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On the Asymptotic Equivalence of L_p Metrics for Convergence to Normality

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Summary. This paper is concerned with the rate of convergence to zero of the L_p metrics Δ_{np}, $1 \leqq p \leqq \infty$, constructed out of differences between distribution functions, for departure from normality for normed sums of independent and identically distributed random variables with zero mean and unit variance. It is shown that the Δ_{np} are, under broad conditions, asymptotically equivalent in the strong sense that, for $1 \leqq p$, $p' \leqq \infty$, $\Delta_{np'}/\Delta_{np}$ is universally bounded away from zero and infinity as $n \to \infty$.

1. Introduction and Results

Let X_i, $i=1,2,\ldots$ be independent and identically distributed random variables with zero mean and variance unity and write $S_n = \sum_{i=1}^{n} X_i$, $n \geqq 1$, $F_n(x) = P(S_n \leqq x n^{\frac{1}{2}})$ and $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du$, $-\infty < x < \infty$. It is well known that the L_p metrics Δ_{np} given by

$$\Delta_{np} = \left(\int_{-\infty}^{\infty} |F_n(x) - \Phi(x)|^p dx \right)^{1/p}, \qquad 1 \leqq p < \infty.$$

$$\Delta_{n\infty} = \sup_x |F_n(x) - \Phi(x)|, \qquad p = \infty,$$

which measure departure from normality, all converge to zero as $n \to \infty$ (e.g. Ibragimov and Linnik [10], Theorem 5.2.1, p. 141) and it is of interest to compare their rates of convergence. Little attempt has been made to obtain a direct comparison although the literature contains considerable circumstantial evidence in favour of the conjecture that they are all asymptotically equivalent in the strong sense that for any $p' \neq p$, $1 \leqq p$, $p' \leqq \infty$, there are *universal* constants C and D such that

$$0 < C < \liminf_{n \to \infty} \Delta_{np'}/\Delta_{np} \leqq \limsup_{n \to \infty} \Delta_{np'}/\Delta_{np} < D < \infty.$$

(The requirement that the constants be univeral is a special feature of our discussion.) In particular, results date back to Esseen [3] from which it can be deduced that asymptotic equivalence holds if $E|X_1|^3<\infty$ and $EX_1^3 \neq 0$ or X_1 has a lattice distribution. More recent circumstantial evidence is of the kind: for $0 \leqq \delta < 1$, $\sum n^{-1+\delta/2} \Delta_{np} < \infty$ if and only if $EX_1^2 \log(1+|X_1|) < \infty$ $(\delta=0)$, $E|X_1|^{2+\delta}<\infty$ $(0<\delta<1)$ (Heyde [7]) and, for $0<\delta<1$, $\Delta_{np}=O(n^{-\delta/2})$ if and only if $EX_1^2<\infty$ and $\int_{|x|>z} x^2 dP(X_1 \leqq x) = O(z^{-\delta})$ as $z \to \infty$ (Ibragimov [8]) and further refinements appear in, for example, Rozovskii [12]. In this paper we will show that the metrics are asymptotically equivalent under a very broad range of conditions.

The question of asymptotic equivalence is treated in the following theorem. Here $a_n \asymp b_b$ for sequences of positive numbers $\{a_n\}$, $\{b_n\}$ means that

$$0 < \liminf_{n \to \infty} a_n/b_n \leqq \limsup_{n \to \infty} a_n/b_n < \infty.$$

Theorem 1. *All the metrics Δ_{np}, $1 \leqq p \leqq \infty$, have the same asymptotic behaviour if any of the following four conditions are satisfied:*

(i) $x^3 P(|X_1|>x) \to \infty$ *as* $x \to \infty$,

(ii) $E|X_1|^3<\infty$ *and* $EX_1^3 \neq 0$,

(iii) X_1 *has a lattice distribution and* $E|X_1|^3<\infty$,

(iv) $\limsup_{|t| \to \infty} |Ee^{itX_1}|<1$ *and* $E|X_1|^r<\infty$, $EX_1^r \neq \int_{-\infty}^{\infty} x^r d\Phi(x)$

for some integer $r \geqq 3$.

In the case of (i), $\Delta_{np} \asymp \delta_n$ *as* $n \to \infty$ *where*

$$\delta_n = EX_1^2 I(|X_1|>n^{\frac{1}{2}}) + n^{-1} EX_1^4 I(|X_1| \leqq n^{\frac{1}{2}}) + n^{-\frac{1}{2}} |EX_1^3 I(|X_1| \leqq n^{\frac{1}{2}})|. \tag{1}$$

For (ii) *and* (iii), $\Delta_{np} \asymp n^{-\frac{1}{2}}$, *and for* (iv), $\Delta_{np} \asymp n^{-(r-2)/2}$ *as* $n \to \infty$.

On the way to Theorem 1 we shall, however, obtain the following result of independent interest. Note particularly that there are universal constants in the upper and lower bounds.

Theorem 2. *Let Y_i, $i=1,2,\ldots,n$ be independent random variables with $EY_i=0$, $EY_i^2=\sigma_i^2<\infty$ and $\sum_{i=1}^n \sigma_i^2=1$. Write*

$$\Delta_{np} = \left(\int_{-\infty}^{\infty} \left| P\left(\sum_{i=1}^n Y_i \leqq x \right) - \Phi(x) \right|^p dx \right)^{1/p}, \qquad 1 \leqq p < \infty,$$

$$\Delta_{n\infty} = \sup_x \left| P\left(\sum_{i=1}^n Y_i \leqq x \right) - \Phi(x) \right|, \qquad p=\infty.$$

Then, there exists a universal constant $C>0$ such that

$$\delta_n \leqq C \left(\Delta_{np} + \sum_{i=1}^n \sigma_i^4 \right) \tag{2}$$

where

$$\delta_n = \sum_{i=1}^{n} E Y_i^2 I(|Y_i|>1) + \sum_{i=1}^{n} E Y_i^4 I(|Y_i| \leqq 1) + \left| \sum_{i=1}^{n} E Y_i^3 I(|Y_i| \leqq 1) \right|.$$

On the other hand, if ε_n is a positive constant then there exists a universal constant $D>0$ such that whenever

$$\sum_{i=1}^{n} E Y_i^2 I(|Y_i|>\varepsilon_n) \leqq \tfrac{1}{8},$$

$$\Delta_{np} \leqq D \left(\delta_n + \varepsilon_n + \sum_{i=1}^{n} \sigma_i^4 \right). \tag{3}$$

Let $\mathscr{S}$ denote the real line or a subset thereof and write

$$\Delta_{np}(\mathscr{S}) = \left(\int_{\mathscr{S}} |F_n(x) - \Phi(x)|^p dx\right)^{1/p}, \qquad 1 \leqq p < \infty,$$

$$\Delta_{n\infty}(\mathscr{S}) = \sup_{x \in \mathscr{S}} |F_n(x) - \Phi(x)|.$$

Asymptotically equivalent behaviour of the L_p metrics appears to be closely linked to the phenomenon of intervals of the form $(-A, A)$, $A>0$, being convergence determining sets. The subset $\mathscr{S}$ of the real line may be called *convergence determining* for the L_p metrics if $(1 \leqq)\ \Delta_{np}/\Delta_{np}(\mathscr{S})$ is universally bounded away from infinity as $n \to \infty$, $1 \leqq p \leqq \infty$. A number of results concerning these sets are given in Hall [5], in which a rather different definition is adopted. Here we shall give a modification of one of the results of [5] to indicate the possibilities.

Theorem 3. *Suppose $x^3 P(|X_1|>x) \to \infty$ as $x \to \infty$. Then, any set $\mathscr{S} = (-A, A)$, $0<A<\frac{1}{2}$, is convergence determining for the L_p metrics, $1 \leqq p \leqq \infty$. We have $\Delta_{np}((-A,A)) \uparrow$ as $p \uparrow$ and $\Delta_{np}((-A,A)) \asymp \delta_n$ as $n \to \infty$ where δ_n is given by* (1).

2. Proofs

As a preliminary to the proofs of the theorems we shall establish the following Lemma which is also of some independent interest.

Lemma. *Let $\mathscr{S}$ denote the real line or a subset thereof and for a function $u \in L_p(\mathscr{S})$, $p \geqq 1$, we write*

$$N_p(u; \mathscr{S}) = \left(\int_{\mathscr{S}} |u(x)|^p dx\right)^{1/p}, \qquad 1 \leqq p < \infty,$$

$$N_\infty(u; \mathscr{S}) = \sup_{x \in \mathscr{S}} |u(x)|.$$

Then, $p \log N_p(u; \mathscr{S})$ is a convex function of p.

This result holds in particular for $u(x) = |F_n(x) - \Phi(x)|$ and if $\mathscr{S} = (-A, A)$, $A<\frac{1}{2}$, then $\Delta_{np}(A) = \Delta_{np}((-A,A)) \uparrow$ as $p \uparrow$.

Proof. Let $1 \leqq p' < p$. Then, Schwarz' inequality gives

$$\int_{\mathscr{S}} |u(x)|^p dx \leqq \left(\int_{\mathscr{S}} |u(x)|^{p+p'} dx \int_{\mathscr{S}} |u(x)|^{p-p'} dx\right)^{\frac{1}{2}}$$

and, upon taking logarithms of both sides,

$$\log N_p^p(u; \mathscr{S}) \leqq \tfrac{1}{2} \log N_{p+p'}^{p+p'}(u; \mathscr{S}) + \tfrac{1}{2} \log N_{p-p'}^{p-p'}(u; \mathscr{S})$$

which gives the convexity of $p \log N_p(u; \mathscr{S})$.

For n sufficiently large, $\log \Delta_{np}(\mathscr{S}) < 0$, while $p \log \Delta_{np}(\mathscr{S}) \to -\infty$ as $p \to \infty$. In the case $\mathscr{S} = (-A, A)$, $0 < A < \frac{1}{2}$, $\Delta_{np}(A) = \Delta_{np}((-A, A))$ can be defined for all $p > 0$ and

$$p \log \Delta_{np}(A) = \log \int_{-A}^{A} |F_n(x) - \Phi(x)|^p dx \to \log 2A < 0$$

as $p \to 0$. Then convexity ensures that the slope $\log \Delta_{np}(A)$ of the line joining the origin to $(p, p \log \Delta_{np}(A))$ is nondecreasing in p. This gives $\Delta_{np}(A) \uparrow$ as $p \uparrow$ and the proof of the Lemma is complete.

Proof of Theorem 2. The result (2) can be obtained by modifying the proof of the theorem of Hall and Barbour [6] which deals with the case $p = \infty$. The modifications required to deal with $1 \leqq p < \infty$ are first to use

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} g(w)\{P(W \leqq w) - \Phi(w)\} dw \right| \\ &\qquad \leqq \left(\int_{-\infty}^{\infty} |g(w)|^q dw \right)^{1/q} \Delta_{np}, \qquad q = (1 - p^{-1})^{-1},\ 1 < p < \infty \\ &\qquad \leqq \sup_w |g(w)| \Delta_{n1}, \qquad p = 1, \end{aligned} \tag{4}$$

with $g(w) = h'(w)$ instead of (2) of [6] and with $g(w) = f''(w + \beta)$ in the bounding of $|t_{2i}|$ of [6]. Then note that

$$\left(\int_{-\infty}^{\infty} |g(w)|^q dw \right)^{1/q} \leqq (\sup_w |g(w)|)^{1/p} \left(\int_{-\infty}^{\infty} |g(w)| dw \right)^{1/q} \tag{5}$$

while simple calculations on the functions g used in the proof of [6] reveal that in each case

$$\sup_w |g(w)| \leqq \int_{-\infty}^{\infty} |g(w)| dw \tag{6}$$

so that, using (5) and (6) in (4),

$$\left| \int_{-\infty}^{\infty} g(w)\{P(W \leqq w) - \Phi(w)\} dw \right| \leqq \Delta_{np} \int_{-\infty}^{\infty} |g(w)| dw, \qquad 1 \leqq p \leqq \infty.$$

The result (2) then follows via the proof of [6] with the same universal constant C as obtained by Hall and Barbour for the case $p = \infty$.

The result (3), on the other hand, is a consequence of Theorem 2.2 (p. 25) and Theorem 2.3 (p. 44) of Hall [5] and this completes the proof of the theorem.

Proof of Theorem 1. Applying the results of Theorem 2 to the case of identically distributed random variables, we can put $Y_i = X_i/n^{\frac{1}{2}}$ so that $\sum_{i=1}^{n} \sigma_i^4 = n^{-1}$ and $\varepsilon_n = \lambda n^{-\frac{1}{2}}$ where

$$\lambda = \min[\mu: EX_1^2 I(|X_1| > \mu) \leqq \tfrac{1}{8}].$$

When $x^3 P(|X_1| > x) \to \infty$ as $x \to \infty$ we have from (2) that $n^{\frac{1}{2}} \Delta_{np} \to \infty$ as $n \to \infty$ since

$$n^{\frac{1}{2}} \delta_n \geqq n^{\frac{1}{2}} EX_1^2 I(|X_1| > n^{\frac{1}{2}}) \geqq n^{\frac{3}{2}} P(|X_1| > n^{\frac{1}{2}}) \to \infty$$

as $n \to \infty$. The results (2) and (3) then give asymptotic equivalence of the L_p metrics and $\Delta_{np} \asymp \delta_n$ as $n \to \infty$. This completes the proof for part (i) of the theorem.

Next suppose that $E|X_1|^3 < \infty$. We have from results due to Esseen [2], [3] (e.g. Ibragimov and Linnik [10], Theorem 5.3.2, p. 147 and Theorem 5.3.3, p. 150) that for $1 \leqq p \leqq \infty$,

$$\Delta_{np} = \Lambda_p |EX_1^3| n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}})$$

where

$$\Lambda_\infty = 1/6(2\pi)^{\frac{1}{2}}$$

$$\Lambda_p = \Lambda_\infty \left(\int_{-\infty}^{\infty} |1 - x^2|^p e^{-\frac{1}{2}px^2} dx \right)^{1/p}, \qquad 1 \leqq p < \infty,$$

if X_1 does not have a lattice distribution, while

$$\Delta_{np} = M_p n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}})$$

where

$$M_\infty = (2\pi)^{-\frac{1}{2}} (\tfrac{1}{6} |EX_1^3| + \tfrac{1}{2} h)$$

$$M_p = (2\pi)^{-\frac{1}{2}} \left(\int_{-\infty}^{\infty} |Q_1(x) + \tfrac{1}{6} EX_1^3 (1 - x^2)|^p e^{-\frac{1}{2}px^2} dx \right)^{1/p}, \qquad 1 \leqq p < \infty \tag{7}$$

with

$$Q_1(x) = hQ(h^{-1} n^{\frac{1}{2}} (x - \xi_n))$$

and

$$Q(x) = [x] - x + \tfrac{1}{2},$$

$[x]$ denoting the integer part of x and

$$\xi_n = hn^{-\frac{1}{2}} (nx_0 h^{-1} - [nx_0 h^{-1}]),$$

the lattice distribution being situated on the points $x_0 + vh$, $v = 0, \pm 1, \pm 2, \ldots$.

To obtain the result for part (ii) in the non-lattice case we first note that

$$\lim_{n \to \infty} \Delta_{np'} / \Delta_{np} = \Lambda_{p'} / \Lambda_p.$$

Furthermore, it is a straightforward exercise to show that $\Lambda_1 = 4e^{-\frac{1}{2}}\Lambda_\infty$ and $\Lambda_p \downarrow \Lambda_\infty$ as $p \uparrow \infty$. Thus, taking $p' > p$ without loss of generality we have

$$\tfrac{1}{4}e^{\frac{1}{2}} \leqq \lim_{n\to\infty} \Delta_{np'}/\Delta_{np} \leqq 1.$$

To deal with parts (ii) (in the lattice case) and (iii) of the theorem we need upper and lower bounds on M_p in (7). Using Minkowski's inequality we have

$$\begin{aligned}&\left(\int_{-\infty}^{\infty} |Q_1(x)+\tfrac{1}{6}EX_1^3(1-x^2)|^p e^{-\frac{1}{2}px^2}dx\right)^{1/p}\\ &\leqq \left(\int_{-\infty}^{\infty} |Q_1(x)|^p e^{-\frac{1}{2}px^2}dx\right)^{1/p} + ch\left(\int_{-\infty}^{\infty} |1-x^2|^p e^{-\frac{1}{2}px^2}dx\right)^{1/p} \qquad (8)\end{aligned}$$

where $hc = \frac{1}{6}|EX_1^3|$. But, as noted previously, for $p \geqq 1$,

$$\left(\int_{-\infty}^{\infty} |1-x^2|^p e^{-\frac{1}{2}px^2}dx\right)^{1/p} \downarrow \quad \text{as } p\uparrow$$

while, since $|Q_1(x)| \leqq \frac{1}{2}h$,

$$\begin{aligned}\left(\int_{-\infty}^{\infty} |Q_1(x)|^p e^{-\frac{1}{2}px^2}dx\right)^{1/p} &\leqq \tfrac{1}{2}h\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}px^2}dx\right)^{1/p}\\ &= \tfrac{1}{2}h(2\pi/p)^{1/2p} \leqq \tfrac{1}{2}h(2\pi)^{\frac{1}{2}},\end{aligned}$$

and hence (8) gives

$$\begin{aligned}M_p &\leqq \tfrac{1}{2}h + hc(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} |1-x^2|e^{-\frac{1}{2}x^2}dx\\ &= \tfrac{1}{2}h + 4hc(2\pi e)^{-\frac{1}{2}} \qquad (9)\end{aligned}$$

after a straightforward calculation.

To obtain a lower bound on M_p we first note that

$$\begin{aligned}&\left(\int_{-\infty}^{\infty} |Q_1(x)+\tfrac{1}{6}EX_1^3(1-x^2)|^p e^{-\frac{1}{2}px^2}dx\right)^{1/p}\\ &\geqq \left(\int_{\xi_n}^{\frac{1}{2}+\xi_n} |hQ(h^{-1}n^{\frac{1}{2}}(x-\xi_n))+\tfrac{1}{6}EX_1^3(1-x^2)|^p e^{-\frac{1}{2}px^2}dx\right)^{1/p}.\end{aligned}$$

Now suppose for definiteness that $EX_1^3 \geqq 0$; the argument for $EX_1^3 \leqq 0$ is similar. We also note that $|\xi_n| \leqq h/n^{\frac{1}{2}}$ and we choose n so large that $h/n^{\frac{1}{2}} < \frac{1}{4}$. Then, we have

$$
\begin{aligned}
(2\pi)^{\frac{1}{2}} M_p &\geqq h\left(\int_0^{\frac{1}{2}} |Q(h^{-1}n^{\frac{1}{2}}x)+c(1-(x+\xi_n)^2)|^p e^{-9p/32}dx\right)^{1/p} \\
&\geqq h e^{-9/32}\left(\sum_{i=0}^{[\frac{1}{2}n^{\frac{1}{2}}h^{-1}]-1}\int_{hin^{-\frac{1}{2}}}^{h(i+\frac{1}{2})n^{-\frac{1}{2}}} |Q(h^{-1}n^{\frac{1}{2}}x)+\tfrac{7}{16}c|^p dx\right)^{1/p} \\
&= h e^{-9/32}\left(\sum_{i=0}^{[\frac{1}{2}n^{\frac{1}{2}}h^{-1}]-1}\int_{hin^{-\frac{1}{2}}}^{h(i+\frac{1}{2})n^{-\frac{1}{2}}} ([h^{-1}n^{\frac{1}{2}}x]-h^{-1}n^{\frac{1}{2}}x+\tfrac{1}{2}+\tfrac{7}{16}c)^p dx\right)^{1/p} \\
&= h e^{-9/32}\left(\sum_{i=0}^{[\frac{1}{2}n^{\frac{1}{2}}h^{-1}]-1} hn^{-\frac{1}{2}}\int_0^{\frac{1}{2}} (\tfrac{1}{2}-y+\tfrac{7}{16}c)^p dx\right)^{1/p} \\
&= h e^{-9/32}\left(\int_0^{\frac{1}{2}} (\tfrac{1}{2}-y+\tfrac{7}{16}c)^p dx\right)^{1/p} (hn^{-\frac{1}{2}}[\tfrac{1}{2}h^{-1}n^{\frac{1}{2}}])^{1/p} \\
&\geqq \tfrac{1}{4}he^{-9/32}(p+1)^{-1/p}((\tfrac{1}{2}+\tfrac{7}{16}c)^{p+1}-(\tfrac{7}{16}c)^{p+1})^{1/p} \\
&\geqq \tfrac{1}{4}he^{-9/32}(p+1)^{-1/p}2^{-1/p}(\tfrac{1}{2}+\tfrac{7}{16}c) \\
&\geqq \tfrac{1}{16}he^{-9/32}(\tfrac{1}{2}+\tfrac{7}{16}c).
\end{aligned}
\tag{10}
$$

From (9) and (10) we obtain

$$R^{-1} \leqq \liminf_{n\to\infty} \Delta_{np'}/\Delta_{np} \leqq \limsup_{n\to\infty} \Delta_{np'}/\Delta_{np} \leqq R$$

where

$$R = 16e^{9/32}(2\pi)^{\frac{1}{2}}\left(\tfrac{1}{2}+\frac{4c}{(2\pi e)^{\frac{1}{2}}}\right)\Big/(\tfrac{1}{2}+\tfrac{7}{16}c).$$

However, note that

$$1 \leqq \left(\tfrac{1}{2}+\frac{4c}{(2\pi c)^{\frac{1}{2}}}\right)\Big/(\tfrac{1}{2}+\tfrac{7}{16}c) \leqq 64/(7(2\pi e)^{\frac{1}{2}})$$

since $4/(2\pi e)^{\frac{1}{2}} > 7/16$. Consequently, whatever the value of c,

$$7(2^{-10})e^{7/32} \leqq \liminf_{n\to\infty} \Delta_{np'}/\Delta_{np} \leqq \limsup_{n\to\infty} \Delta_{np'}/\Delta_{np} \leqq \tfrac{1}{7}2^{10}e^{-7/32}$$

and the proof of part (ii) in the lattice case and part (iii) is complete.

Finally, suppose that $\limsup_{|t|\to\infty} |Ee^{itX_1}| < 1$ (often known as Cramér's condition (C)). We may, in view of the foregoing discussion, confine attention to the case $E|X_1|^3 < \infty$, $EX_1^3 = 0$. Then, if X_1 does not have the unit normal distribution there is a first (integer) moment $\alpha_{k+2} = EX_1^{k+2}$, $k \geqq 2$, which does not match that of the unit normal law. This holds since the normal law is characterized by its moments. By assumption $E|X_1|^{k+2} < \infty$.

Now we can approximate $F_n(x)$ by $\Phi(x) + R_{nk}(x)$ where

$$R_{nk}(x) = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}\sum_{s=1}^{k} P_s(x)n^{-\frac{1}{2}s}$$

is a portion of the Čebyšev-Cramér series corresponding to X_1 (see for example Gnedenko and Kolmogorov [4], Chap. 8), the $P_i(x)$ being polynomials of degree $3j-1$ whose coefficients depend on the first $j+2$ moments of X_1. Furthermore, $P_j(x)\equiv 0$ if the first $j+2$ moments of X_1 match those of the unit normal law.

Now use $\|u(x)\|_p$, $1\leqq p\leqq\infty$, to denote the norm

$$\begin{aligned}\|u(x)\|_p &= \left(\int_{-\infty}^{\infty} |u_n(x)|^p dx\right)^{1/p}, \qquad 1\leqq p<\infty,\\ &= \sup_x |u_u(x)|, \qquad p=\infty.\end{aligned}$$

Then,

$$\Delta_{np}=\|F_n(x)-\Phi(x)\|_p\leqq\|R_{nk}(x)\|_p+\|F_n(x)-\Phi(x)-R_{nk}(x)\|_p \tag{11}$$

and

$$\|F_n(x)-\Phi(x)\|_p\geqq\|R_{nk}(x)\|_p-\|F_n(x)-\Phi(x)-R_{nk}(x)\|_p. \tag{12}$$

But, it follows from Theorem 1 of Erickson [1] for the case $1\leqq p<\infty$ and Theorem 1 of Ibragimov [9] for the case $p=\infty$ that

$$\|F_n(x)-\Phi(x)-R_{nk}(x)\|_p=o(n^{-k/2}) \tag{13}$$

as $n\to\infty$. On the other hand, since the moments EX_1^j, $1\leqq j\leqq k+1$, match those of the unit normal distribution, $P_j(x)\equiv 0$ for $1\leqq j\leqq k-1$ and

$$\|R_{nk}(x)\|_p=n^{-k/2}\,\|(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}P_k(x)\|_p,$$

so that, using (11), (12) and (13),

$$\Delta_{np}\sim\|(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}P_k(x)\|_p\,n^{-k/2}$$

as $n\to\infty$. But, the condition that the moments EX_1^j, $1\leqq j\leqq k+1$, match those of the unit normal law gives

$$P_k(x)=-(EX_1^{k+2}/(k+2)!)\,H_{k+1}(x)$$

where $H_j(x)$ denotes the jth Hermite-Chebyshev polynomial,

$$H_j(x)=(-1)^j e^{\frac{1}{2}x^2}\frac{d^j}{dx^j}e^{-\frac{1}{2}x^2}.$$

Thus,

$$\Delta_{np}\sim(2\pi)^{-\frac{1}{2}}EX_1^{k+2}((k+2)!)^{-1}\,\|e^{-\frac{1}{2}x^2}H_{k+1}(x)\|_p n^{-k/2}$$

and for $p'\neq p$,

$$\Delta_{np'}/\Delta_{np}\to\|e^{-\frac{1}{2}x^2}H_{k+1}(x)\|_{p'}/\|e^{-\frac{1}{2}x^2}H_{k+1}(x)\|_p$$

as $n\to\infty$. However, in view of the Lemma, this limit may be bounded between the universal constants U^{-1} and U (which may depend on k) where

$$U=\max_p\|e^{-\frac{1}{2}x^2}H_{k+1}(x)\|_p/\min_p\|e^{-\frac{1}{2}x^2}H_{k+1}(x)\|_p$$

and this completes the proof.

Proof of Theorem 3. When $x^3 P(|X_1|>x) \to \infty$ as $x \to \infty$, we have from Theorem 2 that there exist universal constants C_1 and C_2 such that

$$0 < C_1 \leqq \liminf_{n\to\infty} \delta_n^{-1} \Delta_{np} \leqq \limsup_{n\to\infty} \delta_n^{-1} \Delta_{np} \leqq C_2 < \infty, \tag{14}$$

and hence the result of the theorem follows (with the aid of the Lemma) if there are universal constants C_3 and C_4 for which

$$0 < C_3 \leqq \liminf_{n\to\infty} \delta_n^{-1} \Delta_{np}(A) \leqq \limsup_{n\to\infty} \delta_n^{-1} \Delta_{np}(A) \leqq C_4 < \infty \tag{15}$$

where $\Delta_{np}(A) = \Delta_{np}((-A, A))$. However,

$$\limsup_{n\to\infty} \delta_n^{-1} \Delta_{np}(A) \leqq \limsup_{n\to\infty} \delta_n^{-1} \Delta_{np} \leqq C_2 < \infty,$$

while, from the Lemma,

$$\liminf_{n\to\infty} \delta_n^{-1} \Delta_{np}(A) \geqq \liminf_{n\to\infty} \delta_n^{-1} \Delta_{n1}(A),$$

and hence (15) is satisfied if $\delta_n^{-1} \Delta_{n1}(A)$ is universally bounded below (away from zero) as $n \to \infty$. This result, however, follows from a minor modification of the proof of Theorem 1.3 of [5] wherein it is necessary to replace (1.14) by

$$\left| \int_{-\infty}^{\infty} \{\alpha^n(t/n^{\frac{1}{2}}) - e^{-\frac{1}{2}t}\} t^{-2}(1 - \cos At)\, dt \right| \leqq \pi \Delta_{n1}(A)$$

and to observe that each constant being used can be made universal.

3. Concluding Remarks

It is clearly a plausible conjecture that all the metrics Δ_{np}, $1 \leqq p \leqq \infty$, have the same asymptotic behaviour in general under $EX_1 = 0$, $EX_1^2 = 1$ without the constraints imposed in Theorem 1. The principal difficulty in establishing this result is in dealing with the class of discrete non-lattice distributions. If $\limsup_{|t|\to\infty} |Ee^{itX_1}| = 1$ and $E|X_1|^r < \infty$ for some integer $r > 3$ it is necessary to augment the Chebyshev-Cramér approximation to $F_n(x)$ with discontinuous terms and these are crucial in determining the asymptotic behaviour of Δ_{np}. In the non-lattice case there appears to be no general and systematic way of accomplishing this aim.

The other difficulty in extending Theorem 1 is in coping with cases such as $\delta_n \asymp n^{-\frac{1}{2}}$ or $n^{-\frac{1}{2}} \delta_n \to 0$ as $n \to \infty$ but $E|X_1|^3 = \infty$. Here the problems are mostly those of establishing universality of bounds for asymptotic equivalence. Non-universal bounds on the ratios are mostly obtainable.

Finally, a small improvement to the result of Theorem 1 may be achievable, by using the ideas of Osipov [11], to extend the conditions (iii) by removing the requirement that $E|X_1|^3 < \infty$ and the condition (iv) to include the case where $EX_1^k = \int_{-\infty}^{\infty} x^k \, d\Phi(x)$, $k = 1, 2, \ldots, r-1$, $E|X_1|^r = \infty$.

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