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On the Asymptotic Behavior of Line Processes and Systems of Non-Interacting Particles

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1. Introduction

In many stochastic particle systems arising in applications, it seems reasonable as a first approximation to neglect the interaction between particles. (We may e.g. think of thin gases or of low density traffic.) The particles will then move with constant velocities, and the evolution of the system will be given in a space-time diagram by a random collection of straight lines, a so called line process. Thus the problem of studying the asymptotic behavior of non-interacting particle systems is equivalent to that of studying the asymptotic properties of line processes under translations. The main purpose of the present paper is to investigate this problem under various types of general assumptions.

A first result in this direction was given by Breiman in [2]. Later on, Stone [19] pointed out that Breiman's theorem follows in extended form from a general result of Dobrushin [4]. The extended version, to be referred to below as the Breiman-Stone theorem, states in essence that, if the initial distribution is stationary under translations and such that the velocities are independent of the positions and independently chosen according to some fixed absolutely continuous distribution, then the process of positions converges in distribution as time tends to infinity to a mixed Poisson process, (cf. [3, 6, 9, 21] and Theorem 6.5.9 in [15]).

If we remove the independence assumption (which is rather artificial since, if independence occurs, it will normally be destroyed immediately [14]), the classical argument fails, and we have to rely on entirely different methods. For the sake of motivation, note that if the state (positions and velocities) of a space stationary particle system converges in distribution, then the limit has to be stationary in time also, and hence must correspond to a line process which is stationary under arbitrary translations. Such processes were considered by Rollo Davidson (§§2.1 and 2.4 in [7]), who conjectured that every stationary second order line process in the plane which has a.s. no pairs of parallel lines is a Cox process, i.e. a mixture of Poisson processes. (He actually considered stationarity under arbitrary rotations, but it is equivalent to consider translations only [11].) Although Davidson's conjecture is false as stated [13], it becomes true under additional regularity

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restrictions, as was shown by Papangelou (§2.7 in [7] as well as [16], see also [17, 11]). Papangelou's method is based on the fact that Cox structure of a line or point process follows from the invariance (under suitable transformations) of the corresponding conditional intensity [16, 12]. Thus the problem of establishing the Cox nature of a line process is reduced to that of proving a.s. invariance of a (sufficiently smooth) random measure on the space of lines. Problems of the latter kind have been treated extensively by Davidson and Krickeberg (§§2.4–2.6 in [7], see also [17, 11]).

The present approach to the corresponding convergence problem is similar. Thus we show in §3 that the asymptotic Cox nature of a line or point process follows from the asymptotic invariance of the corresponding conditional intensity. Given this result, it remains to look for conditions for a random measure on the space of lines to be asymptotically invariant. Three different approaches to this problem are presented in §§4–6, the first depending on local invariance (a smoothness condition of independent interest, to be studied separately in §2), the second on mixing. Our third method uses randomization, in the sense that the system is considered at a sequence of random epochs. One of our main results (Theorem 4.1) provides a common extension of the Breiman-Stone theorem and of the best known conditions for a stationary line process in R^d, $d \geq 3$, to be Cox.

In order not to overload our exposition, we shall only consider the case of lines, although most results admit (usually trivial) extensions to flat processes in general. In most cases, no new proofs are needed, since every flat process can be identified with the corresponding marked line process of intersections with a suitable fixed flat [11]. Thus all we need is to check that our results below remain true for marked line processes. Only results related to Theorem 3.2 in [11] seem to require a direct approach.

We further remark that our results apply to certain one-dimensional systems of *colliding* particles, as defined by Harris [8]. In fact, if the collisions are elastic, the colliding particles will interchange their velocities, so the space-time diagram will be identical with the one for non-interacting particles. Assuming the initial state to be given by a space and hence also time ([11], Lemma 2.2) stationary Poisson process, and the associated independent velocities to have mean 0, Spitzer [18] proved that the path of a fixed particle converges in distribution under appropriate successive scale reductions towards a Brownian motion. Since the stationary Poisson processes are exactly the limiting processes we obtain in the ergodic case, our results imply that a similar convergence takes place under much more general initial conditions, except that we have to consider the path of a particle during the time interval $[T, \infty)$ and to let $T \to \infty$ at a suitable rate along with the scale reductions. (This is because, in an obvious sense, the paths depend continuously on the corresponding line process.) This contrasts with a counterexample in [20] for the case of fixed T.

Throughout the paper we shall assume some familiarity with the basic concepts and terminology of random measure theory, for which we refer to [10]. We shall also use the notations of [10] without further explanation. Note in particular that $\mathscr{B}(S)$ denotes the class of bounded Borel sets in S, and that $(\mathfrak{M}(S), \mathscr{M}(S))$ and $(\mathfrak{N}(S), \mathscr{N}(S))$ denote the measurable spaces of R_+- and Z_+-valued Radon measures on S. If no confusion is likely, we shall write $\mathscr{B}$, $\mathfrak{M}$, $\mathscr{M}$, $\mathfrak{N}$ and $\mathscr{N}$ for brevity. Convergence in $\mathfrak{M}$ and $\mathfrak{N}$ is with respect to the vague topologies ($\xrightarrow{v}$), and convergence in distribution ($\xrightarrow{d}$) is defined accordingly [1]. For random measures, we shall further say that $\xi_n \to \xi$ in L_1 if $\xi_n f \to \xi f$ in L_1 for all $f \in \mathscr{F}_c(S)$, the class of continuous functions on S with bounded support. Note also that $B\mu$ denotes the restriction of the measure μ to the set B. Some further conventions are to write B^c and ∂B for the complement and boundary of B, δ_x for the Dirac measure at x, var for absolute variation, $\ll$ for absolute continuity, $\stackrel{d}{=}$ for equality in distribution, and $\mathscr{L}(\cdot)$ for the linear subspace spanned by $(\cdot)$. We assume all random elements under consideration to be defined on some fixed probability space with probability P and expectation E. Some special notations for line processes will be introduced at the beginning of §4.

2. Local Invariance and Related Concepts

In this section we introduce some classes of measures on R^d, random or not, which apart from being of independent interest will be basic for the subsequent work. We also consider certain classes of ordered sets of measures with index set $T=N$ or R_+. Throughout this section we assume that u is a linear subspace of R^d of dimension $\geqq 1$.

Let us first consider a family $\mu_t \in \mathfrak{M}(R^d)$, $t \in T$, of uniformly totally bounded measures. We shall say that the μ_t are *globally asymptotically u-invariant*, if

$$\lim_{t\to\infty} \operatorname{var}(\mu_t * v - \mu_t * v * \delta_x) = 0, \qquad x \in u,$$

for every absolutely continuous probability measure v on R^d. For $u=R^d$, this coincides with the notion of "weak asymptotic uniformity" in §6.4 of [15], and it is further seen to be equivalent to condition (2.4) in [19]. Choosing the coordinate vectors $x_1, \ldots, x_d$ in R^d such that $u=\mathscr{L}(x_1, \ldots, x_k)$ and writing C_h for the cube spanned by the vectors $hx_1, \ldots, hx_d$, it may be seen from the proof of Satz 6.4.1 in [15] or Lemma 1 in [19] that the above condition is equivalent to

$$\lim_{t\to\infty} \int |\mu_t(C_h+x) - \mu_t(C_h+x+hx_i)|\, dx = 0, \qquad h>0,\ i=1, \ldots, k, \tag{1}$$

and by [19], the integration here may even be replaced by summation over all $x \in (hZ)^d$.

Given any measure $\mu \in \mathfrak{M}(R^d)$, we shall further say that μ is *locally u-invariant* if

$$\lim_{h\to 0} h^{-d} \int_B |\mu(C_h+x) - \mu(C_h+x+hx_i)|\, dx = 0, \qquad i=1, \ldots, k,\ B \in \mathscr{B}. \tag{2}$$

Comparing with (1) and writing $a_t(x) \equiv tx$, it is seen that (2) is equivalent to global asymptotic u-invariance of $(B\mu)\, a_t^{-1}$ for every bounded rectangle B. (By Lemma 2.2 below, this remains true with $B \in \mathscr{B}$ arbitrary.) Note in particular that the equivalent conditions for (1) given in [15] and [19] yield useful criteria for (2).

The last definition carries over to random measures η on R^d as follows. Writing $\|\cdot\|$ for the norm in $L_1(\mathsf{P})$ or $L_2(\mathsf{P})$ respectively, we shall say that η is *first (second)* *order locally u-invariant*, if

$$\limsup_{h\to 0} h^{-d} \int_B \|\eta(C_h+x)\|\, dx < \infty, \qquad B\in\mathscr{B}, \tag{3}$$

and if moreover

$$\lim_{h\to 0} h^{-d} \int_B \|\eta(C_h+x)-\eta(C_h+x+hx_i)\|\, dx = 0, \qquad i=1,\ldots,k,\ B\in\mathscr{B}. \tag{4}$$

Note that, by Jensen's inequality, the L_2 versions of (3) and (4) are more restrictive than those in L_1. In the L_1 case, (3) simply states that $\mathsf{E}\eta\in\mathfrak{M}$. The corresponding L_2 version will be examined in detail at the end of this section.

The classes $\mathfrak{M}_a$, $\mathfrak{M}_l^{(u)}$ and $\mathfrak{M}_d$ of absolutely continuous, locally u-invariant and diffuse measures on R^d are related as follows.

Lemma 2.1. *For any* $u \neq \{0\}$,

$$\mathfrak{M}_a \subset \mathfrak{M}_l^{(u)} \subset \mathfrak{M}_d, \tag{5}$$

and here both inclusions are strict. Similar relations hold for the corresponding classes of random measures satisfying (3).

In the present proof and throughout the paper, we shall use the *Minkowski* (type) *inequality*

$$\left\|\int |X(t)|\, \mu(dt)\right\| \leqq \int \|X(t)\|\, \mu(dt),$$

valid for arbitrary measurable processes X. For a proof in the L_2 case, note that by Fubini's theorem and Schwarz' inequality

$$\begin{aligned}
&\left\|\int |X(t)|\, \mu(dt)\right\| \\
&\quad = \left(\mathsf{E}\left(\int |X(t)|\, \mu(dt)\right)^2\right)^{1/2} = \left(\iint \mathsf{E}\, |X(s)\, X(t)|\, \mu(ds)\, \mu(dt)\right)^{1/2} \\
&\quad \leqq \left(\iint \|X(s)\|\, \|X(t)\|\, \mu(ds)\, \mu(dt)\right)^{1/2} = \int \|X(t)\|\, \mu(dt).
\end{aligned}$$

Proof of Lemma 2.1. The inclusion on the left of (5) is a particular case of Lemma 2.2 below, (see Theorem 5 in [19] or Satz 6.5.10 in [15] for a direct proof in the non-random case), while the one on the right follows (in the random case and hence in general) from the fact that, by Minkowski's inequality and Fatou's lemma,

$$\begin{aligned}
&\liminf_{h\to 0} h^{-d} \int \|\eta(C_h+x)-\eta(C_h+x+hx_i)\|\, dx \\
&\quad \geqq \liminf_{h\to 0} \left\|h^{-d} \int |\eta(C_h+x)-\eta(C_h+x+hx_i)|\, dx\right\| \\
&\quad \geqq \left\|\liminf_{h\to 0} h^{-d} \int |\eta(C_h+x)-\eta(C_h+x+hx_i)|\, dx\right\| \geqq 2 \left\|\sum_x \eta\{x\}\right\|.
\end{aligned}$$

To prove that both inclusions are strict, let us first consider the case when $d=1$. Letting $\xi_1, \xi_2, \ldots$ be independent random variables with $\mathsf{P}\{\xi_i=1\} \equiv \mathsf{P}\{\xi_i=-1\} \equiv 1/2$, and writing

$$\eta_1 = \sum_{n=1}^{\infty} n^{-1}\, 2^{-n}\, \xi_n, \qquad \eta_2 = \sum_{n=1}^{\infty} 3^{-n}\, \xi_n,$$

it may be seen that $\mu_1 = \mathsf{P}\eta_1^{-1}$ and $\mu_2 = \mathsf{P}\eta_2^{-1}$ both belong to $\mathfrak{M}_d \setminus \mathfrak{M}_a$, and that μ_1 is locally invariant while μ_2 is not. Examples for general d and u may be constructed from μ_1 and μ_2 by forming products with Lebesgue measure on R^{d-1}. □

In the next two lemmas, we shall prove two closure properties of $\mathfrak{M}_l^{(u)}$ and its stochastic counterparts which will be of constant use in subsequent sections.

Lemma 2.2. *Let ξ and η be random measures on R^d satisfying* (3), *and suppose that $\eta \ll \xi$. Then η is first (second) order locally u-invariant whenever ξ is.*

Proof. We give a proof in the second order case only, the first order case being similar but simpler. For simplicity of writing, assume without loss that ξ is supported by some fixed bounded set. We may then take $B = R^d$ in (3) and (4).

First we prove that, if $0 \leqq \eta_n \leqq \eta$ with $\eta_n \xrightarrow{v} \eta$ a.s., then

$$\lim_{n\to\infty} \limsup_{h\to 0} h^{-d} \int \|\eta(C_h+x) - \eta_n(C_h+x)\| \, dx = 0. \tag{6}$$

To see this, note first that, in the proof of (6), h may be restricted to the sequence 2^{-k}, $k \in N$, since if $2^{-k-1} < h \leqq 2^{-k} \equiv h'$,

$$h^{-d} \int \|\eta(C_h+x) - \eta_n(C_h+x)\| \, dx \leqq 2^d h'^{-d} \int \|\eta(C_{h'}+x) - \eta_n(C_{h'}+x)\| \, dx.$$

But for such h, Minkowski's inequality shows that the expression on the left of (3) is non-decreasing as $h \to 0$. Letting $\varepsilon > 0$ be arbitrary, and writing c for the limit in (3), it follows that, for $h = 2^{-k}$ small enough,

$$h^{-d} \int \|\eta(C_h+x)\| \, dx > c - \varepsilon.$$

By dominated convergence, we hence obtain for this particular h and for n large enough

$$h^{-d} \int \|\eta_n(C_h+x)\| \, dx > c - \varepsilon.$$

Using the monotonicity once more together with the fact that $\eta_n \leqq \eta$, we get

$$\lim_{n\to\infty} \liminf_{h\to 0} h^{-d} \int \|\eta_n(C_h+x)\| \, dx = c. \tag{7}$$

From the inequality $\eta_n \leqq \eta$ it is further seen that, for any $B \in \mathscr{B}$,

$$\begin{aligned}\|\eta B - \eta_n B\|^2 &= \|\eta B\|^2 + \|\eta_n B\|^2 - 2\,\mathsf{E}\,\eta B\, \eta_n B \leqq \|\eta B\|^2 - \|\eta_n B\|^2 \\ &= (\|\eta B\| + \|\eta_n B\|)(\|\eta B\| - \|\eta_n B\|) \leqq 2\,\|\eta B\|(\|\eta B\| - \|\eta_n B\|),\end{aligned}$$

so by Schwarz' inequality

$$\begin{aligned}&h^{-d} \int \|\eta(C_h+x) - \eta_n(C_h+x)\| \, dx \\ &\quad \leqq \sqrt{2}\, h^{-d} \int \|\eta(C_h+x)\|^{1/2} (\|\eta(C_h+x)\| - \|\eta_n(C_h+x)\|)^{1/2} \, dx \\ &\quad \leqq \sqrt{2}\,(h^{-d} \int \|\eta(C_h+x)\| \, dx)^{1/2} \\ &\qquad \cdot (h^{-d} \int \|\eta(C_h+x)\| \, dx - h^{-d} \int \|\eta_n(C_h+x)\| \, dx)^{1/2}.\end{aligned}$$

By (7) and the definition of c, the expression on the right tends to zero as $h=2^{-k}\to 0$ and then $n\to\infty$, which completes the proof of (6). Note in particular that we may take as η_n a random measure with uniformly bounded ξ-density. By Minkowski's inequality, we may thus assume from now on that $\eta\leqq\xi$.

Let us now consider any fixed bounded measure μ on R^d. We shall prove that the class $\mathscr{C}$ of rationally valued simple functions over the set of rectangles determined by rational coordinates is dense in $L_1(\mu)$. Since every element in $L_1(\mu)$ may be approximated from below by simple functions, it suffices to show that every indicator function may be approximated in $L_1(\mu)$ by elements in $\mathscr{C}$. But this is easily seen by a monotone class argument. In particular, any function in $L_1(\mu)$ which takes values in the interval [0, 1] may be approximated by $\mathscr{C}$-functions with the same property. Let $f_1, f_2, \ldots$ be an enumeration of all such functions in $\mathscr{C}$.

Applying the last result to $L_1(\xi)$ for any fixed outcome of ξ, it is seen that $\inf_n \operatorname{var}(\eta-f_n\xi)=0$ a.s., so writing η_n for the first measure among $f_k\xi$, $k=1,\ldots,n$, minimizing $\operatorname{var}(\eta-f_k\xi)$, we get

$$\lim_{n\to\infty}\operatorname{var}(\eta-\eta_n)=0 \quad \text{a.s.}$$

Here it should be noticed that the minimizing index k is measurable, and hence that η_n is a random measure for each n. Write $\xi_n=\xi-|\eta-\eta_n|$, where $|\eta-\eta_n|B\equiv\operatorname{var}(B\eta-B\eta_n)$, and note that (6) applies to ξ and ξ_n, yielding

$$\begin{aligned}&\lim_{n\to\infty}\limsup_{h\to 0} h^{-d}\int\|\eta(C_h+x)-\eta_n(C_h+x)\|\,dx\\ &\quad\leqq \lim_{n\to\infty}\limsup_{h\to 0} h^{-d}\int\|\xi(C_h+x)-\xi_n(C_h+x)\|\,dx=0.\end{aligned}$$

Hence, by Minkowski's inequality, it suffices to prove (4) with η replaced by η_n for fixed n. By the definition of η_n, it is then enough to show that $I\xi$ satisfies (4) for any fixed rectangle I. But this follows from the fact that ξ has a.s. no mass on any fixed flat which is not u-invariant. (To see this, apply the right-hand inclusion in Lemma 2.1 to a suitable projection of ξ.) □

To state the next closure property, say that a transformation $f\colon R^d\to R^d$ is *locally affine*, if f has a unique inverse f^{-1}, and if f and f^{-1} have continuous first order partial derivatives forming matrices with non-zero determinants. We shall further say, for brevity, that f is *u-preserving*, if the class of u-parallel flats is invariant under f.

Lemma 2.3. *Let $f\colon R^d\to R^d$ be locally affine and u-preserving. Then the random measures η and ηf^{-1} are simultaneously first (second) order locally u-invariant.*

Proof. We shall only consider the non-random case, the random case being similar. Hence suppose that $\mu\in\mathfrak{M}_l^{(u)}$. We have to verify that

$$\lim_{h\to 0} h^{-d}\int_C|\mu f^{-1}(C_h+x)-\mu f^{-1}(C_h+hv+x)|\,dx=0,$$

where v is anyone of the vectors $x_1,\ldots,x_k$, while C is an arbitrary compact set. The Jacobian of f being bounded, it suffices to prove that

$$\lim_{h\to 0} h^{-d}\int_{C'}|\mu f^{-1}(C_h+f(x))-\mu f^{-1}(C_h+hv+f(x))|\,dx=0, \tag{8}$$

where $C' = f^{-1}C$. Fix an $\varepsilon > 0$, and let $x_0 \in C'$ be arbitrary. Let f_0 be the affine transformation with the property that the values at x_0 of f and its first order partial derivatives coincide with the corresponding quantities for f_0. Define

$$C'_h = f_0^{-1}(C_h + f(x_0)) - x_0, \qquad v' = h^{-1}[f_0^{-1}(hv + f(x_0)) - x_0],$$

and write $(\partial C'_h)_{\varepsilon h}$ for the set of points at a distance $\leqq \varepsilon h$ from $\partial C'_h$. Since f is locally affine, the relations

$$f^{-1}(C_h + f(x)) \Delta (C'_h + x) \subset (\partial C'_h)_{\varepsilon h} + x$$

and

$$f^{-1}(C_h + hv + f(x)) \Delta (C'_h + hv' + x) \subset (\partial C'_h)_{\varepsilon h} + hv' + x$$

hold for all x in some neighbourhood B of x_0 and for all sufficiently small $h > 0$. Letting $\{B_i\}$ be a finite disjoint covering of C' by sets B of this type, we get

$$\begin{aligned}
&\Big|h^{-d} \int_{C'} |\mu f^{-1}(C_h + f(x)) - \mu f^{-1}(C_h + hv + f(x))|\, dx \\
&\qquad - h^{-d} \sum_i \int_{B_i} |\mu(C'_h + x) - \mu(C'_h + hv' + x)|\, dx\Big| \\
&\quad \leqq h^{-d} \sum_i \int_{B_i} [\mu((\partial C'_h)_{\varepsilon h} + x) + \mu((\partial C'_h)_{\varepsilon h} + hv' + x)]\, dx \\
&\quad \sim \varepsilon \sum_i (\mu * v_{h,\varepsilon} + \mu * v_{h,\varepsilon} * \delta_{-hv'}) B_i \sim \varepsilon \sum_i \mu B_i = \varepsilon \mu C'
\end{aligned}$$

as $h \to 0$ (provided $\mu \partial B_i \equiv 0$), where $v_{h,\varepsilon}$ denotes the uniform distribution over the set $-(\partial C'_h)_{\varepsilon h}$. Since moreover

$$\lim_{h \to 0} h^{-d} \sum_i \int_{B_i} |\mu(C'_h + x) - \mu(C'_h + hv' + x)|\, dx = 0$$

by (2), it follows that the left-hand side of (8) is bounded by $\varepsilon \mu C'$. To complete the proof, it remains to let $\varepsilon \to 0$. □

We proceed to introduce still another notion of asymptotic invariance. The random measures η_t, $t \in T$, are said to be *first (second) order asymptotically u-invariant*, if for all $f \in \mathscr{F}_c(R^d)$ and $x \in u$,

$$\limsup_{t \to \infty} \|\eta_t f\| < \infty, \qquad \lim_{t \to \infty} \|\eta_t f - (\delta_x * \eta_t) f\| = 0. \tag{9}$$

(The same definition applies to arbitrary subgroups u.) Under (9), the family $\{\eta_t\}$ is automatically relatively compact in distribution as $t \to \infty$, and every limit is a.s. u-invariant. The following simple lemmas will be useful below.

Lemma 2.4. *Let ϑ be a bounded random element in R^d which is independent of η_t for each t. Then η_t and $\delta_\vartheta * \eta_t$ are simultaneously first (second) order asymptotically u-invariant.*

Proof. If η_t is first (second) order asymptotically u-invariant, then so is $\delta_\vartheta * \eta_t$ by Fubini's theorem and dominated convergence. To prove the converse, assume without loss that the support of $\mathsf{P}\vartheta^{-1}$ contains the origin, and suppose for $k=1$ or 2 that $\delta_\vartheta * \eta_t$ is k-th order asymptotically u-invariant while η_t is not. Then (9) fails for some f and x, so we get for some sequence $t_n \to \infty$ and some $\varepsilon > 0$

$$\|\eta_{t_n} f - (\delta_x * \eta_{t_n}) f\| > \varepsilon, \qquad n \in N.$$

Writing $f_y \equiv f(\cdot - y)$, it is further seen from the L_k-boundedness of η_t and the uniform continuity of f that

$$\|\eta_t(f-f_y)\| < \frac{\varepsilon}{4}, \qquad \|(\delta_x * \eta_t)(f-f_y)\| < \frac{\varepsilon}{4},$$

for all t and for all y belonging to some neighbourhood G of the origin. Hence, by combination,

$$\|(\delta_y * \eta_{t_n}) f - (\delta_{y+x} * \eta_{t_n}) f\| > \frac{\varepsilon}{2}, \qquad n \in N, \; y \in G.$$

But by Fubini's theorem, this yields the contradiction

$$\|(\delta_\vartheta * \eta_{t_n}) f - (\delta_{\vartheta+x} * \eta_{t_n}) f\| \geqq \frac{\varepsilon}{2} (\mathsf{P}\{\vartheta \in G\})^{1/k} > 0, \qquad n \in N. \quad \square$$

Lemma 2.5. *Let* $\nu_1, \nu_2, \ldots$ *be probability measures on* R^d *with uniformly bounded supports and such that* $\nu_n \xrightarrow{w} \delta_0$. *Then* η_t *is first (second) order asymptotically u-invariant iff* $\nu_n * \eta_t$ *is so for every n.*

Proof. Since clearly

$$(\delta_x * (\nu_n * \eta_t)) f \equiv (\delta_x * \eta_t)(\nu_n * f),$$

the "if assertion" follows from the L_1 (L_2) boundedness of η_t and the fact that $\nu_n * f$ tends uniformly to f for all $f \in \mathscr{F}_c$, while the "only if assertion" holds since $f \in \mathscr{F}_c$ implies $\nu_n * f \in \mathscr{F}_c$, $n \in N$. $\square$

We conclude this section with a closer study of (3).

Lemma 2.6. *If* (3) *holds in* L_2, *there exists a measure* $\|\eta\| \in \mathfrak{M}$ *satisfying*

$$\|\eta\| f = \lim_{h \to 0} h^{-d} \int \|\eta(C_h + x)\| f(x)\, dx, \qquad f \in \mathscr{F}_c, \tag{10}$$

and moreover, $\|\eta f\| \leqq \|\eta\| f$ *holds for any measurable* $f \geqq 0$. *Finally, diffuseness of* $\|\eta\|$ *implies that* η *is a.s. diffuse.*

If $\eta \ll \mu$ for some non-random measure $\mu \in \mathfrak{M}$ and if $Y = d\eta/d\mu$, then it is easily verified that $\|\eta\| \ll \mu$ with density $\|Y\|$. This shows in particular that (10) is equivalent to $\|\eta * \nu_h\| \xrightarrow{v} \|\eta\|$, where ν_h denotes the uniform probability measure on C_h. If $\|\eta\|$ exists and is diffuse, we shall say that η is L_2-*regular*. Note that second order local u-invariance implies L_2-regularity for any fixed u. We finally point out that Lemmas 2.1–2.3 have obvious counterparts for L_2-regularity in place of local invariance.

Proof of Lemma 2.6. Let $\{C_{nj}\}$ be a null-array of partitions of R^d into cubes of side 2^{-n}, and let $\mathscr{U}$ denote the ring of all finite unions of cubes C_{nj}. Further introduce, for fixed n, the measure $\mu_n \in \mathfrak{M}$ assigning mass $\|\eta C_{nj}\|$ to the midpoint (say) of C_{nj} for all j. By Minkowski's inequality, the sequence $\mu_n U$ is then eventually non-decreasing for every fixed $U \in \mathscr{U}$, and it is easily seen from (3) that the limit is finite. A simple approximation then shows that $\mu_n f$ converges for each $f \in \mathscr{F}_c$. But this means that $\mu_n \xrightarrow{v}$ some $\mu \in \mathfrak{M}$. It may further be seen by a simple approximation argument based on Minkowski's inequality that the right-hand side of (10) is asymptotically equal to $\mu_n f$ as $h \to 0$ and $n \to \infty$. Thus (10) holds with $\|\eta\| = \mu$.

This relation being valid for any choice of origin in R^d, we may assume the C_{nj} to be $\|\eta\|$-continuity sets. For any $U \in \mathscr{U}$ we then obtain $\mu_n U \to \|\eta\| U$, which implies $\|\eta U\| \leqq \|\eta\| U$ by Minkowski's inequality. Now the class $\mathscr{D}$ of $\mathscr{B}$-sets satisfying the latter inequality is clearly closed under monotone limits, so by a standard monotone class theorem (A2.2 in [10]) we get $\mathscr{D} = \mathscr{B}$. Thus, by Minkowski's inequality, $\|\eta f\| \leqq \|\eta\| f$ holds for simple functions $f \geqq 0$ over $\mathscr{B}$, and the final extension to arbitrary measurable functions $f \geqq 0$ is accomplished by monotone convergence.

The last assertion follows easily from Čebyšev's inequality and Theorem 2.5 in [10]. □

Added in Proof. As will be shown elsewhere, $\|\eta\|$ exists iff $\eta \ll E\eta$ a.s. and moreover $\|d\eta/dE\eta\|$ is locally $E\eta$-integrable. In this case, η is second order locally invariant iff $E\eta$ is locally invariant.

3. Conditional Intensities

Let ξ be a simple point process on R^d, and denote by ζ the conditional intensity of ξ, as defined in [12]. As shown by Papangelou [16, 17] and myself [12], a.s. invariance of ζ in some fixed direction implies that ξ is a Cox process directed by ζ. It is actually enough to require ζ to be h-invariant for any fixed $h \in R^d \setminus \{0\}$, provided that ξ satisfies the regularity condition (Σ) in [12]. Define η as in [12], § 3.

Lemma 3.1. *Suppose that ξ satisfies (Σ), and let χ be an a.s. h-invariant random measure on R^d. Then $\zeta = \chi$ a.s. iff ξ is a Cox process directed by χ.*

Proof. Suppose that ζ is a.s. h-invariant, and assume without loss that $h = (1, 0, \ldots, 0)$. Let ξ^* be a point process on $[0, 1) \times R^d$ obtained from ξ by attaching independent marks to its atoms according to the uniform distribution on $[0, 1)$, and note that ξ^* has conditional intensity $\lambda \times \eta$, where λ denotes Lebesgue measure on $[0, 1)$, (cf. the proof of Theorem 4.2 in [12]). Let us further define the mapping f of $[0, 1) \times R^d$ onto itself by

$$f(u, x_1, \ldots, x_d) = (x_1 - [x_1], [x_1] + u, x_2, \ldots, x_d), \qquad u \in [0, 1), \; x_1, \ldots, x_d \in R,$$

and note that $\xi^* f^{-1}$ has conditional intensity $(\lambda \times \eta) f^{-1}$. Now the latter random measure is clearly a.s. x_1-invariant, and so we may conclude from Theorem 5.1 in [12] that $\xi^* f^{-1}$ is a Cox process directed by $(\lambda \times \eta) f^{-1}$. Since f is $1-1$, it follows that ξ^* is a Cox process directed by $\lambda \times \eta$, and so ξ is a Cox process directed by η, and we get $\zeta = \eta$. The converse assertion is obvious. □

The remainder of this section is devoted to some continuity results related to Lemma 3.1. Suppose that $T=N$ or R_+, and let ξ_t, $t\in T$, be simple point processes on R^d with conditional intensities ζ_t, $t\in T$. Given any random measure ζ, we write $\hat{\zeta}$ for a Cox process directed by ζ. We further denote by $\mathscr{Q}_t$ the completed remote σ-field of ξ_t, i.e. the P-completion of $\bigcap_{B\in\mathscr{B}}\sigma(B^c\xi_t)$. The η_t are defined as in [12], § 3.

We first improve and extend Theorem 5.2 in [12]:

Lemma 3.2. *Suppose that the ξ_t satisfy (Σ), and that $\zeta_t\to$ some ζ in L_1, where ζ is a.s. diffuse, $\mathscr{Q}_t$-measurable for each t and such that $\zeta R^d=0$ or ∞ a.s. Then $\xi_t\xrightarrow{d}\hat{\zeta}$.*

In the applications we have in mind, ξ_t is the state (in the phase space) of our particle system at time t, and it is assumed that ξ_t is h-stationary for some $h\in R^d\setminus\{0\}$. Suppose we can show in this case that $\zeta_t\to\zeta$ in L_1 for some a.s. h-invariant random measure ζ. The measurability hypothesis of the lemma is then fulfilled, since every ζ-event A is ξ_t-measurable for arbitrary t and as such a.s. invariant under h-shifts of ξ_t. Hence it can be approximated by sets in $\sigma(B^c\xi_t)$ for any fixed $B\in\mathscr{B}$, which implies that $A\in\mathscr{Q}_t$.

Proof. First suppose that ζ is non-random. Since ζ is diffuse, the atom sizes of ζ_t must tend to zero, so $\eta_t-\zeta_t\xrightarrow{P}0$ by (3.3) in [12], and we get $\eta_t\to\zeta$ in L_1 since $E\eta_t=E\zeta_t\to\zeta$. Defining ξ_t^* by randomization as above and using Theorem 5.2 in [12] it follows that $\xi_t^*\xrightarrow{d}\widehat{(\lambda\times\zeta)}$, and so $\xi_t\xrightarrow{d}\hat{\zeta}$, as asserted.

In the general case, note that the assumed convergence implies that

$$\mathsf{E}[|\zeta_t f-\zeta f||\zeta]\to 0 \quad \text{in } L_1, \quad f\in\mathscr{F}_c.$$

Thus, given any sequence $T'\subset T$ tending to infinity, there exists some subsequence T'' such that, for fixed $f\in\mathscr{F}_c$,

$$\mathsf{E}[|\zeta_t f-\zeta f||\zeta]\to 0 \text{ a.s.} \quad (t\in T'').$$

Since this can be made valid with a common exceptional null-event for any countable family of functions $f\in\mathscr{F}_c$, and hence for all $f\in\mathscr{F}_c$, we may assume that $\zeta_t\to\zeta$ in L_1 $(t\in T'')$ a.s., conditionally on ζ. It may further be seen from the measurability assumption and from the definition of conditional intensities as a.s. limits that ζ_t remains a.s. the conditional intensity of ξ_t even after conditioning on ζ. Applying the assertion for non-random ζ, we may then conclude that $\xi_t\xrightarrow{d}\hat{\zeta}$ $(t\in T'')$ a.s., conditionally on ζ, and by dominated convergence, this implies the corresponding statement for the unconditional distributions. Since T' was arbitrary, it is seen from Theorem 2.3 in [1] that the convergence remains valid along T. □

We shall also establish a condition for ξ_t to be asymptotically Cox.

Lemma 3.3. *Suppose that the ξ_t satisfy (Σ) and that η_t is first order asymptotically h-invariant. Then $\{\xi_t\}$ is relatively compact in distribution, and all its limit points are Cox processes directed by h-invariant random measures.*

Proof. By assumption, $\mathsf{E}\xi_t B=\mathsf{E}\zeta_t B$ is bounded for every $B\in\mathscr{B}$, so the relative compactness of $\{\xi_t\}$ follows by Lemma 4.5 in [10]. Assume that $\xi_t\xrightarrow{d}\chi$ as $t\to\infty$ along some sequence. In proving that χ is Cox, we may assume that the η_t are a.s.

diffuse, since we may otherwise consider the corresponding randomized point processes ξ_t^* defined in the preceding proofs. In that case, η_t solves the integral equation in § 3 of [12], i.e.

$$\mathsf{E}[\xi_t B;\ \xi_t-\delta_{\tau_{t,B}}\in M]=\mathsf{E}[\eta_t B;\ \xi_t\in M],\qquad B\in\mathscr{B},\ M\in\mathscr{N},\ t\in T,$$

where $\tau_{t,B}$ denotes the position of a randomly chosen atom of $B\xi_t$, if any. Hence, writing $B'=B+h$, the asymptotic invariance of η_t is seen to imply

$$\begin{aligned}&|\mathsf{E}[\xi_t B;\ \xi_t-\delta_{\tau_{t,B}}\in M]-\mathsf{E}[\xi_t B';\ \xi_t-\delta_{\tau_{t,B'}}\in M]|\\&\quad=|\mathsf{E}[(\eta_t B-\eta_t B');\ \xi_t\in M]|\leqq \mathsf{E}|\eta_t B-\eta_t B'|\to 0,\end{aligned}\tag{1}$$

at least for rectangular B.

Now suppose that $\chi\,\partial B=\chi\,\partial B'=0$ and $\chi\notin\partial M$ a.s. Then the integrands on the left side of (1) tend in distribution to the corresponding expressions with χ in place of ξ_t and with τ_B and $\tau_{B'}$ in place of $\tau_{t,B}$ and $\tau_{t,B'}$ respectively, τ_B being the position of a randomly chosen atom of $B\chi$. If moreover $M\subset\{\mu\colon \mu(B\cup B')\leqq k\}$ for some $k<\infty$, we hence obtain from (1) by uniform integrability

$$\mathsf{E}[\chi B;\ \chi-\delta_{\tau_B}\in M]=\mathsf{E}[\chi B';\ \chi-\delta_{\tau_{B'}}\in M],$$

or equivalently, since $\mathsf{E}\chi$ exists by the Fatou type lemma of weak convergence theory [1],

$$\int_B \mathsf{P}\{\chi_s-\delta_s\in M\}\,\mathsf{E}\chi(ds)=\int_{B+h}\mathsf{P}\{\chi_s-\delta_s\in M\}\,\mathsf{E}\chi(ds),\tag{2}$$

where χ_s is distributed according to the Palm distribution of χ relative to the point s, (cf. [10]). By monotone convergence, the boundedness assumption imposed on M may now be removed, and (2) may then be extended to arbitrary $B\in\mathscr{B}$ and $M\in\mathscr{N}$ by a standard monotone class argument.

By the a.e. uniqueness of Palm distributions, it is seen from (2) that $\mathsf{P}(\chi_s-\delta_s)^{-1}$ is a.e. h-invariant in s. Since moreover $\mathsf{E}\chi$ is h-invariant by (2), it follows from Lemma 4.3 in [11] that χ is a Cox process of the stated form. □

For our needs in § 6, we extend the last two results to the case of random indices. For this randomization to make sense, we add the assumption that the measure valued random process ξ_t, $t\in T$, be measurable.

Lemma 3.4. *Let $\tau_1,\tau_2,\ldots$ be T-valued random variables independent of $\{\xi_t\}$. Then Lemma 3.2 remains true with $\zeta_t\to\zeta$ and $\xi_t\xrightarrow{d}\hat\zeta$ replaced by $\zeta_{\tau_n}\to\zeta$ and $\xi_{\tau_n}\xrightarrow{d}\hat\zeta$ respectively. Similarly, the conclusion of Lemma 3.3 remains true for ξ_{τ_n}, provided the ξ_t are L_1-bounded and satisfy (Σ) while η_{τ_n} is first order asymptotically h-invariant.*

Proof. In case of Lemma 3.2, we may proceed as in the proof of that lemma to reduce to the case of non-random $\{\tau_n\}$ by conditioning on $\{\tau_n\}$ (rather than on ζ).

To prove the randomized version of Lemma 3.3, let ρ be a metrization of the weak topology in the space of point processes on R^d, and denote by $\mathscr{C}$ the class of Cox processes on R^d directed by h-invariant random measures. Since the sequence

ξ_{τ_n} is L_1-bounded and hence relatively compact in distribution, and since moreover $\mathscr{C}$ is closed (cf. Exercise 4.5 in [10]), it is enough to prove that

$$\rho(\mathsf{P}\,\xi_{\tau_n}^{-1}, \mathscr{C}) \to 0. \tag{3}$$

For this purpose, consider an arbitrary subsequence $N' \subset N$, and proceed as in the proof of Lemma 3.2 to show that, for n belonging to some further subsequence $N'' \subset N'$, η_{τ_n} is a.s. first order asymptotically h-invariant, conditionally on $\{\tau_n\}$. (This is where we need the L_1-boundedness of $\{\xi_t\}$.) Thus it follows by Lemma 3.3 that (3) is conditionally true, in the sense that

$$\rho(P_{\tau_n}, \mathscr{C}) \to 0 \quad \text{a.s.} \qquad (n \in N''), \tag{4}$$

where $P_t = \mathsf{P}\,\xi_t^{-1}$. (Note that $\rho(P_t, \mathscr{C})$ is a measurable function of t, because of the assumed measurability of $\{\xi_t\}$.)

We now choose a specific metric of the form

$$\rho(\mathsf{P}\,\xi^{-1}, \mathsf{P}\,\eta^{-1}) = \sum_{k=1}^{\infty} 2^{-k} |\mathsf{E}\,g_k(\xi) - \mathsf{E}\,g_k(\eta)|, \tag{5}$$

where the g_k are suitable vaguely continuous and uniformly bounded functionals on $\mathfrak{N}$. This ρ being bounded, it is seen from (4) that

$$\mathsf{E}\,\rho(P_{\tau_n}, \mathscr{C}) \to 0 \qquad (n \in N''). \tag{6}$$

Let us further choose a dense sequence $\{Q_i\}$ in $\mathscr{C}$, and write $P_t^{(m)}$ for the first distribution among $Q_1, \ldots, Q_m$ minimizing $\rho(P_t, Q_i)$ for fixed t. Introducing a random element $\xi_t^{(m)}$ with distribution $P_t^{(m)}$, we get by (5)

$$\begin{aligned}\rho(\mathsf{P}\,\xi_{\tau_n}^{-1}, \mathscr{C}) &\leqq \rho(\mathsf{E}\,P_{\tau_n}, \mathsf{E}\,P_{\tau_n}^{(m)}) = \sum_k 2^{-k} |\mathsf{E}[g_k(\xi_{\tau_n}) - g_k(\xi_{\tau_n}^{(m)})]| \\ &\leqq \mathsf{E} \sum_k 2^{-k} |\mathsf{E}[g_k(\xi_{\tau_n}) - g_k(\xi_{\tau_n}^{(m)}) \mid \tau_n]| = \mathsf{E}\,\rho(P_{\tau_n}, P_{\tau_n}^{(m)}),\end{aligned}$$

the measurability requirements needed here being trivially fulfilled. By (6), the right-hand side tends to 0 as $m \to \infty$ and then $n \to \infty$ along N''. Thus (3) holds for $n \in N''$, and since N' was arbitrary, it remains true for $n \in N$. □

We finally remark that the assumption that (Σ) be fulfilled, which was made throughout this section, can usually be removed by considering instead of the processes ξ_t their p-thinnings ξ_t', which will automatically satisfy (Σ), provided that $p < 1$. Moreover, ξ_t and ξ_t' will simultaneously converge in distribution to Cox processes. A similar remark applies to the assumption that all point processes involved be simple. (Cf. the proof of Theorem 4.4 below.)

4. Results in Case of Local Invariance

The remainder of the paper will be devoted to line processes in R^d, where d is a fixed integer $\geqq 2$. By a line process we mean a point process ξ on the space L of lines in R^d, (cf. [7, 11, 16]). Already this definition requires a parametrization of L, the most convenient one for our purposes being the *phase representation* [13, 14], according to which a line is represented by its point of intersection q with a fixed hyperplane u (i.e. a $(d-1)$-dimensional affine subspace of R^d), and by its direction p, the latter being defined as the rate of change in q when u is moved in the direction of its normal. (Note that lines parallel to u have no such representation. This restricts the choice of u.) In this way, ξ may be regarded as a point process in $R^{2(d-1)}$. We shall always assume that ξ is a.s. simple and locally finite, the latter meaning that the set of lines going through a bounded region always carries finite mass.

Regarding ξ as a point process, it is clear how to define the corresponding conditional intensity ζ, and further how properties of ζ like a.s. invariance, first order asymptotic invariance, and L_1-convergence towards an a.s. invariant random measure enable us to draw conclusions about ξ, using the results of § 3. Since conditions ensuring asymptotic invariance etc. have independent interest (cf. §§ 2.4 and 2.6 in [7], and also [11]), we shall forget about the connection with (discrete) line processes, and consider arbitrary (locally finite) random measures η on L. The application of our results to line processes is usually left to the reader. In that connection, note that strict stationarity of ξ carries over to ζ, and further that $\mathsf{E}\zeta = \mathsf{E}\xi$ whenever $\mathsf{E}\xi \in \mathfrak{M}$, (cf. Theorem 4.2 in [12]).

As in [11], we shall write Φ_k for the set of k-dimensional linear subspaces of R^d, and put $\Phi = \cup\, \Phi_k$. Similarly, $\Phi_k(u)$ and $\Phi(u)$ will denote the corresponding sets of subspaces of $u \in \Phi$. For any $x \in L$, we write πx for the element in Φ_1 which is parallel to x, and refer to πx as the *direction* of x. We assume once and for all that $u \in \Phi_{d-1}$ and $v \in \Phi$, and further that H is a closed subgroup of R^d. The letter y is reserved for a unit vector in R^d perpendicular to u. Convolutions correspond by definition to addition in the basic space R^d. Note in particular that convolution by δ_x is equivalent to translation by x. Otherwise, measures on L and Φ should usually be thought of as defined on the phase space $R^{2(d-1)}$ and its p-projection R^{d-1}. This applies in particular to the notion of local invariance. Note that, by Lemma 2.3, the choice of reference plane u is immaterial for the definition, as long as $\mu\pi^{-1}u = 0$ or $\eta\pi^{-1}u = 0$ a.s. respectively. In the general case, local v-invariance is by definition equivalent to local x-invariance for any set of lines x spanning v.

If η is strictly H-stationary for some H spanning u, then the a.s. u-average $\bar{\eta}_u$ of η exists in the sense of the pointwise ergodic theorem (though it may be infinite), and it is easily seen that $\bar{\eta}_u = \mathsf{E}[\eta \mid \mathscr{I}_u]$, where $\mathscr{I}_u$ is the σ-field of all u-invariant η-events. In case of L_2-stationarity, the corresponding L_2-average $\bar{\eta}_u'$ exists according to the mean ergodic theorem [5], and $\bar{\eta}_u'$ equals the projection of η onto the Hilbert space spanned by all u-invariant linear η-functionals. Since these averages must coincide a.s. if both exist, we shall henceforth use $\bar{\eta}_u$ as a common notation.

Most results for general random measures η on L will be given in two versions, one involving strict stationarity, first order local and asymptotic invariance, strong mixing, L_1-convergence, etc., the other involving the corresponding L_2-concepts. In order to avoid repetitions, and also to stress the analogy between the two cases, we state both versions together with the modifications in the L_2-case within parentheses. As in § 2, it will be convenient to use $\|\cdot\|$ as a common notation for the norms in $L_1(\mathsf{P})$ and $L_2(\mathsf{P})$. Two further conventions are to write ρ for Euclidean distance and λ for Lebesgue measure.

The remainder of this section is devoted to results obtainable under local invariance conditions. Our main result for this case is given first:

Theorem 4.1. *Assume that $v \subset u = \mathscr{L}(H)$. Further suppose that η is strictly (second order) H-stationary, and such that $(B\eta)\pi^{-1}$ is first (second) order locally v-invariant for every $B \in \mathscr{B}(L)$. Then $\delta_x * \eta$ is first (second) order asymptotically v-invariant as $\rho(x, u) \to \infty$. In the second order case we have even $\delta_x * \eta \to \bar{\eta}_u$ in L_2, provided that $v = u$ and $\eta\pi^{-1}u = 0$ a.s.*

Note that, by Lemma 2.2, the conditions of local invariance, here and in similar cases below, need only be verified for a fixed covering class of open sets $B \in \mathscr{B}(L)$.

Proof. If ϑ is independent of η and uniformly distributed over the quotient group u/H, then $\delta_\vartheta * \eta$ becomes strictly (second order) u-stationary. (Here $\delta_\vartheta * \eta$ may be defined in different ways, but our statement remains true for any reasonable choice of definition.) To see this, it suffices to consider the case $d = 1$, to take $H = Z$ and to let ϑ be uniformly distributed over the interval $[0, 1]$. If in this case η is strictly H-stationary, we get for any non-negative measurable functional f on $\mathfrak{M}(L)$ and for any $r \in R$,

$$\begin{aligned} \mathsf{E} f(\delta_{r+\vartheta} * \eta) &= \mathsf{E}\,\mathsf{E}[f(\delta_{r+\vartheta} * \eta) | \vartheta] = \mathsf{E}\,\mathsf{E}[f(\delta_{\vartheta'} * \eta) | \vartheta] \\ &= \mathsf{E}\,\mathsf{E}[f(\delta_\vartheta * \eta) | \vartheta] = \mathsf{E} f(\delta_\vartheta * \eta), \end{aligned}$$

where $\vartheta' \equiv r + \vartheta - [r + \vartheta]$, which is seen to be distributed as ϑ, independently of η. Thus $\delta_\vartheta * \eta$ is indeed strictly stationary. In the second order case, we may apply the same argument to functionals f of the form

$$f(\mu) = (\mu B)(\mu C), \qquad B, C \in \mathscr{B}(L).$$

Next note that, by Lemma 2.2, the first (second) order local v-invariance carries over from η to $\delta_\vartheta * \eta$. Hence, if the first assertion of the theorem is known to be true in the case $H = u$, we may conclude that $\delta_{\vartheta+x} * \eta$ is first (second) order asymptotically v-invariant, and by Lemma 2.4, this remains true for $\delta_x * \eta$. A similar argument applies to the second assertion. We may thus assume from now on that $H = u$. Under this assumption, it is further clear from the stationarity of η and the invariance of $\bar{\eta}_u$ that the assertions need to be proved only for x of the form ty with $t > 0$.

Now suppose that the theorem has been proved for $\eta' \equiv (\pi^{-1}u)\eta = 0$ a.s., and consider the general case. We may then apply our theorem to the projection onto u of the restriction $\eta'_B = (u + By)\eta'$ for arbitrary $B \in \mathscr{B}(R)$ to conclude that η' is asymptotically first (second) order u-invariant under u-translations. Since it is further strictly (second order) u-stationary by assumption, it is in fact a.s. u-invariant, and this implies the desired asymptotic invariance of $\delta_{ty} * \eta'$. We may thus assume without loss that $\eta\pi^{-1}u \equiv 0$, which enables us to use the phase representation based on u.

Let ν be an absolutely continuous probability measure on R^{d-1} with density f, interpret ν as a distribution on u, and write for brevity $(\nu \times \delta_0) * \eta = \nu * \eta$. Then clearly

$$(\nu * \eta)(B \times C) = \int_B \eta_q C\, dq, \qquad B, C \in \mathscr{B}(R^{d-1}), \tag{1}$$

where $\{\eta_q\}$ is the measurable, $\mathfrak{M}(R^{d-1})$-valued and strictly (second order) stationary random process on R^{d-1} given by

$$\eta_q C = \int_{R^{d-1}} f(q-s)\,\eta(ds \times C), \qquad q \in R^{d-1},\ C \in \mathscr{B}(R^{d-1}). \tag{2}$$

By a simple geometric argument,

$$(\delta_{ty} * v * \eta)(B \times C) = (v * \eta)\left\{(q,p):\ p \in \frac{-B+q}{t} \cap C\right\},$$

and similarly for $\delta_{ty+s} * v * \eta$, $s \in v$, so by (1), Minkowski's inequality and the stationarity of $\{\eta_q\}$,

$$\begin{aligned}
&\|[(\delta_{ty} - \delta_{ty+s}) * v * \eta](B \times C)\| \\
&\quad = \left\| \int \left[C\eta_q \left(\frac{-B+q}{t}\right) - C\eta_q \left(\frac{-B+s+q}{t}\right)\right] dq \right\| \\
&\quad \leqq \int \left\| C\eta_q \left(\frac{-B+q}{t}\right) - C\eta_q \left(\frac{-B+s+q}{t}\right)\right\| dq \\
&\quad = \int \left\| C\eta_0 \left(\frac{-B+q}{t}\right) - C\eta_0 \left(\frac{-B+s+q}{t}\right)\right\| dq .
\end{aligned} \tag{3}$$

Now it is seen from (2) that η_0 is first (second) order locally v-invariant, and hence so is $C\eta_0$ on account of Lemma 2.2. Thus the right-hand side of (3) tends to zero as $t \to \infty$, at least for rectangular B. A similar argument shows that $\|(\delta_{ty} * v * \eta)(B \times C)\|$ is bounded as $t \to \infty$. Thus $\delta_{ty} * v * \eta$ is first (second) order asymptotically v-invariant, and hence so is $\delta_{ty} * \eta$ by Lemma 2.5, since v was arbitrary.

Turning to the second assertion, suppose we can show that

$$\lim_{r \to \infty} \sup_t \|(\delta_{ty} * v_r * \eta - \bar{\eta}_u) A\| = 0, \qquad A = B \times C, \tag{4}$$

for any rectangles $B, C \in \mathscr{B}(R^{d-1})$, where v_r denotes the uniform probability measure over C_r. Combining (4) with the fact that, by the second order asymptotic invariance,

$$\lim_{t \to \infty} \|(\delta_{ty} * (v_{nh} - v_h) * \eta) A\| = 0, \qquad h > 0,\ n \in N,$$

we obtain

$$\|(\delta_{ty} * v_h * \eta - \bar{\eta}_u) A\| \leqq \|(\delta_{ty} * (v_h - v_{nh}) * \eta) A\| + \|(\delta_{ty} * v_{nh} * \eta - \bar{\eta}_u) A\| \to 0$$

as $t \to \infty$ and then $n \to \infty$, and since h was arbitrary, we may conclude that indeed $\delta_{ty} * \eta \to \bar{\eta}_u$ in L_2.

Next consider an arbitrary null-array $\{C_{nj}\}$ of partitions of C, (cf. [10]), and let $p_{nj} \in C_{nj}$ be fixed. Then

$$\begin{aligned}
&\|(\delta_{ty} * v_r * \eta - \bar{\eta}_u)(B \times C)\| \\
&\quad = \|(v_r * \eta - \bar{\eta}_u)\{(q,p):\ p \in C,\ q \in B + pt\}\| \\
&\quad \leqq \Big\|\sum_j (v_r * \eta - \bar{\eta}_u)[(B + p_{nj} t) \times C_{nj}]\Big\| \\
&\qquad + \Big\|\sum_j (v_r * \eta)\{(q,p):\ p \in C_{nj},\ q \in (B + pt)\,\Delta\,(B + p_{nj} t)\}\Big\| .
\end{aligned}$$

By Minkowski's inequality and the second order u-stationarity of η, the first term on the right is bounded by

$$\sum_j \|(v_r * \eta - \bar{\eta}_u)[(B + p_{nj} t) \times C_{nj}]\| = \sum_j \|(v_r * \eta - \bar{\eta}_u)(B \times C_{nj})\|.$$

If the C_{nj} have diameters $< \varepsilon$, it is further seen that the second term is bounded by

$$\begin{aligned}&\|\sum_j (v_r * \eta)\{(q, p):\ p \in C_{nj},\ q \in (\partial B)_{\varepsilon t} + pt\}\| \\ &\quad = \|\sum_j (\delta_{ty} * v_r * \eta)[(\partial B)_{\varepsilon t} \times C_{nj}]\| = \|(\delta_{ty} * v_r * \eta)[(\partial B)_{\varepsilon t} \times C]\| \\ &\quad \leqq \|(\delta_{ty} * \eta)[(\partial B)_{\varepsilon t} \times C]\|,\end{aligned}$$

where $(\partial B)_{\varepsilon t}$ denotes the εt-neighbourhood of the boundary ∂B. As ε tends to zero for fixed t, the last estimate tends to $\|(\delta_{ty} * \eta)(\partial B \times C)\|$ by dominated convergence, and this expression must be zero for all t, B and C, due to the second order u-stationarity of η. Thus

$$\sup_t \|(\delta_{ty} * v_r * \eta - \bar{\eta}_u)(B \times C)\| \leqq \limsup_{n \to \infty} \sum_j \|(v_r * \eta - \bar{\eta}_u)(B \times C_{nj})\|,$$

and so (4) will follow if we can show that

$$\lim_{r \to \infty} \sup_n \sum_j \|(v_r * \eta - \bar{\eta}_u)(B \times C_{nj})\| = 0. \tag{5}$$

For the sake of brevity, define

$$Y_{nj}(s) = \eta((B + s) \times C_{nj}), \qquad s \in R^{d-1},$$

and note that the processes Y_{nj} are second order stationary. Let λ_{nj} be the corresponding spectral measures, and write (assuming $d = 2$) $Y_{nj}^{(\varepsilon)}$ for the component of Y_{nj} corresponding to the restriction of λ_{nj} to the set $\{x: 0 < |x| < \varepsilon\}$. Then it may be seen from the proof of the mean ergodic theorem in [5], §§ X.6 and XI.6, that

$$\begin{aligned}&\sum_j \|(v_r * \eta - \bar{\eta}_u)(B \times C_{nj})\| \\ &\quad = \sum_j \|(v_r * Y_{nj})(0) - \bar{Y}_{nj}\| \\ &\quad = \sum_j \left\{ \int_{-\infty}^{\infty} \left(\frac{\sin 2\pi r x}{2\pi r x} - 1_{\{0\}}(x) \right)^2 \lambda_{nj}(x) \right\}^{1/2} \\ &\quad \leqq \sum_j \{\lambda_{nj}\{x:\ 0 < |x| < \varepsilon\} + \lambda_{nj}\{x:\ |x| \geqq \varepsilon\} (2\pi r \varepsilon)^{-2}\}^{1/2} \\ &\quad \leqq \sum_j \{\|Y_{nj}^{(\varepsilon)}\|^2 + \|Y_{nj}\|^2 (2\pi r \varepsilon)^{-2}\}^{1/2} \\ &\quad \leqq \sum_j \|Y_{nj}^{(\varepsilon)}\| + (2\pi r \varepsilon)^{-1} \sum_j \|Y_{nj}\|.\end{aligned}$$

Here the right hand side tends to zero uniformly in n as $r \to \infty$ and then $\varepsilon \to 0$, since $\sup_n \sum_j \|Y_{nj}\| < \infty$ by the definition of second order local invariance, and since moreover

$$\lim_{\varepsilon\to 0}\sup_n \sum_j \|Y_{nj}^{(\varepsilon)}\| = 0,$$

as may be seen by arguing as in the first part of the proof of Lemma 2.2. This proves (5) for $d=2$. The proof for general d is similar. □

We next show that, under a somewhat stronger regularity requirement, the convergence assertion of Theorem 4.1 remains valid in L_1.

Theorem 4.2. *Suppose that $\mathscr{L}(H)=u$, and let $\mu\in\mathfrak{M}(\Phi_1)$ be locally u-invariant with $\mu\pi^{-1}u=0$. Further suppose that η is strictly (second order) H-stationary and such that $Y_B\equiv d(B\eta)\pi^{-1}/d\mu$ exists a.s. and satisfies $\int\|Y_B\|\,d\mu<\infty$ for every $B\in\mathscr{B}(L)$. Then $\delta_x*\eta\to\bar{\eta}_u$ in L_1 (L_2) as $\rho(x,u)\to\infty$.*

Proof. For arbitrary $B\in\mathscr{B}(L)$, it is seen from Minkowski's inequality and Fubini's theorem that

$$\begin{aligned}
h^{-d}\int\|(B\eta)\pi^{-1}(C_h+x)\|\,dx &= h^{-d}\int\Big\|\int_{C_h+x} Y_B(t)\,\mu(dt)\Big\|\,dx\\
&\leqq h^{-d}\int dx\int_{C_h+x}\|Y_B(t)\|\,\mu(dt)=h^{-d}\int\|Y_B(t)\|\,\mu(dt)\int_{-C_h+t}dx\\
&=\int\|Y_B(t)\|\,\mu(dt)<\infty.
\end{aligned}$$

Hence it follows from Lemma 2.2 that $(B\eta)\pi^{-1}$ is first (second) order locally u-invariant. Since B was arbitrary, the assertion in the second order case now follows from Theorem 4.1.

In the first order case, note first that the assertion need only be proved for $\nu*\eta$ in place of η, where ν is an arbitrary absolutely continuous probability measure on u. Since $\nu*\eta$ also fulfills the conditions imposed on η, we may hence assume from now on that $\eta\ll\lambda\times\mu$ a.s. By a simple truncation argument, we may further assume that $Y=d\eta/d(\lambda\times\mu)$ is a.s. bounded by some constant. But in this case $\int\|Y_B\|_2\,d\mu<\infty$, so we may conclude from Lemma 2.2 that $(B\eta)\pi^{-1}$ is second order locally u-invariant for all $B\in\mathscr{B}(L)$. The assertion now follows from that in L_2. □

A more direct approach in the L_1 case would be to reduce to the case when $H=u$ while $x=ty$, and note as in the proof of Theorem 3.1 in [11] that Y has a strictly stationary version. Letting $B,\ C\in\mathscr{B}(R^{d-1})$, we then obtain

$$\begin{aligned}
\|(\delta_{ty}*\nu_r*\eta-\bar{\eta}_u)(B\times C)\| &= \Big\|\int_C\mu(dp)\int_{B+pt}[(\nu_r*Y)(q,p)-\bar{Y}(p)]\,dq\Big\|\\
&= \Big\|\int_C\mu(dp)\int_B[(\nu_r*Y)(q+pt,p)-\bar{Y}(p)]\,dq\Big\|\\
&\leqq \int_C\mu(dp)\int_B\|(\nu_r*Y)(q+pt,p)-\bar{Y}(p)\|\,dq\\
&= \int_C\mu(dp)\int_B\|(\nu_r*Y)(q,p)-\bar{Y}(p)\|\,dq,
\end{aligned}$$

and this last expression tends to zero by dominated convergence as $r\to\infty$, since the integrand is bounded by $2\|Y(q,p)\|$. In the L_2 case, the same argument would apply, provided we could show that Y has a second order stationary version. However, I don't know whether the last statement is true in general.

The L_2 version of Theorem 4.2 was shown above to be a special case of the last assertion in Theorem 4.1. In fact, the two statements are essentially equivalent, provided that the stationarity of η is known to be strict. To see this, define

$$\zeta_s \equiv [((B+s)\times C)\eta]\pi^{-1}, \qquad \zeta^{(r)} \equiv r^{-d}\int_{C_r} \zeta_s\, ds, \qquad \bar{\xi} = \lim_{r\to\infty} \zeta^{(r)},$$

and let $\{C_{nj}\}$ be a null-array of partitions of C. By monotone convergence and Jensen's inequality, we get

$$\begin{aligned}\mathsf{E}\,\mathrm{var}(\zeta^{(r)}-\bar{\zeta}) &= \mathsf{E}\lim_{n\to\infty}\sum_j |(\zeta^{(r)}-\bar{\zeta})\, C_{nj}| = \lim_{n\to\infty}\sum_j \|(\zeta^{(r)}-\bar{\zeta})\, C_{nj}\|_1\\ &\leqq \sup_n \sum_j \|(\zeta^{(r)}-\bar{\zeta})\, C_{nj}\|_2.\end{aligned}$$

Thus, by (5) and the second order local invariance,

$$\lim_{r\to\infty} \mathsf{E}\,\mathrm{var}(\zeta^{(r)}-\bar{\zeta}) = 0.$$

Letting $X(s)$ denote the total mass of the $\bar{\zeta}$-singular component of ζ_s, and noting that the corresponding quantity for $\zeta^{(r)}$ is $X^{(r)} = (v_r * X)(0)$, it follows by stationarity that

$$\mathsf{E}\,X(0) \equiv \mathsf{E}\,X^{(r)} = \lim_{r\to\infty} \mathsf{E}\,X^{(r)} \leqq \lim_{r\to\infty} \mathsf{E}\,\mathrm{var}(\zeta^{(r)}-\bar{\zeta}) = 0,$$

and so $\zeta_0 \ll \zeta$ a.s. Thus the ergodic components of η satisfy the hypothesis of Theorem 4.2.

From this argument it is further seen that, if the measure η in Theorem 4.2 is strictly stationary and ergodic, then μ may always be chosen such that $\lambda\times\mu = \bar{\eta}_u$.

The last two theorems yield improvements in various directions of Theorems 3.1 and 3.2 in [11]:

Theorem 4.3. *Let $\mathscr{L}(H) = R^d$, and suppose that η is strictly (second order) H-stationary and such that $(B\eta)\pi^{-1}$ is first (second) order locally v-invariant for every $B \in \mathscr{B}(L)$. Then η is a.s. v-invariant. If moreover*

$$\mathsf{P}\{\mathscr{L}(v, \pi x_1, \pi x_2) = R, \text{[illegible]} L^2 \text{ a.e. } \eta^2\} = 1, \tag{6}$$

then η is a.s. R^d-invariant.

Proof. First proceed as in the proof of Theorem 4.1 to reduce to the case when $H = R^d$. The first assertion is then an obvious consequence of Theorem 4.1, (cf. the second paragraph in the proof of that theorem). It remains to prove the second assertion for $v \neq R^d$. If $v \in \Phi_{d-1}$, our assertion follows from Lemma 2.2 in [11], since by (6)

$$\mathsf{P}\{\pi x \notin u, x \in L \text{ a.e. } \eta\} = 1.$$

Thus by (6) it remains to assume that $v \in \Phi_{d-2}$.

Consider the phase representation based on some $u \in \Phi_{d-1}$ with $u \supset v$. (By (6), any such u is permissible.) Write $(q,p) = (q', q'', p', p'')$, where q'' and p'' are the projections of q and p on v. According to the first part of the theorem, η is a.s. invariant under arbitrary q''-translations, which means that $\eta = \zeta\times\lambda$ a.s. for some random measure ζ on R^d. Interpreting (q', p') as the phase parametrization of the set L' of lines in R^2 and p'' as a mark, ζ becomes a random measure on the space $L' \times R^{d-2}$ of R^{d-2}-marked lines in R^2. (It is instructive to show how ζ may be obtained directly as a suitable projection of η.) We shall apply Theorem 3.2 in [11] to conclude that ζ is a.s. invariant under q'-translations. This will clearly yield the asserted a.s. invariance of η.

First we need to verify that ζ is strictly (second order) stationary under arbitrary translations. In the first order case, let $f \in \mathscr{F}_c(R^d)$ and $t \in R$ be arbitrary, and choose $B \in \mathscr{B}(R^{d-2})$ such that $\lambda B = 1$. By Fubini's theorem and the strict stationarity of η, we get

$$\begin{aligned}(\delta_{ty} * \zeta) f &= \int f(q'-p't, p)\, \zeta(dq'\,dp) = \int f(q'-p't, p)\, \lambda(B+p''t)\, \zeta(dq'\,dp) \\ &= \iint f(q'-p't, p)\, 1_B(q''-p''t)\, \eta(dq\,dp) = (\delta_{ty} * \eta)(f \times 1_B) \\ &\overset{d}{=} \eta(f \times 1_B) = \zeta f \cdot \lambda B = \zeta f,\end{aligned}$$

which implies that ζ is strictly stationary. The proof in the second order case is similar.

It remains to show that the condition

$$\mathsf{P}\{\mathscr{L}(\pi x_1, \pi x_2) = R^2, (x_1, x_2) \in (L' \times R^{d-2})^2 \text{ a.e. } \zeta^2\} = 1$$

in [11] is fulfilled, i.e. that the p'-projections of ζ are a.s. diffuse. Suppose on the contrary that there exists with positive probability some $a \in R$ with $\zeta \pi^{-1}\{a\} > 0$. The latter relation implies (in a self-explanatory notation) that

$$\eta^2\{(q_1, p_1, q_2, p_2): p_1' = p_2' = a\} \neq 0,$$

and from $p_1' = p_2' = a$ it follows that

$$\begin{aligned}\mathscr{L}(v, \pi(q_1, p_1), \pi(q_2, p_2)) &= \mathscr{L}(v, (p_1, 1), (p_2, 1)) \\ &= \mathscr{L}(v, (a, p_1'', 1), (a, p_2'', 1)) = \mathscr{L}(v, (a, 0, 1)) \neq R^d.\end{aligned}$$

Thus we get

$$\mathsf{P}\{\eta^2\{(x_1, x_2): \mathscr{L}(v, \pi x_1, \pi x_2) \neq R^d\} \neq 0\} > 0,$$

which contradicts (6). □

The results for smooth random measures here and below may be combined in an obvious way with the results of § 3 to yield corresponding statements for point processes on the space of lines, i.e. for non-interacting particle systems. The resulting corollaries are omitted. Less obvious is the fact that Theorem 4.2 yields the following strengthened version of the Breiman-Stone theorem. (Though most results in this paper extend to the case of marked random measures and point processes, this is the only place where marks are used explicitly. They are assumed to belong to some fixed locally compact second countable Hausdorff space K.)

Theorem 4.4. *Let $\mathscr{L}(H) = u$, and suppose that a system of marked particles on u is given at time zero by an H-stationary K-marked point process ξ_0 with a.s. finite sample* *intensity. Further suppose that the particles move with constant velocities which are chosen independently according to some mark dependent absolutely continuous distributions* μ_k, $k\in K$, *where* μ_k *is a measurable function of k. Then the resulting process* ξ *of positions and associated marks and velocities converges in distribution as time tends to infinity towards a Cox process directed by* $\bar{\xi}_u$.

The main improvements consist in the consideration of the process of both positions and velocities rather than that of positions alone, and further in the allowance for the velocities to depend on the relative positions of neighbouring particles. As will be seen from the proof, it is enough to require absolute continuity of the μ_k with respect to some fixed locally invariant measure. To attain asymptotic Cox structure, local invariance is then needed in one direction only.

Proof. By Exercise 4.5 in [10], we may replace ξ_0 in our argument by a corresponding p-thinning ξ_0' for some $p\in(0,1)$. We may further attach independent marks to the atoms of ξ_0', e.g. according to the uniform distribution on $[0,1]$, and consider the resulting point process ξ_0'' on $u\times K\times[0,1]$. This simplifies our proof, since ξ_0'' is automatically a.s. simple and satisfies the conditions (Σ) and (Σ^*) in [12], (cf. the proof of Theorem 4.2 in [12]). For the sake of brevity, we assume that ξ_0 itself has these properties. We may further assume that $\mathsf{E}\,\xi_0$ is locally finite, since we may otherwise consider the ergodic components of ξ separately.

Writing ζ_0 for the conditional intensity of ξ_0, the conditional intensity of ξ becomes

$$\zeta B=\int_B \zeta_0(dq,dk)\,\mu_k(dp)\equiv\zeta^* B, \qquad B\in\mathscr{B}(L\times K). \tag{7}$$

Anticipating the proof of (7), we get

$$(B\zeta)\pi^{-1}C=\int_B \zeta_0(dq,dk)(C\mu_k)(dp), \qquad B\in\mathscr{B}(L\times K),\ C\in\mathscr{B}(R^{d-1}),$$

and since μ_k is absolutely continuous for every $k\in K$, it follows that $(B\zeta)\pi^{-1}$ is a.s. absolutely continuous for fixed $B\in\mathscr{B}(L\times K)$. By Lemma 2.2, the $(B\zeta)\pi^{-1}$ are then first order locally invariant, so Theorem 4.2 applies, yielding $\delta_{ty}*\zeta\to\bar{\zeta}_u$ in L_1. Since $\bar{\zeta}_u=\bar{\xi}_u$, the asserted convergence may now be inferred from Lemma 3.3.

To prove (7), let us first assume that $d=2$. Since (Σ) and (Σ^*) hold by assumption, it is seen that ζ_0 coincides with the solution η_0 of the integral equation in §3 of [12] (with ξ replaced by ξ_0). In proceeding from ξ_0 to ξ, we introduce an intermediate process ξ' to be defined like ξ, except that the μ_k are replaced by Lebesgue measure on $[0,1]$. Putting

$$f(p,k)=\sup\{x\colon \mu_k(-\infty,x]\leqq p\}, \qquad p\in[0,1],\ k\in K,$$

we then define $\xi=\xi' f^{-1}$. For this construction to make sense, we have to prove that f is jointly measurable in p and k. Taking this for granted, it is easily verified that ξ has the desired distribution. By the remark concluding §3 in [12], it is further seen that ξ has conditional intensity

$$\zeta=\mathsf{E}[\zeta' f^{-1}\,|\,\xi f^{-1}]=\mathsf{E}[\zeta^*\,|\,\xi]=\zeta^* \quad \text{a.s.},$$

in conformity with (7), (ζ' being the conditional intensity of ξ').

Since f is increasing and right continuous in p for fixed k, the desired measurability will follow if we can show that $f(p,k)$ is measurable in k for fixed p. But this follows from the assumed measurability of $\{\mu_k\}$ and from the fact that, by the diffuseness of μ_k,

$$\{k: f(p,k)<t\} = \{k: \sup\{x: \mu_k(-\infty, x] \leqq p\} < t\} = \{k: \mu_k(-\infty, t] > p\}.$$

For $d \geqq 3$ we may proceed inductively, considering one component at a time of the rate vector $p=(p_1, \ldots, p_{d-1})$. Thus, after m steps, we choose $k'=(k, p_1, \ldots, p_m)$ as our new mark and replace μ_k by its conditional distribution μ'_k, given that the first m components of p are equal to $p_1, \ldots, p_m$. By the measurability of μ_k and the definition of conditional distributions, the measure μ'_k will automatically be measurable in k', and so the preceding argument applies at each step, eventually leading to (7). □

By a similar argument, the main result of Jacobs in [9] follows (in a strengthened form) from our Theorem 4.1. Indeed, the present approach shows that most of her assumptions are redundant.

We conclude this section with a different kind of extension of the Breiman-Stone theorem. It turns out that the conclusion of asymptotic Cox nature remains valid under the weaker assumption of diffuseness of the velocity distribution. The hypothesis of local invariance is only needed to ensure that the limits are mixtures of *stationary* Poisson processes.

Theorem 4.5. *Let $\mathscr{L}(H)=u$, and suppose that a particle system on u is given at time zero by an H-stationary point process ξ_0 with a.s. finite sample intensity. Further suppose that the particles move with constant velocities which are chosen independently according to some diffuse distribution μ. Then the resulting process ξ of positions and associated velocities is relatively compact in distribution as time tends to infinity, and the limit points are all Cox processes.*

Proof. By an obvious truncation argument, it is enough to prove the theorem for processes with bounded sample intensity. Let us first assume that ξ_0 is u-stationary. Then $\mathsf{E}\xi$ becomes invariant under arbitrary translations (cf. Lemma 2.2 in [11]), which implies in particular that $\delta_{ty} * \xi$ is relatively compact.

Next, given ξ_0, let $\pi_1, \pi_2, \ldots$ be the conditional probabilities for the lines of ξ (when regarded as a line process on R^d) to pass through $B_t \equiv B - ty$ for some fixed $B \in \mathscr{B}(R^d)$. Since μ is diffuse, the π_j must be bounded by some constant p_t which tends to zero as $t \to \infty$. Now it follows from item 1.5.8 in [15] that the restriction of ξ to the set $L(B_t)$ of lines hitting B_t differs in variation from a Cox process by at most

$$\begin{aligned} 2\mathsf{E}\sum \pi_j^2 &\leqq 2p_t \mathsf{E}\sum \pi_j = 2p_t \mathsf{E}\,\mathsf{E}[\xi L(B_t) | \xi_0] = 2p_t \mathsf{E}\xi L(B_t) \\ &= 2p_t\, \mathsf{E}(\delta_{ty} * \xi) L(B) = 2p_t \mathsf{E}\xi L(B) \to 0. \end{aligned}$$

Hence, if $\delta_{ty} * \xi \xrightarrow{d}$ some ζ as $t \to \infty$ through some sequence, then $L(B)\zeta$ must be a Cox process for every B, which implies the asserted Cox structure of ζ itself.

When ξ_0 is only known to be H-stationary, we consider the u-stationary process ξ^* obtained from ξ by randomization as in the proof of Theorem 4.1. We may then argue as before, except that the invariance of $\mathsf{E}\xi^*$ plus the estimate $\xi B \leqq \xi^*(B + u/H)$, $B \in \mathscr{B}(L)$, replace the invariance of $\mathsf{E}\xi$. □

5. Results under Mixing Conditions

Our present aim is to show that results similar to those of §4 are obtainable under suitable mixing conditions in place of the previous local invariance assumptions. In addition to the general assumptions and notational conventions introduced in §§1 and 4, we shall need some further definitions.

A random measure η on L is said to be *second order v-mixing*, if

$$\limsup_{\rho(\pi_v B_1, \pi_v B_2)\to\infty} \frac{\mathsf{E}\eta B_1 \eta B_2}{\mathsf{E}\eta B_1 \mathsf{E}\eta B_2} \leqq 1 \qquad (0/0=1),$$

π_v being the v-projection operator in the phase space. (In this connection, v is regarded as a sub-space of the space of positions rather than velocities.) We shall further say that η is *strongly v-mixing*, if

$$\limsup_{r\to\infty} \sup_{A_1, A_2} |\mathsf{P}(A_1 \cap A_2) - \mathsf{P}A_1 \mathsf{P}A_2| = 0$$

with the supremum extending over all pairs of events A_1 and A_2 which may be defined in terms of $B_1\eta$ and $B_2\eta$ for some B_1, $B_2 \in \mathscr{B}(L)$ with $\rho(\pi_v B_1, \pi_v B_2) > r$.

Given any family η_t, $t \geqq 0$, of random measures on L, we shall say that η_t is *first (second) order asymptotically non-random*, if

$$\limsup_{t\to\infty} \|\eta_t f\| < \infty, \qquad \lim_{t\to\infty} \|(\eta_t f)^s\| = 0, \quad f \in \mathscr{F}_c(L), \tag{1}$$

the superscript s denoting symmetrization. Clearly (1) implies that $\{\eta_t\}$ is relatively compact in distribution and that every limit point is a non-random measure.

For brevity, we shall often write Φ_1' for $\Phi_1 \setminus \Phi_1^{(u)}$, the set of directions which are not parallel to u. Whenever convenient, we shall further identify Φ_1' with u without further notice. In particular, the v-projections of $\mu \in \mathfrak{M}(\Phi_1')$ are by definition the projections on v of $B\mu$, $B \in \mathscr{B}(u)$, when regarded as measures on u.

Theorem 5.1. *Let $\mu \in \mathfrak{M}(\Phi_1')$ with diffuse v-projections. Further suppose that η is strongly (second order) v-mixing and that, for some probability measure ν on R^d, the density $d(\eta * \nu)/d(\mu \times \lambda)$ exists a.s. and is uniformly integrable (L_2-bounded). Then $\delta_x * \eta$ is first (second) order asymptotically non-random as $\rho(x, u) \to \infty$. If η is in addition first (second) order H-stationary, then $\delta_x * \eta$ is further first (second) order asymptotically H-invariant.*

Proof. We shall only consider the case when $x = t\,y$, $t \in R$. The modifications required in the general case are obvious.

Suppose it is known that $\delta_{ty} * \eta * \nu$ is first (second) order asymptotically non-random. Then so is $\delta_{ty} * \eta * (B\nu)$ for every $B \in \mathscr{B}(R^d)$, since the assumption on $\eta * \nu$ carries over to $\eta * (B\nu)$. Letting B be open and decrease towards a point in the support of ν, it may be seen as in the proof of Lemma 2.5 that $\delta_{ty} * \eta$ is first (second) order asymptotically non-random. In proving the first assertion, we may thus assume without loss that $\nu = \delta_0$. By Lemma 2.5, the same assumption can be made in the proof of the second assertion.

Let Y be a jointly measurable version of $d\eta/d(\mu\times\lambda)$, and assume that $\|Y(x)\|\leqq c<\infty$ for all x. By Minkowski's inequality, we get for any $f\in\mathscr{F}_c(L)$,

$$\begin{aligned}\|(\delta_{ty}*\eta)f\| &= \|\int Y(x)f(x-ty)(\mu\times\lambda)(dx)\| \leqq \int \|Y(x)\| f(x-ty)(\mu\times\lambda)(dx)\\ &\leqq c\int f(x-ty)(\mu\times\lambda)(dx) = c\int f(x)(\mu\times\lambda)(dx)<\infty,\end{aligned}$$

proving the first relation in (1) for $\eta_t\equiv\delta_{ty}*\eta$.

In proving the second relation in (1), we shall first consider the L_1-case. Since Y is then assumed to be uniformly integrable, there exists for every $\varepsilon>0$ some constant $b>0$ such that $\|(Y(x)-b)_+\|\leqq\varepsilon$ for all x, and it is clearly enough to prove the second half of (1) for the random measure with density $Y\wedge b$, or rather to assume from the beginning that $Y\leqq b$. Put $b=1$ without loss, and note that in this case $\eta_t\leqq\mu\times\lambda$ for all t. Now suppose that $\eta_t \xrightarrow{d}$ some ζ as $t\to\infty$ along some sequence T. Then

$$\mathsf{P}\{\zeta f>(\mu\times\lambda)f\}\leqq\liminf_{t\in T} P\{\eta_t f>(\mu\times\lambda)f\}=0,\qquad f\in\mathscr{F}_c(L),$$

so ζ is also bounded by $\mu\times\lambda$.

We next obtain for any B, $C\in\mathscr{B}(R^{d-1})$

$$(\delta_{ty}*\eta)(B\times C)=\eta\{(q,p)\colon p\in C, q\in B+tp\},\tag{2}$$

which is clearly a (measurable) function of $[(B+tC)\times C]\eta$. Writing ρ_v for distance in the v-direction, we get for any C_1, $C_2\in\mathscr{B}(R^{d-1})$ with $\rho_v(C_1,C_2)>0$

$$\rho_v(B+tC_1, B+tC_2)=t\rho_v(C_1+t^{-1}B, C_2+t^{-1}B)\sim t\rho_v(C_1,C_2).$$

Assuming that $\zeta\partial(B\times C_1)=\zeta\partial(B\times C_2)=0$ a.s., it follows from the strong v-mixing property of η that the events $\{\zeta(B\times C_1)\leqq r_1\}$ and $\{\zeta(B\times C_2)\leqq r_2\}$ are independent for all but at most countably many r_1, $r_2\geqq 0$. Hence $\zeta(B\times C_1)$ and $\zeta(B\times C_2)$ are independent, and since they are further bounded by constants, they must be uncorrelated. Thus the projection ζ_v of the random measure $[(B\times C)\zeta]\pi^{-1}$ onto v has uncorrelated increments. Since ζ_v is further bounded by the corresponding projection of $[(B\times C)(\lambda\times\mu)]\pi^{-1}=(\lambda B)(C\mu)$ and is therefore a.s. diffuse, we may conclude from Exercise 7.16 in [10], p. 96, that ζ_v is a.s. non-random. Hence so is $\zeta(B\times C)$, and B and C being essentially arbitrary, it follows that ζ itself is a.s. non-random. Thus $(\eta_t f)^s \xrightarrow{d} 0$, and so, by the uniform integrability,

$$\lim_{t\in T}\|(\eta_t f)^s\|=0,\qquad f\in\mathscr{F}_c(L).\tag{3}$$

Since $\{\eta_t\}$ is relatively compact and (3) has been shown to hold for every convergent subsequence, (3) must remain true for $T=R_+$.

Turning to the L_2-case and assuming without loss that $c=1$, we get for any $f\in\mathscr{F}_c(L^2)$, writing Y_t for the $(\mu\times\lambda)$-density of η_t,

$$\begin{aligned}\mathsf{E}\eta_t^2 f &= \mathsf{E}\iint Y_t(r)Y_t(s)f(r,s)(\mu\times\lambda)^2(dr\,ds)\\ &=\iint \mathsf{E}\,Y_t(r)Y_t(s)f(r,s)(\mu\times\lambda)^2(dr\,ds)\\ &\leqq\iint\|Y_t(r)\|\cdot\|Y_t(s)\|f(r,s)(\mu\times\lambda)^2(dr\,ds)\leqq(\mu\times\lambda)^2 f.\end{aligned}$$

If $\eta_t \xrightarrow{d} \zeta$ as $t \to \infty$ along some sequence T, it follows by Fatou's lemma that $\mathsf{E}\,\zeta^2 \leqq (\mu \times \lambda)^2$. Since the v-projections of μ are diffuse, it may then be seen from Čebyšev's inequality that the v-projections of $\zeta\pi^{-1}$ are regular in the sense of [10], and hence a.s. diffuse by Theorem 2.5 in [10].

It may next be seen from Fatou's lemma and the uniform integrability of $\eta_t B$ for fixed B (which follows from the L_2-boundedness) that, for arbitrary B, C_1, $C_2 \in \mathscr{B}(R^{d-1})$,

$$\begin{aligned}
&\mathsf{E}\,\zeta(B\times C_1)\,\zeta(B\times C_2) - \mathsf{E}\,\zeta(B\times C_1)\,\mathsf{E}\,\zeta(B\times C_2)\\
&\quad \leqq \liminf_{t\in T} E\eta_t(B\times C_1)\,\eta_t(B\times C_2) - \lim_{t\in T} \mathsf{E}\eta_t(B\times C_1)\,\mathsf{E}\eta_t(B\times C_2)\\
&\quad = \liminf_{t\in T}\{\mathsf{E}\,\eta_t(B\times C_1)\,\eta_t(B\times C_2) - \mathsf{E}\,\eta_t(B\times C_1)\,\mathsf{E}\,\eta_t(B\times C_2)\}.
\end{aligned}$$

Noting that, for $i=1,2$,

$$\begin{aligned}
\mathsf{E}\,\eta_t(B\times C_i) &\leqq \|\eta_t(B\times C_i)\| \leqq [\delta_{ty} * (\lambda\times\mu)](B\times C_i)\\
&= (\lambda\times\mu)(B\times C_i) = \lambda B\cdot\mu C_i < \infty,
\end{aligned}$$

and making use of (2) and the second order v-mixing of η, we obtain

$$\mathsf{E}\,\zeta(B\times C_1)\,\zeta(B\times C_2) \leqq \mathsf{E}\,\zeta(B\times C_1)\,\mathsf{E}\,\zeta(B\times C_2).$$

But this implies in turn that the increments of the v-projection ζ_v of $[(B\times C)\zeta]\,\pi^{-1}$ are non-positively correlated for all B and C. Since ζ_v was shown above to be a.s. diffuse, it follows as in Exercise 7.16 of [10] that ζ_v is a.s. non-random. Hence so is ζ itself.

Let us now consider an arbitrary closed ζ-continuity set $B\in\mathscr{B}(L)$, and define

$$D = \{(x,y)\in B^2 : \rho_v(\pi x, \pi y) = 0\}.$$

Consider B^2 as a topological space in the relative product topology, and note that D is closed. Given any $\varepsilon > 0$, there exists by dominated convergence some open set $G \supset D$ such that $(\mu\times\lambda)^2\,G < \varepsilon$. Any point in $B^2\setminus G$ is contained in the interior of some product set $B'\times C'$, such that $\zeta\,\partial B' = \zeta\,\partial C' = 0$ and the closure of $B'\times C'$ is disjoint from D. Since the set $B^2\setminus G$ is compact, it may be covered by finitely many such sets $B_1\times C_1, \ldots, B_n\times C_n$, say, which may clearly be assumed to be disjoint.

By the second order v-mixing and uniform integrability, we get for $j=1,\ldots,n$

$$\limsup_{t\in T} \mathsf{E}\eta_t^2(B_j\times C_j) \leqq \limsup_{t\in T} \mathsf{E}\eta_t B_j\,\mathsf{E}\,\eta_t C_j = \zeta^2(B_j\times C_j),$$

so

$$\begin{aligned}
\limsup_{t\in T} \mathsf{E}(\eta_t B)^2 &= \limsup_{t\in T} \mathsf{E}\eta_t^2[\textstyle\bigcup_j (B_j\times C_j)\cup G]\\
&\leqq \limsup_{t\in T}[\textstyle\sum_j \mathsf{E}\,\eta_t^2(B_j\times C_j) + \mathsf{E}\,\eta_t^2 G]\\
&\leqq \textstyle\sum_j \zeta^2(B_j\times C_j) + (\mu\times\lambda)^2\,G \leqq \zeta^2 B^2 + \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we obtain

$$\limsup_{t\in T} \mathsf{E}(\eta_t B)^2 \leqq (\zeta B)^2.$$

Conversely, we get by Fatou's lemma

$$\liminf_{t\in T} \mathsf{E}(\eta_t B)^2 \geqq (\zeta B)^2,$$

and so, by combination, $\mathsf{E}(\eta_t B)^2 \to (\zeta B)^2$ $(t\in T)$. Since moreover $(\eta_t B)^2 \xrightarrow{d} (\zeta B)^2$, it follows that $\eta_t B$ is uniformly square integrable. Hence so is $(\eta_t f)^s$ for every $f\in\mathscr{F}_c(L)$. The proof of (1) may now be completed as before.

In proving the second assertion, note that stationarity implies invariance for non-random measures, and conclude that every limiting measure of η_t is H-invariant. The first (second) order asymptotic invariance now follows from the uniform (square) integrability established above. □

If $H=u$ in the last theorem, it may be seen as in case of Theorem 4.1 that $\delta_{ty} * \eta \to \bar{\eta}_u$ in $L_1(L_2)$. However, no uniform integrability is actually needed for the L_1 version of this result, and in the L_2 case, the absolute continuity can essentially be weakened to L_2-regularity:

Theorem 5.2. *Let η be strictly (second order) u-stationary, strongly (second order) v-mixing and of first order. Further suppose that, for some $\mu\in\mathfrak{M}(\Phi_1')$ with diffuse v-projections, $(B\eta)\pi^{-1} \ll \mu$ a.s. (that $\|(B\eta)\pi^{-1}\|$ exists with diffuse v-projections), $B\in\mathscr{B}(L)$. Then $\delta_x * \eta \to \bar{\eta}_u$ in $L_1(L_2)$ as $\rho(x,u)\to\infty$.*

Proof. We may clearly restrict our attention to the case $x=ty$, $t>0$. In the L_1 case, let $A\in\mathscr{B}(R^{d-1})$ with $\lambda A>0$, and put $\nu = A\lambda$. Define $Y_B = d(B\eta)\pi^{-1}/d\mu$. Writing $A_q = (q-A)\times R^{d-1}$, we get by (1) and (2) in §4 for any B, $C\in\mathscr{B}(R^{d-1})$

$$\begin{aligned}(\nu*\eta)(B\times C) &= \int_B \eta[(q-A)\times C]\,dq = \int_B dq \int_C (A_q\eta)\pi^{-1}(dp)\\ &= \int_B dq \int_C Y_{A_q}(p)\,\mu(dp).\end{aligned}$$

This shows that $\nu*\eta \ll \mu\times\lambda$ with density

$$Y(q,p) = Y_{A_q}(p), \qquad q,p\in R^{d-1}.$$

As in [11], we may choose a strictly q-stationary version of Y. In that case, $\|Y(q,p)\| = \|Y_{A_q}(p)\|$ exists a.e. and is independent of q. We shall prove that $\delta_{ty}*\eta$ is first order asymptotically u-invariant. The asserted convergence will then follow as in case of Theorem 4.1.

As in Theorem 5.1, the asymptotic invariance need only be proved for $\nu*\eta$ in place of η, so we may assume without loss that $\nu=\delta_0$. For fixed $r>0$, let $\eta^{(r)}$ be the random measure with $(\mu\times\lambda)$-density $Y\wedge r$, and conclude from Theorem 5.1 that $\delta_{ty}*\eta^{(r)}$ is first order asymptotically u-invariant. This property carries over to $\delta_{ty}*\eta$, provided we can show that $\delta_{ty}*(\eta-\eta^{(r)})$ tends to zero in L_1 uniformly in t as $r\to\infty$. To see this, let B, $C\in\mathscr{B}(R^{d-1})$ be arbitrary, and conclude from Fubini's theorem and the stationarity of Y that

$$\begin{aligned}\|[\delta_{ty}*(\eta-\eta^{(r)})](B\times C)\| &= \Big\|\int_C \mu(dp) \int_{B+tp} (Y(q,p)-r)_+\,dq\Big\|\\ &= \int_C \mu(dp)\int_{B+tp} \|(Y(q,p)-r)_+\|\,dq = \int_C \mu(dp)\int_{B+tp}\|(Y(0,p)-r)_+\|\,dq\\ &= \lambda B\int_C \|(Y(0,p)-r)_+\|\,\mu(dp).\end{aligned}$$

As $r \to \infty$, the last integral tends to zero by dominated convergence. –

In the second order case, let B and C be rectangles in R^{d-1}, let $\varepsilon > 0$ be arbitrary, and choose a finite partition $\{C_j\}$ of C such that all diameters are bounded by ε. Fixing $p_j \in C_j$ for all j and writing $B_{\varepsilon t}$ for an εt-neighbourhood of B, we get for any $t \in R$

$$\begin{aligned}
\|(\delta_{ty} * \eta)(B \times C)\| &= \|(\delta_{ty} * \eta)(B \times \bigcup_j C_j)\| \\
&= \{\sum_i \sum_j \mathsf{E}(\delta_{ty} * \eta)^2 (B^2 \times C_i \times C_j)\}^{1/2} \\
&\leqq \{\sum_i \sum_j \|(\delta_{ty} * \eta)(B \times C_i)\| \, \|(\delta_{ty} * \eta)(B \times C_j)\|\}^{1/2} \\
&= \sum_j \|(\delta_{ty} * \eta)(B \times C_j)\| = \sum_j \|\eta\{p \in C_j, q \in B + pt\}\| \\
&= \sum_j \|\eta\{p \in C_j, q \in B + (p - p_j)t\}\| \leqq \sum_j \|\eta\{p \in C_j, q \in B_{\varepsilon t}\}\| \\
&= \sum_j \|\eta(B_{\varepsilon t} \times C_j)\| \leqq \|((B_{\varepsilon t} \times R^{d-1})\eta)\pi^{-1}\| \, C.
\end{aligned}$$

Choosing $\varepsilon < t^{-1}$, we thus obtain the uniform bound

$$\sup_t \|(\delta_{ty} * \eta)(B \times C)\| \leqq \|((B_1 \times R^{d-1})\eta)\pi^{-1}\| \, C < \infty.$$

We may now proceed as in case of Theorem 5.1 to show that $\delta_{ty} * \eta$ is second order asymptotically u-invariant. The asserted convergence will then follow as in the proof of Theorem 4.1. □

The preceding argument shows incidentally that, if $\|(B\eta)\pi^{-1}\|$ exists for all $B \in \mathscr{B}(L)$, then $\delta_{ty} * \eta$ is uniformly integrable as $t \to \infty$. If η is further known to be u-stationary, then the intensity of $\delta_{ty} * \eta$ is t-invariant by Lemma 2.2 in [11]. Thus $\delta_{ty} * \eta \xrightarrow{d} 0$ is impossible in this case, unless $\eta = 0$ a.s. This remark is somewhat related to the open question of Davidson stated in [7], §3.7, whether (in our notation) $\delta_{ty} * \xi \xrightarrow{d} 0$ can occur for a non-vanishing u-stationary line process ξ. (Of course, many results of this paper give rather precise information about the asymptotic behavior of $\delta_{ty} * \xi$ or $\delta_{ty} * \eta$ under specific assumptions. However, nothing seems to be known in general.)

The present methods yield surprisingly strong results in the case of "time" stationarity:

Theorem 5.3. *Let η be strictly (second order) y-stationary and strongly (second order) v-mixing. Further suppose that $\eta\pi^{-1}u = 0$ a.s. and that the v-projection of $(B\eta)\pi^{-1}$ is a.s. diffuse for every $B \in \mathscr{B}(L)$. Then η is a.s. non-random and y-invariant.*

Proof. Proceeding as in the proof of Theorem 5.1, it is seen that, for any $B \in \mathscr{B}(L)$, the increments over disjoint rectangles of the v-projection of $(B\eta)\pi^{-1}$ are independent (uncorrelated). Since this projection is further a.s. diffuse by assumption, it follows by Exercise 7.16 (7.15) in [10], p. 96, that ηB is a.s. non-random. Hence so is η itself, since B was arbitrary. But in this case, stationarity and invariance are equivalent. □

Again it is obvious how our theorems for "smooth" random measures may be translated into statements about point processes. Note, however, that a direct approach may sometimes yield stronger results:

Theorem 5.4. *Let ξ be a strictly y-stationary and strongly v-mixing point process on L with $\xi\pi^{-1}u=0$ a.s., and suppose that, for every $B\in\mathscr{B}(L)$, the v-projection of $(B\xi)\pi^{-1}$ is a.s. simple and has no fixed atoms. Then ξ is a Poisson process with y-invariant intensity.*

Proof. Proceed as in the proof of Theorem 5.3 to show that, for every fixed $B\in\mathscr{B}(L)$, the v-projection ξ_B of $(B\xi)\pi^{-1}$ has independent increments. We may thus conclude from Corollary 7.4 in [10] that ξ_B is a Poisson process. In particular, the random variable ξB is Poisson for every B, and so it follows from Satz 1.3.12 in [15] that ξ is a Poisson process. Finally, the y-invariance of $\mathsf{E}\,\xi$ follows from the y-stationarity of ξ. □

Added in Proof. Some results related to Theorem 5.4 above have been obtained, independently, by R.L. Dobrushin and Ju. Suchov.

6. The Method of Randomization

In this section we shall investigate the asymptotic behavior of $\delta_{\vartheta_n}*\eta$, where ϑ_1, $\vartheta_2, \ldots$ are random elements of R^d which are independent of η (though not necessarily mutually independent), and whose distributions (or certain projections of them) are globally asymptotically invariant. Results along these lines have some merits from the point of view of applications, since it may often be most natural to consider the evolution of a system along a sequence of random epochs. Note in particular that, in the important case when the ϑ_n form a random walk in R^d, $\mathsf{P}\,\vartheta_n^{-1}$ is globally asymptotically invariant as $n\to\infty$ iff $\mathsf{P}\,\vartheta_1^{-1}$ is non-lattice, (cf. item 6.5.4 in [15]).

However, our main motivation for studying random translations is the fact that results for this case yield interesting information about the case of non-random translations. Let us e.g. suppose that η has bounded intensity and that $\delta_{\vartheta_n}*\eta\to\bar{\eta}_u$ in L_1. Then we get for every fixed $f\in\mathscr{F}_c(L)$

$$\mathsf{E}[|(\delta_{\vartheta_n}*\eta-\bar{\eta}_u)f|\,|\,\vartheta_n]\to 0 \quad \text{in } L_1,$$

and turning to a suitable sub-sequence $N'\subset N$, we can make this hold in the sense of a.s. convergence. By the familiar diagonal procedure, the same type of result holds with a common exceptional null-set for any countable class of functions. A simple approximation argument then yields the same result for the entire class $\mathscr{F}_c(L)$, i.e.

$$\mathsf{E}[|(\delta_{\vartheta_n}*\eta-\bar{\eta}_u)f|\,|\,\vartheta_n]\to 0 \quad (n\in N'),\ f\in\mathscr{F}_c(L), \text{ a.s.}$$

But this is clearly equivalent to

$$\delta_{x_n}*\eta\to\bar{\eta}_u \quad \text{in } L_1 \quad (n\in N') \qquad \{x_n\}\in R^\infty \text{ a.e. } \mathsf{P}\{\vartheta_n\}^{-1}.$$

Specializing to the case when $\vartheta_n\equiv t_n\vartheta$, where the t_n are real numbers while ϑ is a random element of R^d satisfying $\lambda\ll\mathsf{P}\,\vartheta^{-1}\ll\lambda$, we get

$$\delta_{t_n x}*\eta\to\bar{\eta}_u \quad \text{in } L_1, \qquad x\in R^d \text{ a.e. } \lambda,$$

whenever $t_n \to \infty$ rapidly enough. Similar conclusions may be drawn in the cases of L_2-convergence and of first (second) order asymptotic invariance.

In our statements and proofs we shall still adhere, without further comments, to the notational and other conventions listed in §§1 and 4. Given any $w \in \Phi$, we shall further write w' for the orthogonal complement of w.

Theorem 6.1. *Let $\mu \in \mathfrak{M}(\Phi_1)$ be locally v-invariant and such that*

$$\mathscr{L}(v, x_1, x_2) = R^d, \quad (x_1, x_2) \in \Phi_1^2 \text{ a.e. } \mu^2.$$

Further suppose that η is strictly (second order) H-stationary and that, for some probability measure ν on R^d, the density $d(\eta * \nu)/d(\mu \times \lambda)$ exists a.s. and is uniformly (square) integrable. Finally suppose that the $\mathscr{L}(H)'$-projections of $\mathsf{P}\vartheta_n^{-1}$ are globally asymptotically invariant. Then $\delta_{\vartheta_n} * \eta$ is first (second) order asymptotically R^d-invariant. If $\mathscr{L}(H) = u \in \Phi_{d-1}$, we have even $\delta_{\vartheta_n} * \eta \to \bar{\eta}_u$ in $L_1(L_2)$.

Proof. Arguing as in the proofs of Theorems 4.1 and 5.1 respectively, we may first reduce the discussion to the case when $H = \mathscr{L}(H) \equiv w$ while $\nu = \delta_0$. By Lemma 2.5, we may further assume that the v-projections ϑ_n' of ϑ_n, $n \in N$, satisfy

$$\operatorname{var}[\mathsf{P}(\vartheta_n' + x)^{-1} - \mathsf{P}\vartheta_n'^{-1}] \to 0, \quad x \in w'. \tag{1}$$

In the L_1-case, we may finally truncate the density as in the proof of Theorem 5.1, so even in this case we may assume that η is second order w-stationary, and that $Y = d\eta/d(\mu \times \lambda)$ is uniformly square integrable. It is thus enough to prove the L_2-version of the theorem.

First we show that $(\delta_{\vartheta_n} * \eta)B$ is uniformly square integrable for fixed $B \in \mathscr{B}(L)$. By Fubini's theorem, it suffices to prove the uniform square integrability of $(\delta_x * \eta)B$ for non-random x. For fixed $\varepsilon > 0$, choose $c > 0$ so large that $\mathsf{E}[(Y(s))^2; Y(s) > c] < \varepsilon$, $s \in L$, and write η' and η'' for the random measures with densities $Y \wedge c$ and $(Y - c)_+$ respectively. Abbreviating $\eta_x = (\delta_x * \eta)B$ and similarly in case of primes, and putting $b = (\mu \times \lambda)B$, we get for any $r > 0$

$$\begin{aligned}\mathsf{E}[\eta_x^2; \eta_x > r] &\leqq 2\,\mathsf{E}[\eta_x'^2; \eta_x > r] + 2\,\mathsf{E}[\eta_x''^2; \eta_x > r] \\ &\leqq 2b^2 c^2\, \mathsf{P}\{\eta_x > r\} + 2\,\mathsf{E}\,\eta_x''^2 \leqq 2(bc/r)^2\, \|\eta_x\|^2 + 2\,\|\eta_x''\|^2.\end{aligned}$$

Since, by Minkowski's inequality,

$$\|\eta_x\| \leqq \int_B \|\delta_x * Y(s)\| (\mu \times \lambda)(ds) \leqq b \sup_s \|Y(s)\|$$

and similarly for $\|\eta_x''\|$, we thus obtain

$$\mathsf{E}[\eta_x^2; \eta_x > r] \leqq 2b^2[(bc/r)^2 \sup \|Y(s)\|^2 + \varepsilon].$$

As $\varepsilon \to 0$ and $r \to \infty$, the right-hand side tends to zero, which yields the desired conclusion.

By Fubini's theorem and the second order w-stationarity of η, we next obtain for any $B \in \mathscr{B}(L^2)$ and $x \in w'$

$$|\mathsf{E}(\delta_{\vartheta_n+x}*\eta)^2 B-\mathsf{E}(\delta_{\vartheta_n}*\eta)^2 B|=|\mathsf{E}(\delta_{\vartheta'_n+x}*\eta)^2 B-\mathsf{E}(\delta_{\vartheta'_n}*\eta)^2 B|$$
$$\leqq \sup_x \mathsf{E}(\delta_x*\eta)^2 B\cdot \mathrm{var}\{\mathsf{P}(\vartheta'_n+x)^{-1}-\mathsf{P}\vartheta'^{-1}_n\},$$

which tends to zero by (1). By the uniform square integrability of $\delta_{\vartheta_n}*\eta$, it follows that every limit in distribution ζ is second order w'-stationary. By another application of the uniform square integrability, it is seen from the w-stationarity of η that ζ is in fact second order R^d-stationary.

Next we show that $\zeta \ll \mu\times\lambda$ a.s. Indeed, choosing $c>0$ so large that, for fixed $\varepsilon>0$, $\mathsf{E}(Y(s)-c)_+<\varepsilon$, $s\in L$, and defining η' and η'' as before, we get

$$\delta_{\vartheta_n}*\eta'\leqq c(\mu\times\lambda) \text{ a.s.,}\qquad \mathsf{E}(\delta_{\vartheta_n}*\eta'')\leqq\varepsilon(\mu\times\lambda).$$

Choosing a sub-sequence $N'\subset N$ such that $\delta_{\vartheta_n}*(\eta',\eta'')\xrightarrow{d}$ some (ζ',ζ'') with $\zeta'+\zeta''\overset{d}{=}\zeta$, and using Fatou's lemma, we hence obtain

$$\zeta'\leqq c(\mu\times\lambda) \text{ a.s.,}\qquad \mathsf{E}\zeta''\leqq\varepsilon(\mu\times\lambda).$$

Thus, writing ζ_a for the component of ζ which is absolutely continuous with respect to $\mu\times\lambda$, we get

$$\mathsf{E}(\zeta-\zeta_a)\leqq\varepsilon(\mu\times\lambda),$$

and since ε was arbitrary, this yields $\zeta=\zeta_a$ a.s., as desired.

By Lemma 2.2 and Theorem 4.3, the stationarity and absolute continuity established above imply that ζ is a.s. R^d-invariant. The asserted second order asymptotic invariance of $\delta_{\vartheta_n}*\eta$ now follows from the uniform square integrability.

To establish the last assertion, we may essentially proceed as in case of Theorem 4.3. □

For $d=2$, the second order version of the last theorem may be improved considerably:

Theorem 6.2. *Let $d=2$ and $u=\mathscr{L}(H)$. Suppose that η is second order H-stationary and such that $(B\eta)\pi^{-1}$ is L_2-regular for every $B\in\mathscr{B}(L)$. Further suppose that the u'-projections of $\mathsf{P}\vartheta_n^{-1}$ are globally asymptotically invariant. Then $\delta_{\vartheta_n}*\eta\to\bar{\eta}_u$ in L_2.*

We shall use a direct argument which is related to a method employed by Davidson and Krickeberg in [7], pp. 64 and 104.

Proof. As in the proof of Theorem 4.1, we may assume without loss that $H=u$. Let B and C be any finite real intervals. Arguing as in the proof of Theorem 5.2 and using Fubini's theorem, we get

$$\sup_n\|(\delta_{\vartheta_n}*\eta)(B\times C)\|\leqq\sup_t\|(\delta_{t y}*\eta)(B\times C)\|\leqq\|((B_1\times C)\eta)\pi^{-1}\|\,C<\infty.$$

By a simple truncation argument, we may thus assume that ϑ_n is bounded for each n. As in case of Theorem 6.1, we may further assume that (1) holds, and we may finally identify ϑ_n with its u'-projection ϑ'_n and put $\eta_n=\delta_{\vartheta_n}*\eta$.

Consider a finite partition $\{C_j\}$ of C into sub-intervals, and fix in each rectangle

$C_i \times C_j$ a point (p'_{ij}, p''_{ij}), to be chosen later. Then we get for any $s \in u$ (to be identified, for convenience, with its coordinate along u)

$$\begin{aligned}
&\|(\eta_n - \delta_s * \eta_n)(B \times C)\|^2 \\
&\quad = \|\eta_n(B \times C)\|^2 + \|(\delta_s * \eta_n)(B \times C)\|^2 - 2\,\mathsf{E}(\eta_n \times (\delta_s * \eta_n))(B \times C)^2 \\
&\quad = 2\,\mathsf{E}\eta_n \times (\eta_n - \delta_s * \eta_n)(B \times C)^2 \\
&\quad = 2 \sum_j \sum_j \mathsf{E}\eta_n(B \times C_i)(\eta_n - \delta_s * \eta_n)(B \times C_j) \\
&\quad = 2 \sum_i \sum_j \mathsf{E}\eta\{p \in C_i,\ q \in B + p\vartheta_n\}(\eta - \delta_s * \eta)\{p \in C_j,\ q \in B + p\vartheta_n\} \\
&\quad \leqq 2 \sum_i \sum_j |\mathsf{E}\eta\{p \in C_i,\ q \in B + p'_{ij}\vartheta_n\}(\eta - \delta_s * \eta)\{p \in C_j,\ q \in B + p''_{ij}\vartheta_n\}| \\
&\qquad + 2 \sum_i \sum_j |\mathsf{E}[\eta \times (\eta - \delta_s * \eta)]\{(p', p'', q', q''):\ p' \in C_i,\ p'' \in C_j, \\
&\qquad\qquad (q', q'') \in B^2 \Delta [(B + (p'_{ij} - p')\vartheta_n) \times (B + (p''_{ij} - p'')\vartheta_n)] + (p', p'')\vartheta_n\}| \\
&\quad = 2S_1 + 2S_2.
\end{aligned}$$

We now choose the points (p'_{ij}, p''_{ij}) so as to minimize S_1. Writing $\mu = \|((B \times C)\eta)\pi^{-1}\|$, we then obtain

$$\begin{aligned}
S_1 &= \sum_i \sum_j |\mathsf{E}\eta^2[B \times (B + (p'_{ij} - p''_{ij})\vartheta_n) \times C_i \times C_j] \\
&\qquad - \mathsf{E}\eta^2[B \times (B + (p'_{ij} - p''_{ij})\vartheta_n - s) \times C_i \times C_j]| \\
&\leqq \sum_i \sum_j \sup_t \mathsf{E}\eta^2[B \times (B + t) \times C_i \times C_j] \operatorname{var}\left[\mathsf{P}\left(\vartheta_n - \frac{s}{p'_{ij} - p''_{ij}}\right)^{-1} - \mathsf{P}\vartheta_n^{-1}\right] \\
&\leqq \sum_i \sum_j \|\eta(B \times C_i)\|\ \|\eta(B \times C_j)\| \operatorname{var}\left[\mathsf{P}\left(\vartheta_n - \frac{s}{p'_{ij} - p''_{ij}}\right)^{-1} - \mathsf{P}\vartheta_n^{-1}\right] \\
&\leqq \sum_i \sum_j \mu^2(C_i \times C_j) \inf_{p' \in C_i,\, p'' \in C_j} \operatorname{var}\left[\mathsf{P}\left(\vartheta_n - \frac{s}{p' - p''}\right)^{-1} - \mathsf{P}\vartheta_n^{-1}\right] \\
&\leqq \int_{C^2} \operatorname{var}\left[\mathsf{P}\left(\vartheta_n - \frac{s}{p' - p''}\right)^{-1} - \mathsf{P}\vartheta_n^{-1}\right] \mu^2(dp' \times dp'').
\end{aligned}$$

Now the regularity assumption means that μ is diffuse, so $\mu^2\{(p', p''): p' = p''\} = 0$ by Fubini's theorem. Thus, by (1), the integrand tends to zero as $n \to \infty$, a.e. μ^2, and it follows by bounded convergence that $S_1 \to 0$. Note that this holds uniformly with respect to $\{C_j\}$.

Next suppose that the intervals C_j have lengths $\leqq \varepsilon$, and conclude that

$$\begin{aligned}
S_2 &\leqq \sum_i \sum_j \mathsf{E}[\eta_n \times (\eta_n + \delta_s * \eta_n)][(\partial B^2)_{\varepsilon\vartheta_n} \times C_i \times C_j] \\
&= \mathsf{E}[\eta_n \times (\eta_n + \delta_s * \eta_n)][(\partial B^2)_{\varepsilon\vartheta_n} \times C^2].
\end{aligned}$$

By dominated convergence and the stationarity of η_n, we hence obtain for fixed n

$$\limsup_{\varepsilon \to 0} S_2 \leqq \mathsf{E}[\eta_n \times (\eta_n + \delta_s * \eta_n)](\partial B^2 \times C^2) = 0.$$

Thus $\|(\eta_n - \delta_s * \eta_n)(B \times C)\| \to 0$, and since B, C and s were arbitrary, it follows that η_n is second order asymptotically u-invariant. The proof may now be completed as in case of Theorem 4.1. □

References

1. Billingsley, P.: Convergence of probability measures. New York: Wiley 1968
2. Breiman, L.: The Poisson tendency in traffic distribution. Ann. Math. Statist. **34**, 308–311 (1963)
3. Clifford, P., Sudbury, A.: Some results on the limiting behaviour of infinite particle systems. Z. Wahrscheinlichkeitstheorie verw. Gebiete **30**, 273–278 (1974)
4. Dobrushin, R.L.: On Poisson laws for distributions of particles in space (in Russian). Ukrainian Math. J. **8**, 127–134 (1956)
5. Doob, J.L.: Stochastic processes. New York: Wiley 1953
6. Goldman, J.R.: Stochastic point processes: limit theorems. Ann. Math. Statist. **38**, 771–779 (1967)
7. Harding, E.F., Kendall, D.G. (eds.): Stochastic geometry. London: Wiley 1974
8. Harris, T.E.: Diffusions with collisions between particles. J. Appl. Probability **2**, 323–338 (1965)
9. Jacobs, P.A.: A Poisson convergence theorem for a particle system with dependent constant velocities. Stoch. Proc. Appl. **6**, 41–52 (1977)
10. Kallenberg, O.: Random measures. Berlin-London: Akademie-Verlag and Academic Press 1976
11. Kallenberg, O.: On the structure of stationary flat processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **37**, 157–174 (1976)
12. Kallenberg, O.: On conditional intensities of point processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **41**, 205–220 (1978)
13. Kallenberg, O.: A counterexample to R. Davidson's conjecture on line processes. Math. Proc. Cambridge Philos. Soc. **82**, 301 – 307 (1977)
14. Kallenberg, O.: On the independence of velocities in a system of non-interacting particles [to appear in Ann. Probability]
15. Kerstan, J., Matthes, K., Mecke, J.: Unbegrenzt teilbare Punktprozesse. Berlin: Akademie 1974
16. Papangelou, F.: The conditional intensity of general point processes and an application to line processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **28**, 207–226 (1974)
17. Papangelou, F.: Point processes on spaces of flats and other homogeneous spaces. Math. Proc. Cambridge Philos. Soc. **80**, 297–314 (1976)
18. Spitzer, F.: Uniform motion with elastic collision of an infinite particle system. J. Math. Mech. **18**, 973–989 (1969)
19. Stone, C.: On a theorem by Dobrushin. Ann. Math. Statist. **39**, 1391–1401 (1968)
20. Szatzschneider, W.: A more deterministic version of Harris-Spitzer's "Random Constant Velocity" model for infinite systems of particles. In: Probability Winter School, Karpacz. Lecture Notes in Math. No. 472. Berlin: Springer 1975
21. Thedéen, T.: A note on the Poisson tendency in traffic distribution. Ann. Math. Statist. **35**, 1823–1824 (1964)

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