On Recurrence

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Summary. Let T be a non-singular ergodic automorphism of a Lebesgue space $(X, \mathscr{S}, \mu)$ and let $f: X \to \mathbb{R}$ be a measurable function. We define the notion of recurrence of such a function f and introduce the recurrence set $R(f) = \{\alpha \in \mathbb{R}: f - \alpha \text{ is recurrent}\}$. If $\rho = \log \dfrac{d\mu T}{d\mu}$, then $R(\rho) = \{0\}$, but in general recurrence sets can be very complicated. We prove various conditions for a number $\alpha \in \mathbb{R}$ to lie in $R(f)$ and, more generally, for $R(f)$ to be non-empty. The results in this paper have applications to the theory of random walks with stationary increments.

§ 1. Introduction

While the recurrence properties of classical, independent random walks on $\mathbb{R}^n$, $n \geq 1$, are well understood, the corresponding theory for random walks with stationary, not necessarily independent, increments is still in its infancy. A definition of recurrence and transience in a general context is given in [13], but these concepts have been used earlier, if implicitly, both in probability theory and in ergodic theory (e.g. [8, 10, 11, 14]). During the last few years some general results on the recurrence of random walks with stationary increments have appeared ([1–4]), amongst them the following sufficient condition for recurrence: Let $(X_k, k \geq 1)$ be a real valued stationary stochastic process, and let $Y_m = \sum_{k=1}^{m} X_k$, $m \geq 1$, be the associated random walk. If $\lim_m Y_m/m = 0$ in probability, then $(Y_m, m \geq 1)$ is recurrent (i.e. a.e. sample path visits every neighbourhood of zero infinitely often). This result has been obtained independently by a number of mathematicians, including F.M. Dekking [4], B. Weiss and myself. In the special case where the X_k are integrable one obtains the well known result that $(Y_m, m \geq 1)$ is recurrent if and only if $EX_k = 0$ (cf. [1, 11]). As this last example indicates, the random walk $(Y_m, m \geq 1)$ may be transient (= not recurrent) only because of the presence of a 'drift term': there

may exist a constant $\alpha \in \mathbb{R}$ such that $(Y_m - m\alpha, \, m \geq 1)$ is recurrent (this amounts to considering the stationary process $(X_k - \alpha, \, k \geq 1)$). One is thus led to the definition of the recurrence set of $(Y_m, \, m \geq 1)$ consisting of all $\alpha \in \mathbb{R}$ which make $(Y_m - m\alpha, \, m \geq 1)$ recurrent. This recurrence set is not directly related to the objects studied in ([5–7]), but there are some connections.

It is a much harder problem to determine whether $(Y_m, \, m \geq 1)$ has a non-empty recurrence set than to decide whether $(Y_m, \, m \geq 1)$ itself is recurrent. In Sect. 3 we prove the following results. Let $(X_k, \, k \geq 1)$ be an ergodic, stationary, real valued stochastic process, define the random walk $Y_m = \sum_{k=1}^{m} X_k$ as before, and denote by Σ the set of limit points of the distributions of the Y_m/m in the vague topology. If the set Σ is uniformly absolutely continuous, and if the distribution of the Y_m/m are uniformly tight, then $R(f) \neq \phi$ (Corollary 3.13). If any of the measures in Σ is *singular* at a point $\alpha \in \mathbb{R}$, then $(Y_m - m\alpha, \, m \geq 1)$ is recurrent (a finite measure σ on $\mathbb{R}$ is singular at $\alpha \in \mathbb{R}$ if $\limsup_{\varepsilon \to 0} \sigma(\alpha - \varepsilon, \, \alpha + \varepsilon)/2\varepsilon = \infty$) (cf. Corollary 3.10). Finally we show that, if $\alpha \in \mathbb{R}$ does not belong to the recurrence set of (Y_m), there exists, for every $\delta > 0$, a limit point ν of the sequence of distributions of the random variables $(Y_m/m, \, m \geq 1)$ with $\nu(\alpha - \eta, \, \alpha + \eta) < \delta\eta$ for some $0 < \eta < 1$ (Corollary 3.11). This last condition implies in particular the well known result that the recurrence set of the Cauchy random walk is equal to $\mathbb{R}$ (Corollary 3.12). All these statements are obtained in the course of investigating the following problem: Let T be a non-singular, ergodic automorphism of a Lebesgue probability space $(X, \mathscr{S}, \mu)$ and let $f \colon X \to \mathbb{R}^n$ a measurable function. Define, for $m \geq 1$, $f^*(m, \cdot) \colon X \to \mathbb{R}^n$ by $f^*(m, \cdot) = \sum_{k=0}^{m-1} f \cdot T^k$.

In Definition 2.1 we introduce the notion of recurrence for the function f which is equivalent to the usual condition under the assumption that T preserves μ: in this case recurrence means that, for every $\varepsilon > 0$, and for μ-a.e. $x \in X$, $\| f^*(m, x) \| < \varepsilon$ for infinitely many m. The assumption that T is only non-singular means that we are considering random walks obtained from certain non-stationary processes (cf. [12]). In particular we consider the Radon-Nikodym derivative $\rho = \log \dfrac{d\mu T}{d\mu}$ and prove that f is recurrent if and only if the pair $(f, \rho) \colon X \to \mathbb{R}^{n+1}$ is recurrent (Theorem 2.5). This allows us to rephrase the condition of recurrence as follows: f is recurrent if, for every $\varepsilon > 0$, and for μ-a.e. $x \in X$, $\| f^*(m, x) \| + \left| \log \dfrac{d\mu T^m}{d\mu}(x) \right| < \varepsilon$ for infinitely many m (Theorem 2.7). If we define the recurrence set $R(f)$ as before by $R(f) = \{ \alpha \in \mathbb{R}^n \colon f - \alpha$ is recurrent$\}$, then $R(f)$ is always a Borel set (Proposition 2.10), and for the special case of the function $\rho \colon X \to \mathbb{R}$ we obtain $R(\rho) = \{0\}$ (Corollary 2.6). Section 2 ends with a short discussion of the special case of classical, independent, random walks and the additional difficulties arising from the relaxation of independence even when T preserves μ. Two of the proofs in this section (Lemma 2.3 and Proposition 2.4) are taken from [13]: I have included them in order to make the paper essentially independent of reference [13] which may be difficult to obtain. Proposition 2.4 and the basic idea in Lemma 2.3 are both

due to Krieger [10], and Theorems 2.7 and 2.8 are generalizations of the corresponding results for measure preserving T in [13]. The contents of Sect. 3 have already been outlined at the beginning of this introduction. The proofs make use of some tools in ergodic theory, mainly orbit equivalence and cohomological considerations.

Section 4 deals with recurrence sets and gives examples and conditions for recurrence sets to be empty or to consist of a single point. Examples 4.5 and 4.6 show how disconnected recurrence sets can arise in a natural context. Example 4.5 is an expression of the fact that every four-dimensional independent random walk is transient, while Example 4.6 arises from an adding machine through the interchange of two digits.

§2. Recurrence and Radon-Nikodym Derivatives

Throughout this section T will denote a fixed non-singular, aperiodic, conservative automorphism of a non-atomic Lebesgue space $(X, \mathcal{S}, \mu)$. For every measurable function $f: X \to \mathbb{R}^n$, $n \geq 1$, we define a map $f^*: \mathbb{Z} \times X \to \mathbb{R}^n$ by setting

$$f^*(k, x) = \begin{cases} \sum_{m=0}^{k-1} f(T^m x) & \text{if } k \geq 1, \\ 0 & \text{if } k = 0, \\ -f^*(-k, T^k x) & \text{if } k < 0. \end{cases} \tag{2.1}$$

The map f^* satisfies

$$f^*(k, T^l x) + f^*(l, x) = f^*(k+l, x) \tag{2.2}$$

for every $k, l \in \mathbb{Z}$, $x \in X$, and is thus a $(1-)$ *cocycle* for action $(m, x) \to T^m x$ of $\mathbb{Z}$ on $(X, \mathcal{S}, \mu)$. The function f (or the cocycle f^*) is called a *coboundary* if there exists a measurable map $b: X \to \mathbb{R}^n$ with

$$f = b \cdot T - b \qquad \mu\text{-a.e.}, \tag{2.3}$$

and two measurable functions $f_i: X \to \mathbb{R}^n$, $i = 1, 2$, are said to be *cohomologous* if $f_1 - f_2$ is a coboundary.

2.1. Definition. A measurable function $f: X \to \mathbb{R}^n$ is called *recurrent* if, for every $B \in \mathcal{S}$ with $\mu(B) > 0$, and for every $\varepsilon > 0$,

$$\mu(B \cap T^{-m} B \cap \{x: \|f^*(m, x)\| < \varepsilon\}) > 0 \tag{2.4}$$

for some $m \neq 0$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. If f is not recurrent it will be called *transient*. Finally, the recurrence set $R(f)$ of f is defined by

$$R(f) = \{\alpha \in \mathbb{R}^n: f - \alpha \text{ is recurrent}\}. \tag{2.5}$$

An important and useful property of $R(f)$ is that it is a cohomology invariant:

2.2. Proposition. *Let* $f: X \to \mathbb{R}^n$, $g: X \to \mathbb{R}^n$ *be measurable functions. Then* $R(f) = R(f + gT - g)$.

For the (elementary) proof of Proposition 2.2 we refer to [13, Proposition 3.14]. In this section we shall discuss some basic properties of recurrent and transient functions and, in particular, the connection between the recurrence properties of a given function f and the Radon-Nikodym derivative $d\mu T/d\mu$. We start with a short technical discussion. Let $[T]$ denote the *full group* of T, i.e. the group of all non-singular automorphisms V of $(X, \mathscr{S}, \mu)$ satisfying $Vx \in \{T^n x : n \in \mathbb{Z}\}$ for μ-a.e. $x \in X$. For every $V \in [T]$ we can find a measurable, a.e. unique function $n(V, \cdot): X \to \mathbb{Z}$ satisfying

$$T^{n(V, x)} x = Vx \qquad \mu\text{-a.e.} \tag{2.6}$$

If $f: X \to \mathbb{R}^n$ is measurable and $V \in [T]$, put

$$f(V, x) = f^*(n(V, x), x), \qquad x \in X \tag{2.7}$$

where $n(V, \cdot)$ is given by (2.6) and f^* by (2.1). Equation (2.2) shows that, for every $V, W \in [T]$,

$$f(V, Wx) + f(W, x) = f(VW, x) \qquad \mu\text{-a.e.} \tag{2.8}$$

In the special case where $V = T^k$ we have

$$f(T^k, \cdot) = f^*(k, \cdot) \qquad \mu\text{-a.e.} \tag{2.9}$$

The following lemma provides one of the basic tools for our analysis (cf. [13, Lemma 4.1]).

2.3. Lemma. *Let* $f: X \to \mathbb{R}$ *be a transient measurable function. Then there exists a set* $B \in \mathscr{S}$ *with* $0 < \mu(B) < 1$, *an* $\varepsilon > 0$, *a measurable function* $b: X \to \mathbb{R}$, *and an automorphism* $S \in [T]$ *with* $[S] = [T]$ *(i.e. with* $\{S^n x : n \in \mathbb{Z}\} = \{T^n x : n \in \mathbb{Z}\}$ *for* μ-*a.e.* $x \in X$), *such that the following conditions are satisfied:*

$$(1) \quad b(x) = 0 \text{ for } \mu\text{-a.e. } x \in B; \tag{2.10}$$

$$(2) \quad \text{If } f_1 = f + bT - b \text{ and } f_1(S, \cdot) \text{ is defined as in (2.7) then}$$

$$f_1(S, \cdot) = 0 \qquad \mu\text{-a.e. on } X \setminus S^{-1} B \tag{2.11}$$

and

$$f_1(S, \cdot) \geqq \varepsilon \qquad \mu\text{-a.e. on } S^{-1} B. \tag{2.12}$$

Proof. The basic idea of this proof is due to W. Krieger [10]. Since f is transient we can find an $\varepsilon > 0$ and a $B \in \mathscr{S}$ with $0 < \mu(B) < 1$ such that $|f^*(k, x)| \geqq \varepsilon$ for every $k \in \mathbb{Z}$ and every $x \in B \cap T^{-k} B$. For $x \in B$, let

$$A_x = \{f^*(k, x): k \in \mathbb{Z} \text{ and } T^k x \in B\}.$$

Equation (2.2) implies that, for every $x \in B$, and for every $\alpha, \beta \in A_x$ with $\alpha \neq \beta$, we have $|\alpha - \beta| \geqq \varepsilon$. This allows us to define, for every $x \in B$,

$$\psi_+(x) = \begin{cases} \min \{\alpha \in A_x : \alpha > 0\} & \text{if this set is non-empty,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\psi_-(x) = \begin{cases} \max\{\alpha \in A_x : \alpha < 0\} & \text{if this set is non-empty,} \\ 0 & \text{otherwise,} \end{cases}$$

and to conclude that $\psi_\pm : B \to \mathbb{R}$ are measurable functions with $\psi_+ \geqq \varepsilon$ and $\psi_- \leqq -\varepsilon$ μ-a.e. on B. There exist unique measurable functions $m_\pm : X \to \mathbb{Z}$ with $m_\pm = 0$ on $X \setminus B$ and such that $T^{m_\pm(x)} x \in B$ and $f^*(m_\pm(x), x) = \psi_\pm(x)$ for every $x \in B$. Define $V_\pm : X \to X$ by

$$V_\pm x = T^{m_\pm(x)} x, \qquad x \in X. \tag{2.13}$$

Equation (2.2) and the definitions of $\psi_\pm$ show that V_+ and V_- are non-singular endomorphisms of X with $V_+ V_- x = V_- V_+ x = x$ μ-a.e., so that $V_+ \in [T]$. Furthermore,

$$f(V_+, x) = \begin{cases} \psi_+(x) & \text{for } x \in B, \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

and

$$\{V_+^m x : m \in \mathbb{Z}\} = \{T^m x : m \in \mathbb{Z}\} \cap B \tag{2.15}$$

for μ-a.e. $x \in B$. Next we define a measurable function $r_B : X \to \mathbb{Z}$ by

$$r_B(x) = \begin{cases} \min\{m > 0 : T^m x \in B\} & \text{if } x \in B \text{ and if this set is non-empty,} \\ 0 & \text{otherwise,} \end{cases}$$

and define $V_B : X \to X$ by

$$V_B x = T^{r_B(x)} x, \qquad x \in X.$$

Apart from a minor change V_B is the transformation induced by T on B. Note that $V_B \in [T]$ and that

$$V_B^{-1} x = T^{-r_B(V_B^{-1} x)} x \qquad \mu\text{-a.e.} \tag{2.16}$$

Turning now to the definition of S and b we put

$$S = V_+ V_B^{-1} T \tag{2.17}$$

and define $b : X \to \mathbb{R}$ μ-a.e. by setting, for every $x \in B$ and $k \in \mathbb{Z}$ with $0 \leq k < r_B(x)$,

$$b(T^k x) = \begin{cases} -f^*(k-1, T^{-k} x) & \text{if } k \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

From (2.15) and (2.17) one easily concludes that $[S] = [T]$. The function b in (2.18) is obviously measurable and satisfies (2.10). Put $f_1 = f + bT - b$. Then

$$f_1(x) = f^*(r_B(V_B^{-1} Tx), V_B^{-1} Tx) \qquad \mu\text{-a.e.,} \tag{2.19}$$

and, in particular, $f_1 = 0$ on $X \setminus T^{-1} B$. Since $S = T$ on $X \setminus T^{-1} B = X \setminus S^{-1} B$ we also have $f_1(S, \cdot) = f_1 = 0$ on $X \setminus S^{-1} B$, as claimed in (2.11). Finally, if $x \in S^{-1} B = T^{-1} B$, (2.7) and (2.17) show that

$$f_1(S, x) = f_1^*(n(V_+ V_B^{-1} T, x), x),$$

and (2.13) and (2.16) yield that

$$n(V_+ V_B^{-1} T, x) = n(V_+, V_B^{-1} Tx) + n(V_B^{-1}, Tx) + 1$$
$$= m_+(V_B^{-1} Tx) - r_B(V_B^{-1} Tx) + 1.$$

Equations (2.2), (2.10) and (2.19) now imply that

$$f_1(S, x) = f^*(m_+(V_B^{-1} Tx), V_B^{-1} Tx) - f^*(r_B(V_B^{-1} Tx), V_B^{-1} Tx)$$
$$+ f^*(r_B(V_B^{-1} Tx), V_B^{-1} Tx) = \psi_+(V_B^{-1} Tx) \geqq \varepsilon.$$

This proves (2.12).   □

Before we can discuss the connection between recurrent functions and Radon-Nikodym derivatives we apply Lemma 2.3 to the recurrence of Radon-Nikodym derivatives.

2.4. Proposition [10, 13]. *Let*

$$\rho = \log \frac{d\mu T}{d\mu} \qquad (2.20)$$

Then the function $\rho: X \to \mathbb{R}$ is recurrent.

Proof. Assume that ρ is transient and apply Lemma 2.3 to find $B \in \mathscr{S}$, $\varepsilon > 0$, $S \in [T]$ and $b: X \to \mathbb{R}$ with the properties stated there. We define a σ-finite measure ν on $(X, \mathscr{S})$ by $d\nu(x) = e^{b(x)} d\mu(x)$ and conclude from (2.10) that $0 < \mu(B) = \nu(B) < 1$, and that $\rho_1(S, \cdot) = \log \dfrac{d\nu S}{d\nu}$ satisfies $\rho_1(S, \cdot) = 0$ on $X \setminus S^{-1} B$ and $\rho_1(S, \cdot) \geqq \varepsilon$ on $S^{-1} B$. In particular, the induced transformation S_B of S on B satisfies

$$\frac{d\nu S_B}{d\nu} \geqq e^\varepsilon > 1 + \varepsilon$$

ν-a.e. on B, so that

$$\nu(B) = \nu(S_B B) = \int_B \frac{d\nu S_B}{d\nu} d\nu > (1 + \varepsilon) \nu(B),$$

which is impossible. Hence ρ is recurrent.   □

2.5. Theorem. *Let T be a non-singular, aperiodic, conservative automorphism of a non-atomic Lebesgue space $(X, \mathscr{S}, \mu)$, and let let $\rho: X \to \mathbb{R}$ be defined by (2.20). For every measurable function $f: X \to \mathbb{R}^n$, $n \geqq 1$, the following conditions are equivalent.*

(1) *f is recurrent;*
(2) *The function $(f, \rho): X \to \mathbb{R}^{n+1}$, given by*

$$(f, \rho)(x) = (f(x), \rho(x)),$$

is recurrent.

Proof. Clearly (2) implies (1). If f is recurrent, but (f, ρ) transient, we can find a set $B \in \mathscr{S}$ with $\mu(B) > 0$ and an $\varepsilon > 0$ such that $|\rho^*(m, x)| = \left| \log \dfrac{d\mu T^m}{d\mu}(x) \right| > \varepsilon$

whenever $x \in B \cap T^{-m}B$ and $\|f^*(m, x)\| < 2\varepsilon$, $m \neq 0$. Consider the skew product transformation W on $Y = X \times \mathbb{R}^n$ given by

$$W(x, \alpha) = (Tx, \alpha + f(x))$$

for every $x \in X$, $\alpha \in \mathbb{R}^n$. Define a measure $\tilde{\mu}$ on Y by setting $d\tilde{\mu}(x, \alpha) = d\mu(x) d\alpha$, where $d\alpha$ stands for integration with respect to Lebesgue measure on $\mathbb{R}^n$. The transformation W is a non-singular, aperiodic automorphism of $(Y, \tilde{\mu})$, and the recurrence of f is easily seen to imply that W is conservative. By Proposition 2.4, the function

$$\log \frac{d\tilde{\mu}W}{d\tilde{\mu}}(x, \alpha) = \log \frac{d\mu T}{d\mu}(x), \quad (x, \alpha) \in Y,$$

is recurrent for the automorphism W on $(Y, \tilde{\mu})$. Put $C = B \times \{\alpha \in \mathbb{R}^n : \|\alpha\| < \varepsilon\}$ and choose an integer $m_0 \neq 0$ with

$$\tilde{\mu}\left(C \cap W^{-m_0} C \cap \left\{(x, \alpha) : \left|\log \frac{d\tilde{\mu}W^{m_0}}{d\tilde{\mu}}(x, \alpha)\right| < \varepsilon\right\}\right) > 0.$$

For this integer m_0 we obviously have

$$\mu(B \cap T^{-m_0} B \cap \{x : \|f^*(m_0, x)\| < 2\varepsilon\} \cap \{x : |\rho^*(m_0, x)| < \varepsilon\}) > 0,$$

contrary to our assumption concerning B and ε. This contradiction shows that (f, ρ) is recurrent. $\square$

The remarkable property of the function ρ described in Theorem 2.5 has several interesting consequences. The first one concerns the recurrence set $R(\rho)$ of ρ.

2.6. Corollary. *Let ρ be given by (2.20). The $R(\rho) = \{0\}$.*

Proof. Proposition 2.4 implies that $0 \in R(\rho)$. If $\alpha \in R(\rho)$ for some real number $\alpha \neq 0$ we can apply Theorem 2.5 to see that the function $F : X \to \mathbb{R}^2$, given by $F(x) = (\rho(x) - \alpha, \rho(x))$, is recurrent. Put $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and note that $A \cdot F : X \to \mathbb{R}^2$, given by $A \cdot F(x) = (\alpha, \rho(x))$, must again be recurrent, which is absurd. This contradiction proves the corollary. $\square$

The remainder of this section will be devoted to characterising recurrence or transience of a function $f : X \to \mathbb{R}^n$ in terms of the asymptotic behaviour of the sequences $((f^*(k, x), \rho^*(k, x)), k \in \mathbb{Z})$ for individual points $x \in X$. This characterization will enable us to see that the notion of recurrence introduced here coincides with the classical concept of recurrence for random walks.

2.7. Theorem. *Let T be a non-singular, aperiodic, conservative automorphism of a non-atomic Lebesgue probability space $(X, \mathscr{S}, \mu)$ and let ρ be defined by (2.20). A measurable function $f : X \to \mathbb{R}^n$ is recurrent if and only if*

$$\liminf_{m \to \infty}(\|f^*(m, x)\| + |\rho^*(m, x)|) = 0 \quad \mu\text{-a.e.} \tag{2.21}$$

Proof. If f is recurrent, Theorem 2.5 shows that (f, ρ) is recurrent, and (2.21) is an easy consequence of that. To prove the converse we define a skew product transformation W on $Y = X \times \mathbb{R}^{n+1}$ by setting

$$W(x, \alpha, t) = (Tx, \alpha + f(x), t + \rho(x)) \tag{2.22}$$

for every $x \in X$, $\alpha \in \mathbb{R}^n$, $t \in \mathbb{R}$. Let $\tilde{\mu}$ denote the measure $d\tilde{\mu}(x, \alpha, t) = d\mu(x) \cdot d\alpha \cdot e^{-t} dt$ on Y. Then W is a measure preserving, aperiodic automorphism of the infinite measure space $(Y, \tilde{\mu})$. Put

$$C = X \times \{(\alpha, t) \in \mathbb{R}^n \times \mathbb{R} : \|\alpha\| + |t| < 1\}$$

and denote by v the restriction of $\tilde{\mu}$ to C. Condition (2.21) implies that the induced transformation W_C of W on C is well defined. Since W_C is an aperiodic automorphism of the finite measure space (C, v) which preserves v, W_C must be conservative. This in turn implies the recurrence of the pair (f, ρ), and, in particular, the recurrence of f. $\square$

If T is ergodic, the following result gives a simple characterization of transient functions.

2.8. Theorem. *Let T be a non-singular, ergodic automorphism of a non-atomic Lebesgue probability space $(X, \mathscr{S}, \mu)$, and let $f: X \to \mathbb{R}^n$ be a measurable function. Then f is transient if and only if*

$$\lim_{m \to \infty} (\|f^*(m, x)\| + |\rho^*(m, x)|) = \infty \qquad \mu\text{-a.e.,} \tag{2.23}$$

where ρ is defined in (2.20).

Proof. In view of Theorem 2.7 we only have to prove that the transience of f implies (2.23). Assume therefore that f is transient and consider the automorphism W of the measure space $(Y, \tilde{\mu})$ defined in (2.22). Since f is transient, W is not conservative, and the ergodicity of T is easily seen to imply that W is completely dissipative: there exists a measurable set $D \subset Y$ with $W^k D \cap D = \phi$ for all $k \in \mathbb{Z}$, $k \neq 0$, and such that $\tilde{\mu}(Y \setminus \bigcup_{k \in \mathbb{Z}} W^k D) = 0$. If (2.23) is not satisfied we can find an $N > 0$ and a set $B \in \mathscr{S}$ with $\mu(B) > 0$ such that

$$\liminf_{m \to \infty} (\|f^*(m, x)\| + |\rho^*(m, x)|) < N$$

for every $x \in B$. Let $E = B \times \{(a, t) \in \mathbb{R}^{n+1} : \|\alpha\| + |t| < 1\}$, $F = X \times \{(\alpha, t) \in \mathbb{R}^{n+1} : \|\alpha\| + |t| < N+1\}$ and assume (without loss in generality) that $\tilde{\mu}(D \cap E) > 0$. For every $y \in D \cap E$, F contains infinitely many elements of $\{W^m y : m \geq 0\}$. Hence

$$\infty = \int \sum_{m \geq 0} \chi_{W^m(D \cap E) \cap F} \, d\tilde{\mu} = \sum_{m \geq 0} \tilde{\mu}(W^m(D \cap E) \cap F) \leq \tilde{\mu}(F) < \infty,$$

where χ_A denotes the indicator function of a set A. This contradiction shows that (2.23) is satisfied whenever f is transient. $\square$

2.9. Corollary. *Let T be a non-singular, aperiodic, conservative automorphism of the probability space $(X, \mathscr{S}, \mu)$ and let $f: X \to \mathbb{R}^n$ be a measurable function. Then f*

is transient if and only if

$$\mu(\{x: \lim_{m\to\infty} (\|f^*(m, x)\| + |\rho^*(m, x)|) = \infty\}) > 0. \tag{2.24}$$

Proof. Apply Theorem 2.8 to the ergodic components of μ. $\square$

2.10. Proposition. *Let T be a non-singular, aperiodic, conservative automorphism of a non-atomic Lebesgue space $(X, \mathscr{S}, \mu)$ and let $f: X \to \mathbb{R}^n$ be a measurable function. Then the recurrence set $R(f)$ of f is a Borel subset of $\mathbb{R}^n$.*

Proof. Without loss in generality we may assume that $\mu(X) = 1$. By Theorem 2.7,

$$R(f) = \{\alpha \in \mathbb{R}^n: \liminf_{m\to\infty} (\|f^*(m, x) - m\alpha\| + |\rho^*(m, x)|) = 0 \text{ a.e.}\}$$

and hence obviously a Borel set. $\square$

If T is a measure preserving, ergodic automorphism of a probability space $(X, \mathscr{S}, \mu)$ and $f: X \to \mathbb{R}^n$ a measurable function, Theorems 2.7 and 2.8 yield the following dichotomy: either $\lim_m \|f^*(m, x)\| = \infty$ a.e., in which case f is transient, or $\liminf_m \|f^*(m, x)\| = 0$ a.e., if f is recurrent. In the special case where $X = (\mathbb{R}^n)^{\mathbb{Z}}$, $\mu = \nu^{\mathbb{Z}}$ for some probability measure ν on $\mathbb{R}^n$, $\mathscr{S}$ the product σ-algebra, T the shift on X given by $(Tx)_k = x_{k+1}$ for every $x = (\ldots x_{-1}, x_0, x_1, \ldots) \in X$, and where $f: X \to \mathbb{R}^n$ is the function $f(x) = x_0$, this dichotomy reduces to the classical definitions of transience and recurrence of the independent random walk on $\mathbb{R}^n$ with distribution ν. In order to determine which of these two possibilities occurs, classical probability theory looks at the distributions of the random variables $(f^*(m, \cdot), m \geq 1)$: for every $m \geq 1$, define a probability measure ν_m on $\mathbb{R}^n$ by setting, for every Borel set $C \subset \mathbb{R}^n$, $\nu_m(C) = \mu\{x: f^*(m, x) \in C\}$. In the special case described here, $\nu_m = \nu^{*m}$, the m-th convolution power of ν, and f is recurrent if and only if $\sum_{m=1}^{\infty} \nu_m(C) = \infty$ for some *bounded* set C. Unfortunately this is no longer true if one removes the condition that the functions $f, f \cdot T, \ldots$ are mutually independent: if T is a measure preserving, ergodic automorphism of a probability space $(X, \mathscr{S}, \mu)$ and $f: X \to \mathbb{R}^n$ is measurable, f may be transient even though $\sum_{m=1}^{\infty} \nu_m(C) = \infty$ for some bounded Borel set $C \subset \mathbb{R}^n$, where ν_m is defined as above (cf. [1]). It is not difficult to prove that the classical result can be salvaged if one allows f to be modified by a coboundary: f is transient if and only if there exists a measurable function $b: X \to \mathbb{R}^n$ such that, for $f_1 = f + bT - b$, and for $\nu_m^{(1)}(C) = \mu\{x: f_1^*(m, x) \in C\}$, $\sum_{m=1}^{\infty} \nu_m^{(1)}(C) < \infty$ for every bounded Borel set $C \subset \mathbb{R}^n$. This fact may be moderately interesting, but it does not lead to any useful criteria for recurrence. In the next section we choose a different approach and study the distributions of the functions $f^*(m, \cdot)/m$, $m \geq 1$, for a given real valued function f on X in order to obtain some information about the recurrence set of f.

§3. Recurrence in the Measure Preserving Case

In the general framework of Sect. 2 it seems difficult to find any conditions for recurrence which go beyond the criterion given in Theorem 2.7. The much more challenging problem of describing the recurrence set $R(f)$ of a given function f has to be left completely open in this case. Even for a bounded function f it is not clear under which conditions $R(f) \neq \phi$ (cf. Proposition 4.2 and 4.4 for some special cases). If T is measure preserving and ergodic one obtains a further criterion described in the concluding remarks of Sect. 2, but once again the set $R(f)$ remains a mystery. In this section we restrict the scope of our investigation still further and assume that T is a measure preserving, ergodic automorphism of a non-atomic Lebesgue probability space $(X, \mathscr{S}, \mu)$ and that f is a measurable *real valued* function on X. We start with a slight improvement of Lemma 2.3.

3.1. Lemma. *Assume that* $f: X \to \mathbb{R}$ *is transient and let* $f_1 = f + bT - b$ *and* $S \in [T]$ *with* $[S] = [T]$ *satisfy* (2.10)–(2.12). *Put*

$$\Delta_0 = \int f_1(S, \cdot) \, d\mu. \tag{3.1}$$

Then we can find, for every $\delta \in \mathbb{R}$ *with* $0 < \delta < \Delta_0$, *a measurable function* $b_2: X \to \mathbb{R}$ *with*

$$f(S, x) + b_2(Sx) - b_2(x) > \delta \qquad \mu\text{-a.e.} \tag{3.2}$$

Proof. By (2.11) and (2.12), $\Delta_0 \geqq \varepsilon \mu(B) > 0$, and we can apply [9, Theorem 1 and Corollary 3] to find the required function b_2 satisfying (3.3). $\square$

3.2. Remark. To understand the geometrical significance of Δ_0, consider the skew product transformation W on $(X \times \mathbb{R}, \mu \times \lambda)$ (λ is the Lebesgue measure on $\mathbb{R}$) given by

$$W(x, t) = (Tx, t + f(x))$$

for every $(x, t) \in X \times \mathbb{R}$. As in the proof of Theorem 2.8 the transience of f implies the existence of a measurable set $D \subset X \times \mathbb{R}$ with $W^k D \cap D = \phi$ for $k \neq 0$ and such that $\tilde{\mu}(X \times \mathbb{R} \setminus (\bigcup_{k \in \mathbb{Z}} W^k D)) = 0$, where $\tilde{\mu} = \mu \times \lambda$. The measure $\tilde{\mu}(D)$ is independent of the actual choice of D, and it is easy to see that one of these possible choices is

$$D = \{(x, t) \in X \times \mathbb{R} : 0 \leqq t < f_1(S, x)\}.$$

Hence we have

$$\Delta_0 = \int f_1(S, \cdot) \, d\mu = \tilde{\mu}(D). \tag{3.3}$$

If $S \in [T]$ is the automorphism of $(X, \mathscr{S}, \mu)$ described in Lemma 2.3 we can define, as in (2.6), a measurable function $a: \mathbb{Z} \times X \to \mathbb{Z}$ by setting

$$S^{a(k, x)} x = T^k x \qquad \mu\text{-a.e.} \tag{3.4}$$

for every $k \in \mathbb{Z}$. For every $k, l \in \mathbb{Z}$,

$$a(k, T^l x) + a(l, x) = a(k + l, x) \qquad \mu\text{-a.e.} \tag{3.5}$$

Since $[S] = [T]$, one can find a null set $N \subset X$ with the following property: for every $x \in X \setminus N$ and every $i \in \mathbb{Z}$ there exists a unique $k_i \in \mathbb{Z}$ with

$$a(k_i, x) = i.$$

Using this fact we can prove the following lemma.

3.3. Lemma. *For every $m \geq 1$, $\eta > 0$ we have*

$$\frac{1}{2m} \sum_{1 \leq |k| \leq m} \mu\{x: |a(k, x)| \leq |k| \eta\} \leq 2\eta \tag{3.6}$$

and

$$\frac{1}{m} \sum_{n \geq 1} \sum_{1 \leq |k| \leq m \cdot 2^n} \mu\{x: |a(k, x)| \leq |k| \eta/2^n\} \leq 8\eta. \tag{3.7}$$

Proof. For a.e. $x \in X$ we have

$$\#\{k \in \mathbb{Z}: 1 \leq |k| \leq m \text{ and } |a(k, x)| \leq |k| \eta\}$$
$$\leq \#\{k \neq 0: |a(k, x)| \leq m\eta\} \leq 2m\eta,$$

and (3.6) follows by integrating with respect to μ. To prove (3.7), note that

$$\frac{1}{m} \sum_{n \geq 0} \#\{k \in \mathbb{Z}: 1 \leq |k| \leq m \cdot 2^n, |a(k, x)| \leq |k| \eta/2^n\}$$
$$= \frac{1}{m} \sum_{1 \leq |i| \leq \eta m} \#\{n \geq 0: |k_i| \leq m \cdot 2^n \leq m |k_i| \eta/|i|\}$$
$$\leq \frac{1}{m} \sum_{1 \leq |i| \leq \eta m} (\#\{n \geq 0: 1 \leq m \cdot 2^n \leq m\eta/|i|\} + 1)$$
$$\leq 8\eta,$$

and the result follows once again by integration. $\square$

We define probability measures $\{\sigma_l: l \in \mathbb{Z}, \ l \neq 0\}$ and $\{\tau_k: k \geq 1\}$ on $\mathbb{R}$ by setting, for every Borel set $C \subset \mathbb{R}$, $l \neq 0$, $k \geq 1$,

$$\sigma_l(C) = \mu\{x: f^*(l, x)/l \in C\} \tag{3.8}$$

and

$$\tau_k(C) = \frac{1}{2k} \sum_{1 \leq |l| \leq k} \sigma_l(C). \tag{3.9}$$

Our next lemma implies that the asymptotic behaviour of these measures is unaffected if we replace f by a cohomologous function. Let $b': X \to \mathbb{R}$ be a measurable function and define probability measures σ_l', $l \neq 0$, on $\mathbb{R}$ by

$$\sigma_l'(C) = \mu\{x: (f^*(l, x) + b'(T^l x) - b'(x))/l \in C\}$$

for every Borel set $C \subset \mathbb{R}$.

3.4. Lemma. *For all $a < b$ in $\mathbb{R}$ and $\varepsilon > 0$,*

$$\liminf_{|l| \to \infty} (\sigma_l[a - \varepsilon, b + \varepsilon] - \sigma'_l[a, b]) \geqq 0$$

and

$$\liminf_{|l| \to \infty} (\sigma'_l[a - \varepsilon, b + \varepsilon] - \sigma_l[a, b]) \geqq 0.$$

Proof. This is obvious from the definitions.   □

3.5. Lemma. *Assume that $f: X \to \mathbb{R}$ is transient and define $\Delta_0 > 0$ as in (3.1). Then we have, for every $\eta > 0$,*

$$\limsup_{k \to \infty} \tau_k([-\eta, \eta]) \leqq 2\eta/\Delta_0, \tag{3.10}$$

and

$$\limsup_{m \to \infty} \sum_{n=1}^{N} 2^n \cdot \tau_{m \cdot 2^n}([-\eta/2^n, \eta/2^n]) \leqq 8\eta/\Delta_0 \tag{3.11}$$

for every $N \geqq 1$.

Proof. For every $\delta \in \mathbb{R}$ with $0 < \delta < \Delta_0$, Lemma 3.1 allows us to find a measurable function $b_2: X \to \mathbb{R}$ such that $f_2 = f + b_2 T - b_2$ satisfies

$$f_2(S, x) = f(S, x) + b_2(Sx) - b_2(x) > \delta$$

μ-a.e. From this one concludes easily that

$$|f_2^*(k, x)| \geqq \delta |a(k, x)| \qquad \mu\text{-a.e.},$$

for every $k \in \mathbb{Z}$, where $a: \mathbb{Z} \times X \to \mathbb{Z}$ is defined in (3.4). Define, for every $l \in \mathbb{Z}$, $l \neq 0$, and $k \geqq 1$, σ'_l and τ'_k by (3.8) and (3.9), respectively, with f replaced by f_2. For every $l \neq 0$ we have

$$\sigma'_l([-\eta, \eta]) = \mu\{x: |f_2(l, x)| \leqq |l| \eta\}$$
$$\leqq \mu\{x: |a(l, x)| \leqq |k| \eta/\delta\}.$$

Hence

$$\tau'_k([-\eta, \eta]) \leqq 2\eta/\delta$$

for every $k \geqq 1$, $\eta > 0$, by (3.6). Lemma 3.4 yields that

$$\limsup_k \tau_k([-\eta, \eta]) \leqq 2\eta/\delta$$

for every $\eta > 0$ and every δ with $0 < \delta < \Delta_0$, and this implies (3.10).

In the same manner one proves that, for every $m \geqq 1$, $\eta > 0$, $0 < \delta < \Delta_0$,

$$\sum_{n \geqq 1} 2^n \cdot \tau'_{m \cdot 2^n}([-\eta/2^n, \eta/2^n]) \leqq 8\eta/\delta,$$

by using (3.7). From Lemma 3.4 it follows that

$$\limsup_{m \to \infty} \sum_{n=1}^{N} 2^n \tau_{m \cdot 2^n}([-\eta/2^n, \eta/2^n]) \leqq 8\eta/\delta$$

for every $N \geqq 1$ and every δ with $0 < \delta < \Delta_0$, which implies (3.11).   □

We can now state the main result of this section which will allow us to obtain various conditions for recurrence and transience of $f - \alpha$, $\alpha \in \mathbb{R}$, in terms of the asymptotic behaviour of the measures τ_k, $k \geqq 1$.

3.6. Theorem. *Let T be a measure preserving, ergodic automorphism of the probability space $(X, \mathscr{S}, \mu)$ and let $f: X \to \mathbb{R}$ be a measurable function with recurrence set $R(f)$. Then $R(f)$ is a Borel set, and there exists a Borel map $\Delta: \mathbb{R} \setminus R(f) \to \mathbb{R}^+$ such that, for every $\alpha \in \mathbb{R} \setminus R(f)$ and every $\eta > 0$,*

$$\limsup_{k \to \infty} \tau_k([\alpha - \eta, \alpha + \eta]) \leqq 2\eta \cdot \Delta(\alpha) \tag{3.13}$$

and

$$\limsup_{m \to \infty} \sum_{n=1}^{N} 2^n \tau_{m \cdot 2^n}([\alpha - \eta/2^n, \alpha + \eta/2^n]) \leqq 8\eta \cdot \Delta(\alpha) \tag{3.14}$$

for every $N \geqq 1$. Here $(\tau_k, k \geqq 1)$ is the sequence of probability measures on $\mathbb{R}$ defined by (3.8)–(3.9).

Proof. The only statement in this theorem which goes beyond Lemma 3.5 is the trivial assertion that the map $\alpha \to \Delta(\alpha)$ in (3.13) and (3.14) may be chosen to be Borel on $\mathbb{R} \setminus R(f)$ (cf. Remark 3.7). $\square$

3.7. Remark. For every $\alpha \in \mathbb{R} \setminus R(f)$, define Δ_α as the measure of the sweep-out set D_α for the skew product transformation

$$W^\alpha(x, t) = (Tx, t + f(x) - \alpha) \tag{3.15}$$

on $(X \times \mathbb{R}, \mu \times \lambda)$ (cf. Remark 3.2). It is not difficult to verify that the map $\alpha \to \Delta_\alpha = (\mu \times \lambda)(D_\alpha)$ is Borel on $\mathbb{R} \setminus R(f)$, and we may, of course, choose

$$\Delta(\alpha) = \Delta_\alpha^{-1} \tag{3.16}$$

for every $\alpha \in \mathbb{R} \setminus R(f)$. This choice of $\Delta(\cdot)$ yields the following corollary.

3.8. Corollary. *Assume that there exists an $\alpha \in \mathbb{R} \setminus R(f)$ for which the sweep-out set D_α for the transformation W^α (cf. (3.15)) has infinite measure. Then*

$$\lim_k \tau_k = 0$$

in the vague topology or, equivalently,

$$\lim_{|k| \to \infty} |f^*(k, \cdot)/k| = \infty$$

in measure.

The next (and probably more appealing) corrollary has been obtained independently by F.M. Dekking [4] and B. Weiss.

3.9. Corollary. *If $\lim_m f^*(m, \cdot)/m = 0$ in measure then f is recurrent.*

Proof. In this case $\lim_k \tau_k = \delta_0$, the measure concentrated at 0. Now apply (3.13). $\square$

3.10. Corollary. *Let* Σ *denote the set of limit points of* $(\tau_k: k \geq 1)$ *in the vague topology. For every* $v \in \Sigma$ *we write* $v = v_{ac} + v_s$ *for the decomposition of* v *into its absolutely continuous and singular parts, respectively. Then the following holds for every* $v \in \Sigma$.

(1) $v((\alpha - \eta, \alpha + \eta)) \leq 2\eta \Delta(\alpha)$ *for every* $\alpha \in \mathbb{R} \setminus R(f)$ *and* $\eta > 0$;   (3.17)

(2) $v_s(\mathbb{R} \setminus R(f)) = 0$.   (3.18)

Corollary 3.10 implies in particular that $R(f) \neq \phi$ if the measures in Σ are not all absolutely continuous. The following corollary goes even further.

3.11. Corollary. *For every* $\alpha \in \mathbb{R} \setminus R(f)$ *we have*

$$\sup_{v \in \Sigma} \sup_{\eta > 0} v([\alpha - \eta, \alpha + \eta])/2\eta < \infty \qquad (3.19)$$

and

$$\inf_{v \in \Sigma} \inf_{0 < \eta < 1} v([\alpha - \eta, \alpha + \eta])/2\eta = 0. \qquad (3.20)$$

Proof. The first inequality is clear from (3.17). To prove (3.20) choose an increasing sequence $(m_k, k \geq 1)$ of natural numbers such that

$$\lim_k \tau_{m_k \cdot 2^n} = v_n \qquad (3.21)$$

exists for every $n \geq 0$ in the vague topology. Put $A = \{t \in \mathbb{R}: v_n(\{t\}) > 0$ for some $n \geq 0\}$. The set A is countable, hence $H = \{\eta > 0: \alpha \pm \eta/2^n \notin A$ for all $n \geq 0\}$ has a countable complement. For every $\eta \in H$ we get that

$$\lim_k \tau_{m_k \cdot 2^n}([\alpha - \eta/2^n, \alpha + \eta/2^n])$$
$$= v_n([\alpha - \eta/2^n, \alpha + \eta/2^n]),$$

so that

$$\sum_{n=0}^{\infty} 2^n v_n([\alpha - \eta/2^n, \alpha + \eta/2^n])/2\eta \leq 4\Delta(\alpha), \qquad (3.22)$$

by (3.14). For every $N \geq 1$ we can thus find an integer n with $0 \leq n \leq N$ such that

$$v_n([\alpha - \eta/2^n, \alpha + \eta/2^n]) \leq 8\eta \Delta(\alpha)/N \cdot 2^n,$$

and this implies (3.20). $\square$

In the case of classical, one-dimensional, independent random walks the acid test for any sufficient condition for recurrence has always been the Cauchy Random Walk based on the probability measure $dv(\alpha) = (\pi \cdot (1 + \alpha^2))^{-1} d\alpha$ on $\mathbb{R}$ (we refer to the discussion at the end of Sect. 2). In this case $\sigma_l = v$ for every $l \neq 0$, hence $\Sigma = \{v\}$, and Corollary 3.11 yields the following well known result.

3.12. Corollary. *If* f *is the Cauchy Random Walk then* $R(f) = \mathbb{R}$.

All the corollaries obtained so far (with the exception of condition (3.18)) are 'local' in the sense that they describe recurrence or transience at a specific

point $\alpha \in \mathbb{R}$. We conclude this section with a 'global' consequence of Theorem 3.6 which gives a sufficient condition for $R(f)$ to be non-empty.

3.13. Corollary. *Let $f: X \to \mathbb{R}$ be a measurable function with the following properties.*

(1) *The measures τ_k, $k \geq 1$, are uniformly tight;*

(2) *The set Σ of vague limit points of $(\tau_k, k \geq 1)$ is uniformly absolutely continuous.*

Then $R(f) \neq \phi$.

Proof. The two conditions of this corollary may be rephrased as follows: every $v \in \Sigma$ is an absolutely continuous probability measure on $\mathbb{R}$, and we can find, for every $\varepsilon > 0$, a $\delta > 0$ with $v(E) < \varepsilon$ for every $v \in \Sigma$ and every Borel set $E \subset \mathbb{R}$ with $\lambda(E) < \delta$. Now assume that $R(f) = \phi$ and consider the sequence $(v_n, n \geq 0) \subset \Sigma$ defined in (3.21). We denote by ϕ_n the density function $\dfrac{dv_n}{d\lambda}$ of v_n. By letting $\eta \to 0$ in (3.22) we see that

$$\sum_{n=0}^{\infty} \phi_n(\alpha) \leq \Delta(\alpha) < \infty \tag{3.23}$$

λ-a.e. on $\mathbb{R}$. The uniform tightness of $(\tau_k, k \geq 1)$ and hence of Σ implies that there exists an $M \geq 0$ with

$$v_n(\{\alpha \in \mathbb{R} : |\alpha| \leq M\}) = \int\limits_{\{\alpha : |\alpha| \leq M\}} \phi_n \, d\lambda > \tfrac{1}{2} \tag{3.24}$$

for every $n \geq 0$. Using the uniform absolute continuity of Σ we choose a $\delta > 0$ with $v_n(E) < \tfrac{1}{8}$ for every $n \geq 0$ and every $E \subset \mathbb{R}$ with $\lambda(E) < \delta$. From (3.23) one concludes the existence of an $N \geq 0$ with

$$\lambda \left\{ \alpha \in \mathbb{R} : |\alpha| \leq M \text{ and } \phi_n(\alpha) > \frac{1}{8M} \right\} < \delta$$

for every $n \geq N$. By combining these inequalities we get that

$$v_n(\{\alpha \in \mathbb{R} : |\alpha| \leq M\}) = v_n \left(\left\{ \alpha : |\alpha| \leq M \text{ and } \phi_n(\alpha) \leq \frac{1}{8M} \right\} \right)$$

$$+ v_n \left(\left\{ \alpha : |\alpha| \leq M \text{ and } \phi_n(\alpha) > \frac{1}{8M} \right\} \right) < \frac{1}{4},$$

which contradicts (3.24). Hence $R(f) \neq \phi$. $\square$

3.14. Remark. It seems reasonable to conjecture that Corollary 3.13 should be true without condition (2), i.e. that the uniform tightness of the measures τ_k, $k \geq 1$, alone should imply that $R(f) \neq \phi$. By using a proof similar to that of Corollary 3.13 one only obtains the following: if f satisfies $R(f) = \phi$, and if the measures τ_k, $k \geq 1$, are uniformly tight, then there exists a $v \in \Sigma$ and a set $C \subset \mathbb{R}$ of first category, but of positive Lebesgue measure, with $v(C) = 1$ (note that v must be absolutely continuous, by Corollary 3.10). The main difficulty in

proving the conjectured version of Corollary 3.13 lies in the fact that the function $\Delta(\cdot)\colon \mathbb{R} \to \mathbb{R}^+$ may not be locally integrable. The situation improves if one considers the 'symmetrized' function $F\colon X \times X \to \mathbb{R}$ given by $F(x, y) = f(x) - f(y)$ for the automorphism $T \times T$ on $(X \times X,\ \mathscr{S} \times \mathscr{S},\ \mu \times \mu)$. If the measures $\tau_k, k \geqq 1$, arising from the function f are uniformly tight, then our techniques yield that the function F is recurrent for the automorphisms $T \times T$.

§ 4. Examples of Recurrence Sets

Let T be a non-singular, ergodic automorphism of a non-atomic Lebesgue space $(X, \mathscr{S}, \mu)$ and let $f\colon X \to \mathbb{R}$ be measurable. From Proposition 2.10 we know that $R(f)$ is a Borel set, but some of the following examples will show that $R(f)$ can be quite complicated. Corollary 3.9 provides an example of a finite measure preserving, ergodic automorphism T and of a function f with $R(f) = \mathbb{R}$. In order to obtain a few more examples we start with some elementary facts, most of which are quite well known.

4.1. Lemma. *Let T be a non-singular, aperiodic, conservative automorphism of a non-atomic Lebesgue space $(X, \mathscr{S}, \mu)$ and let $f\colon X \to \mathbb{R}$ be measurable. If*

$$\lim_{n} |f^*(n, x)/n| = \infty \qquad \mu\text{-a.e.} \tag{4.1}$$

then $R(f) = \phi$. If, on the other hand,

$$\lim_{n} |f^*(n, x)/n| = 0 \qquad \mu\text{-a.e.} \tag{4.2}$$

then we either have $R(f) = \{0\}$ or $R(f) = \phi$.

Proof. This is obvious from the definitions. □

4.2. Proposition. *If $(X, \mathscr{S}, \mu)$ is a non-atomic Lebesgue probability space, T a measure preserving, ergodic automorphism of $(X, \mathscr{S}, \mu)$, and $f \in L^1(X, \mathscr{S}, \mu)$ then $R(f) = \{\int f\, d\mu\}$.*

Proof. Let $\alpha = \int f\, d\mu$. The ergodic theorem shows that

$$\lim_{n} f^*(n, x)/n = \alpha \qquad \mu\text{-a.e.}$$

and hence in measure, and Corollary 3.5 implies the recurrence of $f - \alpha$, while Lemma 4.1 yields that $R(f) \subset \{\alpha\}$. □

4.3. Remark. Assume that T is a measure preserving, ergodic automorphism of the probability space $(X, \mathscr{S}, \mu)$. If $f\colon X \to \mathbb{R}$ is measurable and cohomologous to an integrable function, then card $R(f) = 1$, by Propositions 2.2 and 4.2. If a measurable function $f\colon X \to \mathbb{R}$ satisfies card $R(f) \geqq 2$ then this implies in particular that f is not cohomologous to any L^1-function. For a discussion of some related problems we refer to [9].

4.4. Proposition. *If $(X, \mathscr{S}, \mu)$ is an infinite, non-atomic Lebesgue space, T a measure preserving, ergodic automorphism of $(X, \mathscr{S}, \mu)$, and $f \in L^1(X, \mathscr{S}, \mu)$, then*

$$R(f) = \begin{cases} \{0\} & \text{if } \int f d\mu = 0, \\ \phi & \text{otherwise.} \end{cases} \tag{4.3}$$

Proof. Let $B \in \mathscr{S}$ with $\mu(B) = 1$ and define $r_B \colon X \to \mathbb{Z}$, $b \colon X \to \mathbb{R}$ and $f_1 = f + bT - b$ as in the proof of Lemma 2.3. The function f_1 satisfies (2.19), and, in particular, $f_1 = 0$ on $X \setminus T^{-1}B$. An easy argument shows that $\int f_1 d\mu = \int f d\mu$. Now consider the restriction ϕ of f_1 to the set $C = T^{-1}B$ and the automorphism T_C induced by T on the set C. The restriction μ_C of μ to C is a probability measure, and Proposition 4.2 shows that the function ϕ on C is recurrent for the automorphism T_C of (C, μ_C) if and only if $\int \phi d\mu_C = \int f_1 d\mu = \int f d\mu = 0$. One immediately concludes that f_1, and hence f, is recurrent for T if and only if $\int f d\mu = 0$. Lemma 4.1, together with the individual ergodic theorem for infinite measure preserving transformations, shows that, on the other hand, $R(f) \subset \{0\}$, and this completes the proof of this proposition. $\quad\square$

In all the examples described so far, $R(f)$ is either empty, or a single point, or equal to $\mathbb{R}$. We now show how functions with a more exotic recurrence behaviour can arise in a natural context.

4.5. An Example Where $R(f) \cap \mathbb{Q} = \{-1, 1\}$. This example is based on the fact that the cartesian product of two independent, two-dimensional, recurrent random walks is a four-dimensional random walk and hence transient. Let S on (Y, v) be the uniformly distributed 4-shift: $Y = \{0, 1, 2, 3\}^{\mathbb{Z}}$, $v = v_0^{\mathbb{Z}}$ with $v_0(i) = \frac{1}{4}$ for all $i = 0, \dots, 3$, and let S be the shift on (Y, v) given by $(Sy)_n = y_{n+1}$ for every $y = (\dots y_{-1}, y_0, y_1, \dots) \in Y$. Let $\phi \colon Y \to \mathbb{Z}^2$ be defined by

$$\phi(y) = \begin{cases} (1, 0) & \text{if } y_0 = 0, \\ (-1, 0) & \text{if } y_0 = 1, \\ (0, 1) & \text{if } y_0 = 2, \\ (0, -1) & \text{if } y_0 = 3. \end{cases}$$

We denote by λ_n the counting measure on $\mathbb{Z}^n$ and consider the skew product transformation V on $(Y \times \mathbb{Z}^2, v \times \lambda_2)$ given by

$$V(y, p) = (Sy, p + \phi(y)), \quad (y, p) \in Y \times \mathbb{Z}^2. \tag{4.4}$$

As a consequence of the well-known recurrence of the two-dimensional, independent random walk based on the probability measure $\theta(1, 0) = \theta(-1, 0) = \theta(0, 1) = \theta(0, -1) = \frac{1}{4}$ on $\mathbb{Z}^2$, V is conservative and, in fact, ergodic on the infinite measure space $(Y \times \mathbb{Z}^2, v \times \lambda_2)$. Now consider the space $Z = Y \times Y \times \mathbb{Z}^2 \times \mathbb{Z}^2$ with product measure $\pi = v \times v \times \lambda_2 \times \lambda_2$, and define measure preserving automorphisms V_i, $i = 1, 2$, of (Z, π) by setting

$$V_1(y, y', p, p') = (Sy, S^{-1}y', p + \phi(y), p' - \phi(S^{-1}y'))$$

and

$$V_2(y, y', p, p') = (Sy, Sy', p + \phi(y), p' + \phi(y')),$$

for every $y, y' \in Y$, $p, p' \in \mathbb{Z}^2$. We are now dealing with four-dimensional random walks, and this is well known to imply that V_1 and V_2 are dissipative. As in the proof of Theorem 2.8 we can find a measurable set $X \subset Z$ with $V_1^k X \cap X = \phi$ for $k \neq 0$, $k \in \mathbb{Z}$, and such that $\pi(Z \setminus \bigcup_{k \in \mathbb{Z}} V_1^k X) = 0$. Apart from a null set we may thus write $Z = X \times \mathbb{Z}$, $\pi = \mu \times \lambda_1$, where μ is the restriction of π to X, and represent V_1 on $(X \times \mathbb{Z}, \mu \times \lambda_1)$ by

$$V_1(x, n) = (x, n+1)$$

for every $(x, n) \in X \times \mathbb{Z}$. The automorphism V_2, considered as a map on $X \times \mathbb{Z}$, commutes with V_1 and is therefore of the form

$$V_2(x, n) = (Tx, n+f(x)), \qquad (x, n) \in X \times \mathbb{Z},$$

where T is a measure preserving, and necessarily ergodic, automorphism of the non-atomic Lebesgue space (X, μ) and $f: X \to \mathbb{Z}$ a measurable function. Since V_2 is not conservative, f is transient.

In order to find out about the recurrence of $f+1$ and $f-1$ we note that $W_1 = V_1 V_2$ and $W_2 = V_1^{-1} V_2$ are given as follows: on $(X \times \mathbb{Z}, \mu \times \lambda_1)$,

$$W_1(x, n) = (Tx, n+f(x)+1)$$

and

$$W_2(x, n) = (Tx, n+f(x)-1),$$

while on (Z, τ) they are represented as

$$W_1(y, y', p, p') = (S^2 y, y', p+\phi(y)+\phi(Sy), p')$$

and

$$W_2(y, y', p, p') = (y, S^2 y', p, p'+\phi(y')+\phi(Sy')).$$

The recurrence of the function ϕ for the automorphism S of (Y, v) (or, equivalently, the fact that V in (4.4) is conservative) implies that both W_1 and W_2 are conservative and hence that $f+1$ and $f-1$ are recurrent.

To see that no other rational number lies in $R(f)$, assume that $m/q \in R(f)$ with $m \neq \pm q$. Then $V_1^{\pm m} V_2^q$ would have to be conservative, which we know not to be the case. This proves that $R(f) \cap \mathbb{Q} = \{-1, 1\}$. Although I would conjecture that $R(f) = \{-1, 1\}$ I do not know how to deal with irrational numbers in this example. $\square$

4.6. An Example where Both $R(f)$ and $\mathbb{R} \setminus R(f)$ Are Dense. Let X be the additive group of tri-adic integers with its usual compact topology and with Haar measure μ. Every element $x \in X$ can be written as a formal power series

$$x = \sum_{i=0}^{\infty} c_i(x) \cdot 3^i \qquad (4.5)$$

with $c_i(x) \in \{0, 1, 2\}$ for every $i \geq 0$. The natural numbers $\mathbb{N}$ are embedded in X in the usual way as the (dense) set of terminating power series. Define the 'tri-adic adding machine' transformation $T: X \to X$ by

$$Tx = x+1, \qquad x \in X.$$

Then T is a measure preserving, ergodic automorphism of the non-atomic probability space (X, μ). For every $x \in X$, put

$$Vx = \tilde{x} = \sum_{i=0}^{\infty} \tilde{c}_i(x) \cdot 3^i, \tag{4.6}$$

where

$$\tilde{c}_i(x) = \begin{cases} 0 & \text{if } c_i(x) = 0, \\ 2 & \text{if } c_i(x) = 1, \\ 1 & \text{if } c_i(x) = 2, \end{cases} \tag{4.7}$$

and put $S = VTV$. Clearly $[S] = [T]$, and we can find a measurable function $f: X \to \mathbb{Z}$ with

$$S^{f(x)} x = Tx \qquad \mu\text{-a.e.} \tag{4.8}$$

(cf. the discussion preceding Lemma 3.2, and formula (2.6)). We shall now prove that

$$R(f) \cap \mathbb{Q} = \left\{ \frac{\tilde{m} - \tilde{n}}{m - n} : m, n \in \mathbb{N} \text{ and } m \neq n \right\}, \tag{4.9}$$

where $\tilde{m}$ and $\tilde{n}$ are given by (4.6). Indeed, let

$$\alpha \in \left\{ \frac{\tilde{m} - \tilde{n}}{m - n} : m, n \in \mathbb{N}, m \neq n \right\} = D.$$

We write α as $(\widetilde{(n+k)} - \tilde{n})/k$ for some $k \geq 1$, $n \geq 0$, and choose $L > 0$ with $c_i(n+k) = c_i(n) = 0$ for all $i \geq L$. If $B \subset X$ is a measurable set with $\mu(B) > 0$ then there exists an integer $M \geq 0$ and a point $x \in B \cap T^{-k \cdot 3^M} B$ with

$$c_{M+i}(x) = c_i(n)$$

and

$$c_{M+i}(T^{k \cdot 3^M} x) = c_i(n+k)$$

for all $i = 0, \ldots, L$. From (4.6) and (4.8) it is now clear that $f^*(k \cdot 3^M, x) = 3^M(\widetilde{(n+k)} - \tilde{n})$ and hence that $\alpha \in R(f)$. This shows that $D \subset R(f)$, and an elementary argument yields that $R(f) \cap \mathbb{Q} \subset D$.

   In order to see that $\mathbb{R} \setminus R(f)$ is dense in $\mathbb{R}$ it suffices to observe that $3k/l \notin D$, and hence not in $R(f)$, if $k \in \mathbb{Z}$ and $l \geq 1$, $l \not\equiv 0 \pmod 3$.

   Finally we turn to the problem of showing that D, and hence $R(f)$, is dense in $\mathbb{R}$. One can verify that, for every $p \geq 1$ and every $k = 0, 1, \ldots, (3^p - 1)/2$, the following four numbers are elements of D:

$$2 \cdot (3k+1)/(3^{p+1}+1), \qquad (3^{p+1}+1)/2(3k+1),$$
$$-2 \cdot (3k+1)/(3^{p+1}-1), \qquad -(3^{p+1}-1)/2 \cdot (3k+1).$$

Hence D is dense in $\mathbb{R}$. Once again the set $R(f) \cap \mathbb{Q}$ can be described quite easily in this example, while the irrational elements of $R(f)$ are more elusive. *I shall restrict myself to constructing an uncountable set of irrational numbers in*

$R(f)$. Let $\Omega = \mathbb{Z}_2^{\mathbb{N}}$ and define, for every $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, a real number α_ω by

$$\alpha_\omega = \left(\sum_{k=1}^\infty (1+\omega_k) \cdot 3^{-k!} \right) \Big/ \left(\sum_{k=1}^\infty (2-\omega_k) \cdot 3^{-k!} \right).$$

The numbers α_ω and $\alpha_{\omega'}$ are distinct whenever $\omega \neq \omega'$. To see that $\alpha_\omega \in R(f)$ for every $\omega \in \Omega$, put, for every $n \geq 1$,

$$m_n = 3^{n!} \sum_{k=1}^n (2-\omega_k) \cdot 3^{-k!},$$

$$\alpha_\omega^{(n)} = \tilde{m}_n / m_n \quad \text{(cf. (4.6))},$$

and note that

$$|\alpha_\omega - \alpha_\omega^{(n)}| \leq 6 \cdot 3^{-n!} \cdot m_n^{-1}.$$

For every $l \geq 0$, $n \geq 1$, define sets $B_l^{(n)} \subset X$ by

$$B_l^{(n)} = \{ x \in X : c_{l+k!}(x) = 0 \text{ for } k = 1, 2, \ldots, n \}.$$

From the definition of S and from (4.8) it follows that, for every $x \in B_l^{(n)}$,

$$f^*(3^l m_n, x) = 3^l \tilde{m}_n.$$

Hence

$$|f^*(3^l m_n, x) - 3^l m_n \alpha_\omega| = 3^l m_n |\alpha_\omega^{(n)} - \alpha_\omega| \leq 6 \cdot 3^{l-n!}.$$

Finally we note that $C^{(n)} = \bigcup_{l=n}^{n!-n} B_l^{(n)}$ satisfies

$$\lim_n \mu(C^{(n)}) = 1, \tag{4.10}$$

and that we can find, for every $x \in C^{(n)}$, an integer $l \geq n$ with

$$|f^*(3^l m_n, x) - 3^l m_n \alpha_\omega| \leq 6 \cdot 3^{-n}. \tag{4.11}$$

(4.10) and (4.11) imply that

$$\liminf_k |f^*(k, x) - k\alpha_\omega| = 0 \quad \mu\text{-a.e.},$$

and hence that $\alpha_\omega \in R(f)$ (cf. Theorem 2.7). This completes the proof that the uncountable set $\{\alpha_\omega : \omega \in \Omega\}$ is contained in $R(f)$. $\square$

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