C} E(x_t). \tag{17}$$

Proof. Let $t_1, t_2, \ldots$ be any sequence of s.v.'s such that

$$\sup_i E(x_{t_i}) = \sup_{t \in C} E(x_t). \tag{18}$$

By Lemma 1 we may suppose that the t_i are regular and therefore that

$$E(x_{t_i}) \geqq E(x_1) > -\infty.$$

Set

$$\tau_i = \max(t_1, \ldots, t_i), \quad \tau = \sup_i t_i = \lim_{i\to\infty} \tau_i.$$

By Lemma 2,

$$E(x_{\tau_i}) \geqq E(x_{t_i}) \geqq E(x_1),$$

and $\tau_1 \leqq \tau_2 \leqq \cdots$. By Lemma 3, $P(\tau < \infty) = 1$. Hence by Lemma 2,

$$E(x_\tau) \geqq \sup_i E(x_{t_i}), \tag{19}$$

and (17) follows from (18) and (19).

The main results so far may be summarized in the following theorem.

Theorem 1. *Assume that* $E(\sup_n x_n^+) < \infty$.

(i) *Choose any sequence* $t_1, t_2, \ldots$ *of regular s.v.'s such that*

$$\sup_i E(x_{t_i}) = \sup_{t \in C} E(x_t) \tag{20}$$

(this can always be done), and define the r.v.

$$\tau = \sup_i t_i. \tag{21}$$

Then $P(\tau < \infty) = 1$ *if and only if there exists a s.v.* τ_0 *such that*

$$E(x_{\tau_0}) = \sup_{t \in C} E(x_t), \tag{22}$$

and τ *is then the minimal s.v. satisfying* (22).

(ii) *Assumptions* (13), (14), (15) *are sufficient to ensure that* $P(\tau < \infty) = 1$.

Proof.

(i) If $P(\tau < \infty) = 1$ then by the argument of Lemma 4,

$$E(x_\tau) = \sup_{t \in C} E(x_t).$$

And if any s.v. τ_0 exists satisfying (22), then $P(\tau < \infty) = 1$ by Corollary 2 of Lemma 2, and $\tau \leqq \tau_0$.

(ii) Follows from Lemma 4.

The main defect of Theorem 1 is that it gives no indication of how to choose a sequence of regular s.v.'s $t_1, t_2, \ldots$ satisfying (20). We now turn our attention to this problem.

3. The rules s_N and s

Let C_N denote the class of all s.v.'s t for which $t \leqq N$. We shall first show (cf. [3]) how to construct a certain regular s.v. s_N in C_N such that

$$E(x_{s_N}) = \sup_{t \in C_N} E(x_t). \tag{23}$$

To do this we define for each $N \geqq 1$ a finite sequence of r.v.'s $\beta_1^N, \ldots, \beta_N^N$ by recursion backwards, starting with β_N^N, using the formula

$$\beta_n^N = \max[x_n, E(\beta_{n+1}^N | \mathscr{F}_n)], \quad n = 1, \ldots, N; \quad \beta_{N+1}^N = -\infty. \tag{24}$$

Thus

$$\beta_N^N = \max[x_N, -\infty] = x_N, \tag{25}$$

and β_n^N is measurable ($\mathscr{F}_n$). We now define

(26) $$s_N = \text{first } n \geqq 1 \text{ such that } \beta_n^N = x_n .$$

Note that

(27) $$\beta_{s_N}^N = x_{s_N},$$

and, since $\beta_N^N = x_N$,

(28) $$s_N \leqq N,$$

so that $s_N \in C_N$. Moreover,

(29) $$s_N > n \Rightarrow E(\beta_{n+1}^N | \mathscr{F}_n) = \beta_n^N > x_n,$$

and

(30) $$E(\beta_{n+1}^N | \mathscr{F}_n) \leqq \beta_n^N, \quad \text{all} \quad n = 1, \ldots, N .$$

From [*1*, Lemmas 1, 2, 3] applied to the finite sequence $\beta_1^N, \ldots, \beta_N^N$ it follows that s_N is regular, since

(31) $$s_N > n \Rightarrow E(x_{s_N} | \mathscr{F}_n) = E(\beta_{s_N}^N | \mathscr{F}_n) \geqq \beta_n^N > x_n,$$

and that

(32) $$E(x_{s_N}) = E(\beta_{s_N}^N) \geqq E(\beta_t^N) \geqq E(x_t) \quad \text{all} \quad t \in C_N .$$

Thus the sequence $s_1, s_2, \ldots$ has the following properties:

(33) $$s_N \text{ is regular}, \quad s_N \leqq N, \quad (23) \text{ holds},$$

and, since $C_1 \subset C_2 \subset \ldots$, it follows that

(34) $$E(x_1) = E(x_{s_1}) \leqq E(x_{s_2}) \leqq \cdots \to \lim_{N\to\infty} E(x_{s_N}) .$$

It is easy to show by induction from (24) and (25) that

(35) $$x_N = \beta_N^N \leqq \beta_N^{N+1} \leqq \cdots .$$

Hence from (26) we have

(36) $$1 = s_1 \leqq s_2 \leqq \cdots,$$

and we define

(37) $$s = \sup_N s_N = \lim_{N\to\infty} s_N \leqq +\infty .$$

Lemma 5. *If* $P(s < \infty) = 1$, *then*

(38) $$E(x_s) \geqq \lim_{N\to\infty} E(x_{s_N}) .$$

Proof. By (33) and Lemma 2 applied to the sequence $s_1, s_2, \ldots$.

Lemma 6. *If t is any s.v. such that*

(39) $$\liminf_{n\to\infty} \int_{\{t>n\}} x_n^- = 0$$

then

(40) $$\lim_{N\to\infty} E(x_{s_N}) \geqq E(x_t) .$$

Proof. Set

(41) $$t_N = \min(t, N) \in C_N .$$

Then

(42) $$\int_{\{t \leqq N\}} x_t = E(x_{t_N}) - \int_{\{t > N\}} x_N \leqq E(x_{s_N}) - \int_{\{t > N\}} x_N \leqq E(x_{s_N}) + \int_{\{t > N\}} x_N^- .$$

Letting $N \to \infty$ it follows from (39) that (40) holds.

Corollary. *If $x_n^- \leqq c n^\alpha$ for some $c, \alpha \geqq 0$, and if $E(t^\alpha) < \infty$, then*

$$\lim_{N\to\infty} E(x_{s_N}) \geqq E(x_t) .$$

Proof. From Lemma 6 and the relation

$$\int_{\{t > n\}} x_n^- \leqq c \int_{\{t > n\}} n^\alpha \leqq c \int_{\{t > n\}} t^\alpha \to 0 .$$

Theorem 2. *Assume that*

$$x_n = x_n' - x_n'' = x_n^* - x_n^{**} ,$$

where all the components are measurable $(\mathscr{F}_n)$ and

(43) $$E[\sup_n (x_n')^+] = B < \infty ,$$

(44) $$0 \leqq x_1'' \leqq x_2'' \leqq \cdots, \quad \lim_{n\to\infty} x_n'' = \infty ,$$

(45) *the $(x_n^*)^-$ are uniformly integrable for all n,*

and

(46) $$x_n^{**} \leqq c x_n'' \quad \text{for some} \quad 0 < c < \infty .$$

Then $s = \sup_N s_N$ is a s.v. and

$$E(x_s) = \sup_{t\in C} E(x_t) = \lim_{N\to\infty} E(x_{s_N}) .$$

Proof. For any s.v. t we have from (44) and (46) that for $t > N$,

$$x_N = (x_N^*)^+ - (x_N^*)^- - x_N^{**} \geqq -[(x_N^*)^- + c x_t''] ,$$

so that

(47) $$\int_{\{t > N\}} x_N^- \leqq \int_{\{t > N\}} [(x_N^*)^- + c x_t''] .$$

Now if $E(x_t) \neq -\infty$ then from (43)

$$E(x_t) = E(x_t') - E(x_t'') \leqq E(x_t'^+) \leqq B < \infty ,$$

so that $E(x_t')$ and hence $E(x_t'')$ is finite. From (47) and (45) it follows that (39) holds. From Lemma 3, $P(s < \infty) = 1$, and hence from Lemmas 5 and 6,

$$E(x_s) \geqq \lim_{N\to\infty} E(x_{s_N}) \geqq E(x_t) .$$

Since this is trivially true when $E(x_t) = -\infty$ the result follows.

It is of interest to express $E(x_{s_N})$ explicitly. To do this we observe that by the submartingale property (30),

$$E(x_{s_N}) = E(\beta^N_{s_N}) = \sum_{n=1}^{N} \int_{\{s_N=n\}} \beta^N_n = \sum_{n=1}^{N-1} \int_{\{s_N=n\}} \beta^N_n + \int_{\{s_N>N-1\}} \beta^N_N$$

$$\tag{48} \leqq \sum_{n=1}^{N-2} \int_{\{s_N=n\}} \beta^N_n + \int_{\{s_N=N-1\}} \beta^N_{N-1} + \int_{\{s_N>N-1\}} \beta^N_{N-1}$$

$$= \sum_{n=1}^{N-2} \int_{\{s_N=n\}} \beta^N_n + \int_{\{s_N>N-2\}} \beta^N_{N-1} \leqq \cdots \leqq \int_{\{s_N>0\}} \beta^N_1 = E(\beta^N_1).$$

But since $E(\beta^N_{s_N}) \geqq E(\beta^N_1)$ it follows that

$$\tag{49} E(x_{s_N}) = E(\beta^N_1).$$

Thus under the conditions on the x_n of Theorem 2,

$$\tag{50} E(x_s) = \lim_{N\to\infty} E(x_{s_N}) = \lim_{N\to\infty} E(\beta^N_1).$$

From (35) the limits

$$\tag{51} \beta_n = \lim_{N\to\infty} \beta^N_n$$

exist. By the theorem of monotone convergence for conditional expectations [*2*, p. 348] it follows from (35) that

$$\tag{52} E(\beta^n_n \mid \mathscr{F}_n) \leqq E(\beta^{n+1}_n \mid \mathscr{F}_n) \leqq \cdots \to E(\beta_n \mid \mathscr{F}_n),$$

and hence from (24) that the β_n satisfy the relations

$$\tag{53} \beta_n = \max[x_n, E(\beta_{n+1} \mid \mathscr{F}_n)], \quad n = 1, 2, \ldots.$$

Define for the moment

$$\tag{54} s^* = \text{first } i \geqq 1 \text{ such that } x_i = \beta_i.$$

We shall show that

$$\tag{55} s^* = \sup_N s_N = s.$$

For if $s^* = n$, then by (54) $x_i < \beta_i$ for $i = 1, \ldots, n-1$, and hence for sufficiently large N, $x_i < \beta^N_i$ for $i = 1, \ldots, n-1$, so that $s_N \geqq n$. Hence $s \geqq n$ and therefore $s \geqq s^*$. Conversely, if $s = n$ then for sufficiently large N, $s_N = n$, and hence $x_i < \beta^N_i$ for $i = 1, \ldots, n-1$ so that $x_i < \beta_i$ for $i = 1, \ldots, n-1$ and therefore $s^* \geqq n$. Thus $s^* \geqq s$.

We may now restate Theorem 2 in the following form.

Theorem 2′. *Assume that the hypotheses on the x_n of Theorem 2 are satisfied. For each $N \geqq 1$ define $\beta^N_1, \beta^N_2, \ldots, \beta^N_N$ by (24) and set*

$$\tag{56} s = \textit{first } i \geqq 1 \textit{ such that } x_i = \beta_i = \lim_{N\to\infty} \beta^N_i.$$

Then s is a s.v. and

$$\tag{57} E(x_s) = \lim_{N\to\infty} E(\beta^N_1) = \sup_{t\in C} E(x_t).$$

This generalizes a theorem of Arrow, Blackwell, and Girshick [*3*].

4. The monotone case

If the sequence of r.v.'s $x_1, x_2, \ldots$ is such that for every $n = 1, 2, \ldots,$

(58) $$E(x_{n+1} | \mathscr{F}_n) \leqq x_n \Rightarrow E(x_{n+2} | \mathscr{F}_{n+1}) \leqq x_{n+1},$$

we shall say that we are in the *monotone case* (to which [1] is devoted). In this case the calculation of the s_N defined by (26), and of $s = \sup_N s_N$, become much simpler.

Lemma 7. *In the monotone case we may compute s_N and s by the formulas*

(59) $$s_N = \min[N, \textit{ first } n \geqq 1 \textit{ such that } E(x_{n+1} | \mathscr{F}_n) \leqq x_n],$$

and

(60) $$s = \sup_N s_N = \textit{first } n \geqq 1 \textit{ such that } E(x_{n+1} | \mathscr{F}_n) \leqq x_n.$$

Proof. (a) we begin by proving that in the monotone case, for $n = 1, 2, \ldots, N-1$,

(61) $$E(x_{n+1} | \mathscr{F}_n) \leqq x_n \Rightarrow E(\beta_{n+1}^N | \mathscr{F}_n) \leqq x_n.$$

For $n = N-1$ this is trivial, since $\beta_N^N = x_n$. Assume therefore that (61) is true for $n = j+1$. Then

$$E(x_{j+1} | \mathscr{F}_j) \leqq x_j \Rightarrow E(x_{j+2} | \mathscr{F}_{j+1}) \leqq x_{j+1} \Rightarrow$$
$$E(\beta_{j+2}^N | \mathscr{F}_{j+1}) \leqq x_{j+1} \Rightarrow \beta_{j+1}^N = x_{j+1} \Rightarrow$$
$$E(\beta_{j+1}^N | \mathscr{F}_j) = E(x_{j+1} | \mathscr{F}_j) \leqq x_j,$$

which establishes (61) for $n = j$.

(b) Recall that by (26),

(62) $$s_N = \text{first } n \geqq 1 \text{ such that } \beta_n^N = x_n.$$

Define for the moment

(63) $$s_N' = \min[N, \text{ first } n \geqq 1 \text{ such that } E(x_{n+1} | \mathscr{F}_n) \leqq x_n].$$

(c) Suppose $s_N' = n < N$. Then by (61), $E(\beta_{n+1}^N | \mathscr{F}_n) \leqq x_n$, so that $\beta_n^N = x_n$ and hence $s_N \leqq n = s_N'$. If $s_N' = N$ then also $s_N \leqq s_N'$. Thus $s_N \leqq s_N'$ always.

(d) Suppose $s_N = n \leqq N$. Then $E(\beta_{n+1}^N | \mathscr{F}_N) \leqq x_n$. Since $\beta_{n+1}^N \geqq x_{n+1}$ it follows that $E(x_{n+1} | \mathscr{F}_n) \leqq x_n$. Hence $s_N' \leqq n$ and therefore $s_N' \leqq s_N$.

It follows from (c) and (d) that $s_N' = s_N$, which proves (59), and (60) is immediate.

5. An example

Let $y, y_1, y_2, \ldots$ be independent r.v.'s with a common distribution, let $\mathscr{F}_n$ be the σ-algebra generated by $y_1, \ldots, y_n$, and let

(64) $$x_n = \max(y_1, \ldots, y_n) - a_n,$$

where we assume to begin with only that the a_n are constants such that

(65) $$0 \leqq a_1 < a_2 < \cdots$$

and that $Ey^+ < \infty$. Set

(66) $$m_n = \max(y_1, \ldots, y_n), \quad b_n = a_{n+1} - a_n > 0.$$

Then

$$x_{n+1} = m_{n+1} - a_{n+1} = m_{n+1} - a_n - b_n = x_n + (y_{n+1} - m_n)^+ - b_n.$$

Hence

(67) $$E(x_{n+1} | \mathscr{F}_n) - x_n = E[(y - m_n)^+] - b_n.$$

Define constants γ_n by the relation

(68) $$E[(y - \gamma_n)^+] = b_n$$

(graphically, b_n is the area in the z, y-plane to the right of $y = \gamma_n$ and between $z = 1$ and $z = F(y)$). Then it is easy to see from (67) and (68) that

(69) $$E(x_{n+1} | \mathscr{F}_n) \leqq x \quad \text{if and only if} \quad m_n \geqq \gamma_n.$$

We are in the monotone case when

(70) $$b_1 \leqq b_2 \leqq \cdots.$$

For if (70) holds, and if $E(x_{n+1} | \mathscr{F}_n) \leqq x_n$, then by (68), $m_n \geqq \gamma_n$ and hence $m_{n+1} \geqq m_n \geqq \gamma_n \geqq \gamma_{n+1}$, so that $E(x_{n+2} | \mathscr{F}_{n+1}) \leqq x_{n+1}$. We can therefore assert that when (70) holds

(71) $$s_N = \min[N, \text{ first } n \geqq 1 \text{ such that } m_n \geqq \gamma_n],$$

and

(72) $$s = \sup_N s_N = \text{first } n \geqq 1 \text{ such that } m_n \geqq \gamma_n.$$

An example of the monotone case is given by choosing $a_n = cn^\alpha$ with $c > 0$, $\alpha \geqq 1$. When $\alpha = 1$ all the γ_n coincide and have the value γ given by

(73) $$E[(y - \gamma)^+] = c.$$

For $0 < \alpha < 1$ we are not in the monotone case and no simple evaluation of s_N and s is possible.

It is interesting to note that if we set

(74) $$\tilde{x}_n = y_n - a_n$$

instead of (64), then, setting $\mu = Ey$,

(75) $$E(x_{n+1} | \mathscr{F}_n) - \tilde{x}_n = \mu - b_n - y_n,$$

and we are never in the monotone case. However, for $a_n = cn$ we have by the above,

(76) $$s = \text{first } n \geqq 1 \text{ such that } m_n \geqq \gamma$$
$$= \text{first } n \geqq 1 \text{ such that } y_n \geqq \gamma.$$

Thus

(77) $$x_s = m_s - cs = y_s - cs = \tilde{x}_s,$$

while for any s.v. t, since $\tilde{x}_n \leqq x_n$, we have

(78) $$\tilde{x}_t \leqq x_t .$$

It follows that

$$\sup_{t \in C} E(\tilde{x}_t) \leqq \sup_{t \in C} E(x_t),$$

and that if the distribution of the y_n is such that

(79) $$E(x_s) = \sup_{t \in C} E(x_t),$$

then also

$$E(\tilde{x}_s) = \sup_{t \in C} E(\tilde{x}_t).$$

We shall now investigate whether in fact (79) holds, and for this we shall use Theorem 2. Write

$$x_n = \max(y_1, \ldots, y_n) - a_n = x_n' - x_n'' = x_n^* - x_n^{**}$$

where we have set

(80) $$\begin{cases} x_n' = \max(y_1, \ldots, y_n) - a_n/2, & x_n'' = a_n/2, \\ x_n^* = \max(y_1, \ldots, y_n), & x_n^{**} = a_n . \end{cases}$$

Assume that the constants a_n are such that

(81) $$0 \leqq a_1 \leqq a_2 \leqq \cdots \to \infty .$$

Then (44) and (46) hold, and to apply Theorem 2 it will suffice to show that

(82) $$E \sup_n [\max(y_1, \ldots, y_n) - a_n/2] < \infty$$

and that the r.v.'s

(83) $$[\max(y_1, \ldots, y_n)]^- \text{ are uniformly integrable.}$$

The latter relation is trivial as long as $E|y| < \infty$, since

$$[\max(y_1, \ldots, y_n)]^- \leqq y_1^- .$$

It remains only to verify (82).

To find conditions for the validity of (82) in the case

$$a_n = c n^{\alpha}, \quad c, \alpha > 0,$$

we shall need the following lemma, the proof of which will be deferred until later.

Lemma 8. *Let $w, w_1, w_2, \ldots$ be independent, identically distributed, non-negative r.v.'s and for any positive constants c, α set*

$$z = \sup_n [\max(w_1, \ldots, w_n) - c n^{\alpha}] .$$

Then

(84) $$P(z < \infty) = 1 \text{ if and only if } E(w^{1/\alpha}) < \infty,$$

and

(85) $$\text{for any } \beta > 0, \quad E(z^{1/\beta}) < \infty \quad \text{if and only if} \quad E(w^{1/\alpha + 1/\beta}) < \infty .$$

Now suppose that the common distribution of the y_n is such that

(86) $$E|y| < \infty, \quad E[(y^+)^{1+\alpha}] < \infty.$$

Then
$$\sup_n \left[\max(y_1, \ldots, y_n) - \frac{c n^\alpha}{2}\right] \leqq \sup_n \left[\max(y_1^+, \ldots, y_n^+) - \frac{c n^\alpha}{2}\right],$$
so that by (85) for $\beta = 1$, $w = y^+$,
$$E \sup_n \left[\max(y_1, \ldots, y_n) - \frac{c n^\alpha}{2}\right] < \infty,$$
verifying (82). Thus, *if* $a_n = c n^\alpha$ $(c, \alpha > 0)$ *and if* $E|y| < \infty$ *and* $E[(y^+)^{1+\alpha}] < \infty$, *then defining the s.v. s by* (56) *we have*
$$E(x_s) = \sup_{t \in C} E(x_t).$$

This generalizes a result of [*1*], where it was assumed that $\alpha \geqq 1$, to the more general case $\alpha > 0$. See also [*5*, *6*, *7*] for the case $\alpha = 1$.

A similar argument holds for the sequence
$$x_n = y_n - c n^\alpha,$$
replacing $\max(y_1, \ldots, y_n)$ by y_n in (80).

We may summarize these results in

Theorem 3. *Let* $y, y_1, y_2, \ldots$ *be independent and identically distributed random variables, let* c, α *be positive constants, and let*
$$x_n = \max(y_1, \ldots, y_n) - c n^\alpha, \quad \tilde{x}_n = y_n - c n^\alpha.$$
Then if
$$E|y| < \infty, \quad E[(y^+)^{1+\alpha}] < \infty$$
there exist s.v.'s s and $\tilde{s}$ *such that*
$$E(x_s) = \sup_{t \in C} E(x_t), \quad E(\tilde{x}_{\tilde{s}}) = \sup_{t \in C} E(\tilde{x}_t).$$
For $\alpha \geqq 1$,
$$s = \text{first } n \geqq 1 \text{ such that } \max(y_1, \ldots, y_n) \geqq \gamma_n,$$
where γ_n *is defined by*
$$E[(y - \gamma_n)^+] = c[(n+1)^\alpha - n^\alpha].$$

Proof of Lemma 8. If w is any r.v. with distribution function F, then $E(w) < \infty$ is equivalent to $\sum_1^\infty [1 - F(n)] < \infty$, which in turn is equivalent to the convergence of $\prod_1^\infty F(n)$. Hence $E(w^{1/\alpha}) < \infty$ if and only if $\prod_1^\infty F(n^\alpha)$ converges. Now for $u > 0$ let
$$G(u) = P(z \leqq u) = P[\bigcap_{n=1}^\infty \bigcap_{i=1}^\infty \{w_i \leqq n^\alpha + u\}] = P[\bigcap_{i=1}^\infty \bigcap_{n=i}^\infty \{w_i \leqq n^\alpha + u\}]$$
$$= P[\bigcap_{i=1}^\infty \{w_i \leqq n^\alpha + u\}] = \prod_1^\infty F(n^\alpha + u).$$
It follows that $\lim_{u \to \infty} G(u) = 1$ if and only if $\prod_1^\infty F(n^\alpha)$ converges; thus (84) holds.

To prove (85), we have $E(z) < \infty$ if and only if $\prod_1^\infty G(n)$ converges. Hence [4, p. 223], $E(z^{1/\beta}) < \infty$ is equivalent to

$$(87) \qquad \sum_{m=1}^{\infty} \sum_{n=n_0}^{\infty} \log F[n^\beta + m^\alpha] > -\infty \quad \text{for some } n_0 \text{ such that} \quad F(n_0^\beta) > 0 .$$

Now

$$\int_1^\infty dm \int_{n_0}^\infty \log F(n^\beta + m^\alpha)\, dn = \int_{1+n_0^\beta}^\infty \log F(u)\, du \int_{n_0^\beta}^{u-1} \frac{1}{\alpha\beta} v^{1/\beta-1}(u-v)^{1/\alpha-1}\, dv .$$

Hence (87) is equivalent to

$$-\infty < \int_{n_0^\beta}^\infty \log F(u)\, du \int_0^u v^{1/\beta-1}(u-v)^{1/\alpha-1}\, dv = B\left(\frac{1}{\alpha}, \frac{1}{\beta}\right) \int_{n_0}^\infty u^{1/\alpha+1/\beta-1} \log F(u)\, du ,$$

But $E(w^{1/\alpha+1/\beta}) < \infty$ is equivalent to

$$-\infty < \int_{n_0^{1/\alpha+1/\beta}}^\infty \log F(t^{\alpha\beta/\alpha+\beta})\, dt = \frac{\alpha+\beta}{\alpha\beta} \int_{n_0^{1/\alpha+1/\beta}}^\infty u^{1/\alpha+1/\beta-1} \log F(u)\, du ,$$

which proves (85).

6. Application to the sequential probability ratio test

The following problem in statistical decision theory has been treated in [8, 9, 3, 10, 11]. We shall consider it here as an illustration of our general method.

Let $y_1, y_2, \ldots$ be independent, identically distributed random variables with density function f with respect to some σ-finite measure μ on the line. It is desired to test the hypothesis $H_0 : f = f_0$ versus $H_1 : f = f_1$ where f_0 and f_1 are two specified densities. The loss due to accepting H_1 when H_0 is true is assumed to be $a > 0$ and that due to accepting H_0 when H_1 is true is $b > 0$; the cost of taking each observation y_i is unity. A sequential decision procedure (δ, N) provides for determining the sample size N and making the terminal decision δ; the expected loss for (δ, N) is

$$a\alpha_0 + E_0(N) \quad \text{when } H_0 \text{ is true},$$

$$b\alpha_1 + E_1(N) \quad \text{when } H_1 \text{ is true}$$

where

$$\alpha_0 = P_0(\text{accepting } H_1), \quad \alpha_1 = P_1(\text{accepting } H_0).$$

If there is an a priori probability π that H_0 is true (and hence probability $1-\pi$ that H_1 is true) the global "risk" for (δ, N) is given by

$$r(\pi, \delta, N) = \pi[a\alpha_0 + E_0(N)] + (1-\pi)[b\alpha_1 + E_1(N)] .$$

For a given sampling variable N it is easy to determine the terminal decision rule δ which minimizes $r(\pi, \delta, N)$ for fixed values of a, b, and π. For the part of

$r(\pi, \delta, N)$ that depends on δ is (omitting symbols like $d\mu(y_1)\ldots d\mu(y_n)$)

$$\pi a \alpha_0 + (1-\pi) b \alpha_1 = \pi a \sum_{n=1}^{\infty} \int_{\{N=n,\,\text{accept } H_1\}} f_0(y_1)\ldots f_0(y_n) +$$

$$+ (1-\pi) b \sum_{n=1}^{\infty} \int_{\{N=n,\,\text{accept } H_0\}} f_1(y_1)\ldots f_1(y_n)$$

$$\geqq \sum_{n=1}^{\infty} \int_{\{N=n\}} \min[\pi a f_0(y_1)\ldots f_0(y_n),\quad (1-\pi) b f_1(y_1)\ldots f_1(y_n)]$$

$$= \sum_{n=1}^{\infty} \int_{\{N=n\}} \min[\pi_n a, (1-\pi_n) b]\,[\pi f_0(y_1)\ldots f_0(y_n) +$$

$$+ (1-\pi) f_1(y_1)\ldots f_1(y_n)],$$

where

$$\pi_n = \pi_n(y_1, \ldots, y_n) = \frac{\pi f_0(y_1)\ldots f_0(y_n)}{\pi f_0(y_1)\ldots f_0(y_n) + (1-\pi) f_1(y_1)\ldots f_1(y_n)}.$$

For the given sampling rule N define δ' by

$$\begin{cases} \text{accept } H_1 \text{ if } N = n \quad \text{and} \quad \pi_n a \leqq (1-\pi_n) b, \\ \text{accept } H_0 \text{ if } N = n \quad \text{and} \quad \pi_n a > (1-\pi_n) b. \end{cases}$$

Then

$$\pi a \alpha_0(\delta, N) + (1-\pi) b \alpha_1(\delta, N) \geqq \pi a \alpha_0(\delta', N) + (1-\pi) b \alpha_1(\delta', N).$$

Hence to find a pair (δ, N) which for given π minimizes $r(\pi, \delta, N)$ (a "Bayes" decision procedure) amounts to solving the following problem: for given $0 < \pi < 1$ let $y_1, y_2, \ldots, y_n, \ldots$ have the joint density function for each n equal to

$$\pi f_0(y_1)\ldots f_0(y_n) + (1-\pi) f_1(y_1)\ldots f_1(y_n),$$

where f_0, f_1 are given univariate density functions. For given $a, b > 0$ let

$$h(t) = \min[a t, b(1-t)] \qquad (0 \leqq t \leqq 1),$$

$$\begin{cases} \pi_0 = \pi \\ \pi_n = \pi_n(y_1, \ldots, y_n) = \dfrac{\pi f_0(y_1)\ldots f_0(y_n)}{\pi f_0(y_1)\ldots f_0(y_n) + (1-\pi) f_1(y_1)\ldots f_1(y_n)} \quad (n \geqq 1), \\ x_n = x_n(\pi_n) = -h(\pi_n) - n \qquad (n \geqq 0). \end{cases}$$

We want to find a s. v. s such that $E(x_s) =$ maximum. The problem is trivial if a or b is $\leqq 1$ since then $h(t) < 1$ and $x_0 < x_n$ for all n, so that $E(x_s) =$ max. for $s = 0$. We shall therefore assume that $a > 1$, $b > 1$.

We observe that the assumptions of Theorem 2 are satisfied by setting

$$x_n = x_n' - x_n'' = x_n^* - x_n^{**}$$

with

$$\begin{cases} x_n' = x_n^* = -h(\pi_n), \qquad \left(0 \leqq h(\pi_n) \leqq \dfrac{ab}{a+b}\right), \\ x_n'' = x_n^{**} = n, \end{cases}$$

so that $s = \sup_N s_N$ is the desired s.v. Thus Theorem 2 guarantees the existence of a Bayes solution of our decision problem.

To find the (minimal) Bayes sampling variable s requires that we compute the quantities $\beta_0^N, \beta_1^N, \ldots, \beta_N^N$ for each $N \geqq 0$ (note that in the present context we are allowed to take no observations on the y_i and to decide in favor of H_0 or H_1 with $x_0 = -h(\pi)$). We have

$$\beta_n^N = \max[x, E(\beta_{n+1}^N | \mathscr{F}_n)], \quad n = 0, 1, \ldots, N; \quad \beta_{N+1}^N = -\infty,$$

and by Theorem 2′

$$s = \text{first} \quad n \geqq 0 \quad \text{such that} \quad x_n = \beta_n = \lim_{N\to\infty} \beta_n^N.$$

Observing that

$$\pi_{n+1} = \frac{\pi_n f_0(y_{n+1})}{\pi_n f_0(y_{n+1}) + (1-\pi_n) f_1(y_{n+1})}$$

it follows easily that

$$\beta_n^N(y_1, \ldots, y_n) = \gamma_n^N(\pi_n), \qquad n = 0, 1, \ldots, N+1,$$

where

$$\gamma_n^N(t) = \max\left\{-h(t) - n, \int_{-\infty}^{\infty} \gamma_{n+1}^N \left(\frac{t f_0(y)}{t f_0(y) + (1-t) f_1(y)}\right) \quad [t f_0(y) + (1-t) f_1(y)]\right\}$$

$$(n = 0, 1, \ldots, N); \quad \gamma_{N+1}^N(t) = -\infty.$$

Now set

$$g_n^N(t) = -\gamma_n^N(t) - n, \quad n = 0, 1, \ldots, N+1.$$

Then

$$g_n^N(t) = \min[h(t), G_{n+1}^N(t) + 1],$$

where

$$G_n^N(t) = \int_{-\infty}^{\infty} g_n^N \frac{t f_0(y)}{t f_0(y) + (1-t) f_1(y)} [t f_0(y) + (1-t) f_1(y)]$$

for $n = 0, 1, \ldots, N$ with

$$g_{N+1}^N(t) = \infty.$$

Obviously,

$$g_n^N(t) = g_{n+1}^{N+1}(t), \quad g_n^N(t) \geqq g_n^{N+1}(t) \quad \text{for} \quad n = 0, 1, \ldots, N+1,$$

so that

$$\lim_{N\to\infty} g_n^N(t) = g_n(t) = g(t) \quad \text{exists.}$$

By the Lebesgue theorem of dominated convergence,

$$g(t) = \min[h(t), G(t) + 1]$$

where

$$G(t) = \int_{-\infty}^{\infty} g\left(\frac{t f_0(y)}{t f_0(y) + (1-t) f_1(y)}\right) [t f_0(y) + (1-t) f_1(y)].$$

And

$$\beta_n^N(y_1, \ldots, y_n) = \gamma_n^N(\pi_n) = -g_n^N(\pi_n) - n,$$

so that

$$\beta_n = \lim_{N\to\infty} \beta_n^N = -g(\pi_n) - n$$

and hence

$$s = \text{first} \quad n \geqq 0 \quad \text{such that} \quad g(\pi_n) = h(\pi_n); \quad E(x_s) = \beta_0 = -g(\pi).$$

We shall now investigate the nature of the function $g(t)$ which characterizes s.

If a function $a(t)$ is concave for $0 \leqq t \leqq 1$ and if

$$A(t) = \int_{-\infty}^{\infty} a\left(\frac{t f_0(y)}{t f_0(y) + (1-t) f_1(y)}\right) [t f_0(y) + (1-t) f_1(y)],$$

then it is an easy exercise to show that $A(t)$ is also concave on $0 \leqq t \leqq 1$. Since $h(t)$ is concave, $g_N^N(t) = h(t)$ is concave, and hence $G_N^N(t)$ is concave. Hence by induction all the $g_n^N(t)$ and $G_n^N(t)$ are concave, as are therefore $g(t)$ and $G(t)$. Note also that

$$g(0) = G(0) = g(1) = G(1) = 0.$$

Now put

$$\alpha_1(t) = at - G(t) - 1,$$
$$\alpha_2(t) = b(1-t) - G(t) - 1,$$
$$\alpha(t) = h(t) - G(t) - 1 = \min[\alpha_1(t), \alpha_2(t)].$$

Then for $a, b > 1$,

$$\alpha_1(0) = \alpha_2(1) = -1 < 0,$$
$$\alpha_1(1) = a - 1 > 0,$$
$$\alpha_2(0) = b - 1 > 0.$$

Since $G(t)$ is concave, $G(0) = G(1) = 0$, and at is linear, there exists a unique number $\pi' = \pi'(a, b)$ such that

$$\alpha_1(t) \begin{cases} < 0 & \text{for} \quad t < \pi' \\ = 0 & \text{for} \quad t = \pi' \\ > 0 & \text{for} \quad t > \pi' \end{cases} \qquad \left(\frac{1}{a} \leqq \pi' < 1\right).$$

Similarly, there exists a unique number $\pi'' = \pi''(a, b)$ such that

$$\alpha_2(t) \begin{cases} > 0 & \text{for} \quad t < \pi'' \\ = 0 & \text{for} \quad t = \pi'' \\ < 0 & \text{for} \quad t > \pi'' \end{cases} \qquad \left(0 < \pi'' \leqq 1 - \frac{1}{b}\right).$$

Hence

$$\begin{aligned} s &= \text{first} \quad n \geqq 0 \quad \text{such that} \quad g(\pi_n) = h(\pi_n) \\ &= \text{first} \quad n \geqq 0 \quad \text{such that} \quad h(\pi_n) \leqq G(\pi_n) + 1 \\ &= \text{first} \quad n \geqq 0 \quad \text{such that either} \quad \alpha_1(\pi_n) \quad \text{or} \quad \alpha_2(\pi_n) \leqq 0 \\ &= \text{first} \quad n \geqq 0 \quad \text{such that} \quad \pi_n \leqq \pi' \quad \text{or} \quad \pi_n \geqq \pi''. \end{aligned}$$

If $\pi'' \leqq \pi'$ then $s \equiv 0$. If $\pi' < \pi''$ then s is the first $n \geqq 0$ for which π_n does not lies in the open interval (π', π''), and the decision procedure is a Wald sequential probability ratio test.

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