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Finite Nearest Particle Systems

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Summary. Finite nearest particle systems are certain continuous time Markov chains on the collection of finite subsets of Z^1. In this paper, we give a sufficient condition for such a system to survive, in the sense that the probability of absorption at $\emptyset$ is less than one. This theorem generalizes earlier results for the one-dimensional contact process.

1. The Result

Let $\beta(l,r)$ be nonnegative for $1 \leqq l, r \leqq \infty$, and let Y be the collection of all finite subsets of Z^1. The finite nearest particle system with birth rates $\beta(l,r)$ is a continuous time Markov chain on Y. The transitions which it can make are:

$A \to A \setminus \{x\}$ at rate 1 for each $x \in A$, and

$A \to A \cup \{x\}$ at rate $\beta(l_A(x), r_A(x))$ for each $x \notin A$, where

$$l_A(x) = x - \max\{y : y \leqq x \quad \text{and} \quad y \in A\}$$

and

$$r_A(x) = \min\{y : y \geqq x \quad \text{and} \quad y \in A\} - x,$$

and $l_A(x)$ or $r_A(x)$ is $+\infty$ if the max or min is not defined. In this paper, we will make the following additional assumptions on $\beta(l,r)$:

$$\beta(\infty,\infty) = 0, \tag{1.1}$$

so that $\emptyset$ is a trap for A_t,

$$\beta(l,r) = \beta(r,l) \qquad \text{for all } 1 \leqq l, r \leqq \infty, \tag{1.2}$$

$$\sum_{n=1}^{\infty} \beta(\infty, n) < \infty, \tag{1.3}$$

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so that the chain stays in Y at all times, and finally that enough of the $\beta(l,r)$'s are strictly positive so that

$$P^A(A_t = B) > 0 \tag{1.4}$$

for all $t>0$ and all A, $B \in Y$ with $A \neq \emptyset$.

We will say that the system survives if

$$\sigma(A) = P^A(A_t \neq \emptyset \text{ for all } t) > 0$$

for some (and hence all) $A \neq \emptyset$. Otherwise, say that the system dies out. The problem of interest here is to determine which of these systems survive and which die out.

Two subclasses of finite nearest particle systems have received a considerable amount of attention. The first is the contact process which was introduced by Harris [7]. It is the special case in which

$$\beta(l,r) = \begin{cases} 2\lambda & \text{if } l=r=1, \\ \lambda & \text{if } l=1, r>1 \text{ or if } l>1, r=1, \\ 0 & \text{otherwise.} \end{cases}$$

An excellent survey of the contact process has been given by Griffeath [5]. The two results which are relevant here are that the contact process dies out if $\lambda \leqq 1.18$ and survives if $\lambda \geqq 2$. The first of these is due to Harris [7], and the second to Holley and Liggett [8]. Nonrigorous computations by Brower, Furman and Moshe [2] indicate that the critical value λ_c at which the transition from dying out to surviving occurs is approximately 1.65. They actually looked at a system which is known as the reggeon spin model. The required connection between it and the contact process was later established by Grassberger and de la Torre in [4].

The second class of finite nearest particle systems which has been studied is the collection of those which are attractive and reversible. Without defining these terms, it suffices here to say that they are the ones in which

$$\begin{aligned} \beta(l,r) &= \frac{\beta(l)\,\beta(r)}{\beta(l+r)} \text{ if } l,r<\infty, \quad \text{and} \\ \beta(\infty,n) &= \beta(n), \end{aligned} \tag{1.5}$$

where $\beta(n)>0$, $\sum_{n=1}^{\infty} \beta(n) < \infty$, and $\beta(n)/\beta(n+1) \downarrow 1$. For this class, Griffeath and Liggett [6] showed that the system survives if and only if

$$\sum_{n=1}^{\infty} \beta(n) > 1. \tag{1.6}$$

The reversibility assumption was indispensable in their analysis.

Of course, more general finite nearest particle systems can be shown to die out or survive in some cases by making simple comparisons with the two special classes described above. For example, comparison with the contact process implies that the system with birth rates $\beta(l,r)$ survives whenever

$$\beta(1,1) \geqq 4, \text{ and}$$
$$\beta(1,n) \geqq 2 \text{ for } n \geqq 2.$$

Such an approach cannot be used to prove survival for many interesting systems, however. For example, these comparisons cannot be used if

$$\lim_{n\to\infty} \beta(1,n) = 0,$$

since this limit is positive for both the contact process and the attractive reversible systems.

More reasonable criteria for survival in the general case should be given not in terms of the individual $\beta(l,r)$'s being sufficiently large, but rather in terms of the total birth rate in a connected subset of the complement of A being sufficiently large. Therefore, we will define b_n for $2 \leqq n \leqq \infty$ by

$$b_n = \sum_{l+r=n} \beta(l,r) \quad \text{if} \quad 2 \leqq n < \infty,$$

and

$$b_\infty = \sum_{n=1}^{\infty} \beta(\infty,n) + \sum_{n=1}^{\infty} \beta(n,\infty).$$

Note that $b_n = 2\lambda$ for all $2 \leqq n \leqq \infty$ in the case of the contact process, and

$$\liminf_{n\to\infty} b_n \geqq b_\infty = 2 \sum_{n=1}^{\infty} \beta(n)$$

in case the rates are given by (1.5). Thus in the reversible attractive case, the system survives if and only if $b_\infty > 2$ by criterion (1.6).

One way of formulating the main problem is this. For each $b > 0$, determine which of the following statements is true:

(a) Every finite nearest particle system satisfying $b_n = b$ for all $2 \leqq n \leqq \infty$ dies out.

(b) Some nearest particle systems satisfying $b_n = b$ for all $2 \leqq n \leqq \infty$ die out, and others survive.

(c) Every nearest particle system satisfying $b_n = b$ for all $2 \leqq n \leqq \infty$ survives.

The theorem stated below implies that the answer is (a) if $b \leqq 1$, (b) if $2 < b < 2\lambda_c$, and (c) if $b \geqq 4$. A reasonable conjecture is that the answer is (c) if $b > 2\lambda_c$. If so, that would close one of the gaps left above (except for determining what happens at $b = 2\lambda_c$, which is a very hard open problem). Reasons for believing this conjecture and a possible approach to its proof are discussed at the end of this paper. The other gap left above is to determine which of (a) or (b) is correct for $1 < b \leqq 2$. There appears to be no compelling evidence one way or the other.

(1.7) **Theorem.** (i) *If $b_n \leqq 1$ for all $2 \leqq n \leqq \infty$, then the process dies out.*

(ii) *For every $b > 2$, there is a nearest particle system which survives and satisfies $b_n = b$ for all $2 \leqq n < \infty$.*

(iii) *If $b_n \geqq 4$ for all $2 \leqq n \leqq \infty$, then the process survives.*

The proof of this theorem will be given in the next section. The first statement is elementary. The second statement is a rather simple consequence of criterion (1.6), which is available in case the rates are given by (1.5). The main new result is the third one, which generalizes the bound $\lambda_c \leqq 2$ for the contact process to the general nearest particle context. In fact, the proof of (iii) depends heavily on the proof of that result for the contact process. An additional idea is needed, however. It involves the use of a convexity argument, which may turn out to be an important tool in other related contexts.

Infinite nearest particle systems have been studied also. They are defined in an analogous way, except that the state space of the process is the collection of all subsets of Z^1 which contain infinitely many positive and infinitely many negative points (only the $\beta(l,r)$ with finite l,r are relevant in this case). The process is now said to survive or die out according to whether there is or is not an invariant measure for the process. Infinite nearest particle systems were introduced by Spitzer [10], who obtained necessary and sufficient conditions for them to have reversible invariant measures. One version of his theorem asserts that if the rates are given by (1.5), then the infinite process survives if and only if either

$$(1.8) \qquad \sum_{n=1}^{\infty} \beta(n) > 1, \text{ or } \sum_{n=1}^{\infty} \beta(n) = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} n\beta(n) < \infty.$$

This version appears as Theorem 1.4 in [9].

In the infinite nonreversible context, the following results are known. The infinite version of the contact process has the same critical value as the finite version by duality (see [5] for example). Bramson and Gray [1] considered two other examples. Their results are that the infinite process survives if

$$\beta(l,r) = \frac{b}{r+l-1}$$

with $b > 4\log 2 \approx 2.77$, or if

$$\beta(l,r) = \begin{cases} b \text{ if } l = r \\ \frac{b}{2} \text{ if } |l-r| = 1 \\ 0 \text{ if } |l-r| \geqq 2 \end{cases}$$

with $b > 2$. Note that in each of these cases, $b_n = b$ for all $2 \leqq n < \infty$.

The infinite analogue of part (ii) of Theorem 1.7 follows immediately from the results of either Spitzer [10] or Bramson and Gray [1] which were quoted above. The infinite analogue of part (i) is again elementary. There is at this point no known infinite analogue of part (iii) of Theorem 1.7. One way to prove such a result would of course be to show that whenever the finite system survives, so does the infinite system. This statement is true in the cases of the contact process (by duality) and attractive reversible systems (since (1.6) implies (1.8)). It is not known to be true for more general nearest particle systems, however.

The problems considered in this paper were raised in conversations with R. Holley a number of years ago. He pointed out that Spitzer's results gave reversible examples of infinite systems which survive with $b_n = b > 2$ for all n, and proposed the question of whether there are non-reversible examples which survive with $b_n = b \in (1, 2]$ for all n. This question remains unanswered.

2. The Proof.

This section contains the proof of Theorem 1.7.

Proof of (i). Assume that $b_n \leqq 1$ for all $2 \leqq n \leqq \infty$. Then the cardinality $|A_t|$ decreases by one at rate $|A_t|$ and increases by one at a rate which is at most $|A_t|$. Therefore $|A_t|$ is a (non-negative) supermartingale, so that

$$\lim_{t \to \infty} |A_t|$$

exists a.s. By (1.4), the only possible limit for $|A_t|$ is 0. Therefore the process dies out.

Proof of (ii). Take $b > 2$. The idea of the proof is to find a choice of $\beta(n)$ so that if $\beta(l, r)$ is given by (1.5), then $b_n = b$ for all $2 \leqq n \leqq \infty$. The process will survive by criterion (1.6) since

$$\sum_{n=1}^{\infty} \beta(n) = \sum_{n=1}^{\infty} \beta(\infty, n) = \frac{1}{2} b_\infty = \frac{1}{2} b > 1.$$

The requirement that $b_n = b$ can be rewritten as

$$\sum_{l+r=n} \beta(l)\, \beta(r) = b\, \beta(n) \quad \text{for} \quad 2 \leqq n < \infty, \tag{2.1}$$

and

$$2 \sum_{n=1}^{\infty} \beta(n) = b. \tag{2.2}$$

Summing (2.1) on $n \geqq 2$ and using (2.2) gives $\beta(1) = b/4$. Then (2.1) can be used to compute $\beta(n)$ for $n \geqq 2$ recursively. In fact, it is not hard to check (using generating functions for example) that

$$\beta(n) = b \frac{(2n-2)!}{n!\,(n-1)!} 4^{-n}.$$

Therefore

$$\frac{\beta(n)}{\beta(n+1)} = \frac{2n+2}{2n-1},$$

which decreases to 1 as $n \uparrow \infty$ as required.

Proof of (iii). Let $f(n)$ be the probability density on $\{1,2,\dots\}$ which is defined by $f(n)=F(n)-F(n+1)$, where

$$F(n)=\frac{(2n-2)!}{(n-1)!\,n!}\,4^{-n+1},\ n\geqq 1.$$

Then $\sum_{n=1}^{\infty} F(n)=\sum_{n=1}^{\infty} nf(n)=2$ by (2.2), since $F(n)=\beta(n)$ if b is taken to be 4. Let ν be the stationary renewal measure on $\{0,1\}^{Z^1}$ corresponding to f. It is the measure whose finite dimensional distributions are given by

$$\begin{aligned}&\nu\{\eta:\eta(x_i)=1 \text{ for } 1\leqq i\leqq n, \eta(x)=0 \text{ for } x\in[x_1,x_n]\setminus\{x_i, 1\leqq i\leqq n\}\}\\&=\frac{1}{2}\prod_{i=1}^{n-1} f(x_{i+1}-x_i)\end{aligned}$$

whenever $x_1<x_2<\dots<x_n$ and $n\geqq 1$. For $A\in Y$, let

$$h(A)=\nu\{\eta:\eta(x)=1 \text{ for some } x\in A\}.$$

In [8], it was shown that the contact process with $\lambda=2$ satisfies

$$\text{(2.3)}\qquad \frac{d}{dt}E^A h(A_t)|_{t=0}\geqq 0 \qquad \text{for all } A.$$

In other words, h is a subharmonic function for that chain. This implies that the chain survives, since

$$\sigma(A)=\lim_{t\to\infty} E^A h(A_t)\geqq h(A)>0 \quad \text{for} \quad A\neq\emptyset.$$

We will prove part (iii) of Theorem 1.7 by showing that (2.3) is valid for any nearest particle chain, provided that $b_n\geqq 4$ for all $2\leqq n\leqq\infty$. This cannot be done by simply imitating the proof in the contact process case, since that proof uses in an essential way that births can only occur at the endpoints of the connected components of A^c. Instead, we will deduce (2.3) for the general process from that result for the contact process by a type of comparison. In order to carry out the comparison, we need to know that h satisfies a certain convexity property. This is proved in the next two lemmas. For $A\in Y$ and $x\notin A$, define

$$\begin{aligned}g_A(x)&=h(A\cup\{x\})-h(A)\\&=\nu\{\eta:\eta(x)=1 \quad\text{and}\quad \eta(y)=0 \quad \text{for all } y\in A\}.\end{aligned}$$

(2.4) **Lemma.** *Suppose that $y\geqq x+2$ for all $y\in A$. Then*

$$\text{(2.5)}\qquad 2g_A(x)\geqq g_A(x+1)+g_A(x-1) \quad \textit{and} \quad g_A(x)\geqq g_A(x+1).$$

Proof. In [8], it was shown that

$$\text{(2.6)}\qquad \nu\{\eta:\eta(0)=1,\ \eta(n)=0\}=\frac{1}{8}\sum_{k=1}^{n} F(k)$$

for $n \geqq 1$ (see Eq. (2.13) there). Therefore, if $A=\{y\}$, we can compute

$$2g_A(x)-g_A(x+1)-g_A(x-1)=\tfrac{1}{8}[F(y-x)-F(y-x+1)]=\tfrac{1}{8}f(y-x)>0,$$

and

$$g_A(x)-g_A(x+1)=\tfrac{1}{8}F(y-x)>0.$$

The proof in general is by induction on the cardinality of A. Given A with $|A| \geqq 2$, let y be the left most element of A. Let $B=A\backslash\{y\}$. Then $|B|=|A|-1$, and using elementary properties of renewal measures we see that

$$\begin{aligned} g_A(x) &= g_B(x)-\nu\{\eta:\eta(x)=\eta(y)=1 \text{ and } \eta(z)=0 \text{ for } z\in B\}\\ &= g_B(x)-2g_B(y)\,\nu\{\eta:\eta(x)=\eta(y)=1\}\\ &= g_B(x)-g_B(y)[1-2g_{\{y\}}(x)]\\ &= g_B(x)+2g_{\{y\}}(x)\,g_B(y)-g_B(y). \end{aligned}$$

Thus the required concavity and monotonicity of g_A follows from that of g_B and $g_{\{y\}}$.

(2.7) **Lemma.** *Suppose that* $x-1$, x, $x+1\in A^c$. *Then*

$$2g_A(x) \geqq g_A(x+1)+g_A(x-1). \tag{2.8}$$

Proof. Let $B=A\cap(-\infty,x-1)$ and $C=A\cap(x+1,\infty)$. By basic properties of the renewal measure ν,

$$g_A(x)=2g_B(x)\,g_C(x). \tag{2.9}$$

Therefore

$$\begin{aligned} &2g_A(x)-g_A(x+1)-g_A(x-1)\\ &\quad=4g_B(x)\,g_C(x)-2g_B(x+1)\,g_C(x+1)-2g_B(x-1)\,g_C(x-1)\\ &\quad=[g_B(x)+\tfrac{1}{2}g_B(x+1)+\tfrac{1}{2}g_B(x-1)][2g_C(x)-g_C(x+1)-g_C(x-1)]\\ &\qquad+[g_C(x)+\tfrac{1}{2}g_C(x+1)+\tfrac{1}{2}g_C(x-1)][2g_B(x)-g_B(x+1)-g_B(x-1)]\\ &\qquad+[g_B(x-1)-g_B(x+1)][g_C(x+1)-g_C(x-1)], \end{aligned}$$

which is nonnegative by Lemma (2.4).

We now return to the proof of (2.3) for the general nearest particle chain A_t with birth rates $\beta(l,r)$ which satisfy $b_n \geqq 4$ for all $2 \leqq n \leqq \infty$. Let B_t be the contact process with $\lambda=2$, and denote its birth rates by $\tilde{\beta}(l,r)$. Then

$$\frac{d}{dt}E^A h(A_t)|_{t=0}-\frac{d}{dt}E^A h(B_t)|_{t=0}=\sum_{x\notin A} g_A(x)[\beta(l_A(x),r_A(x))-\tilde{\beta}(l_A(x),r_A(x))].$$

There are no terms in this sum corresponding to $x\in A$ since A_t and B_t have the same death rates. Since (2.3) is satisfied for the contact process, it suffices to prove that the above sum is nonnegative. This will be done by showing that the sum over those x in each connected component of A^c is nonnegative. By Lemmas (2.4) and 2.7, it suffices to show that if g is a concave function on $\{1,\ldots,n-1\}$ with $n \geqq 3$, then

$$\sum_{l+r=n} g(l)\,\beta(l,r) \geqq 2[g(1)+g(n-1)],$$

while if g is an increasing function on $\{1,2,\ldots\}$, then

$$\sum_{l=1}^{\infty} g(l)\,\beta(l,\infty) \geqq 2g(1).$$

The second statement is an immediate consequence of the assumption $b_\infty \geqq 4$. For the first statement, use the concavity of g to write

$$g(l) \geqq \frac{n-l-1}{n-2}\, g(1) + \frac{l-1}{n-2}\, g(n-1).$$

Therefore

$$\sum_{l+r=n} g(l)\,\beta(l,r) \geqq g(1) \sum_{l+r=n} \beta(l,r)\frac{n-l-1}{n-2} + g(n-1) \sum_{l+r=n} \beta(l,r)\frac{l-1}{n-2}.$$

But by the symmetry assumption (1.2),

$$\sum_{l+r=n} \beta(l,r)\frac{n-l-1}{n-2} = \sum_{l+r=n} \beta(l,r)\frac{l-1}{n-2} = \frac{1}{2}\, b_n \geqq 2.$$

This completes the proof of part (iii) of Theorem 1.7.

It would be nice to replace the assumption that $b_n \geqq 4$ in part (iii) of Theorem (1.7) by $b_n > 2\lambda_c$. The above proof would yield this improvement provided that the monotonicity and convexity assertions in the two lemmas could be proved with $g_A(x)$ replaced by

$$\mu\{\eta : \eta(x)=1 \quad \text{and} \quad \eta(y)=0 \quad \text{for all } y \in A\}$$

where μ is the upper invariant measure for the contact process with parameter $\lambda > \lambda_c$. This is much more difficult, of course, because μ is not known explicitly, while v is. The monotonicity statement in (2.5) is easy to prove for μ (see for example Eq. (16) in [3]). The convexity statement in (2.5) is harder, but does not seem to be out of reach. The real difficulty lies in proving the analogue of Lemma (2.7). since the factorization (2.9) is no longer available.

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Note Added in Proof

At the end of the first section, it was stated that the exact analogue of part (iii) of Theorem 1.7 for the infinite system is not known to be true. The results of Bramson and Gray [1] can be generalized, however, to obtain a similar, but not identical, sufficient condition for survival of the infinite system. This generalization appears in Chapter VII of the book "Interacting Particle Systems" by Thomas M. Liggett, which will be published by Springer-Verlag in early 1985.