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A Limit Theorem for Discounted Sums

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Summary. This note is concerned with the weak convergence of discounted sums in case the variance of the underlying probability distribution may be infinite.

1. Introduction and Results

Let $\{X_i\}_{i=0}^{\infty}$ be a sequence of i.i.d. random variables. For $0<r<1$ we shall study weak convergence of the discounted sum

$$Y(r)=\sum_{n=0}^{\infty} r^n X_n.$$

When $E|X_0|<\infty$ and $EX_0=\mu$ it is known [4] that as $r\to 1-$,

$$(1-r)\,Y(r)\to\mu \quad \text{a.s.}$$

In this paper we ask for conditions under which there exists a function $A(r)$: $[0,1[\to\mathbb{R}^+$ such that as $r\to 1-$,

$$A(r)\,Y(r)\overset{\mathcal{D}}{\Rightarrow} Y$$

where $\overset{\mathcal{D}}{\Rightarrow}$ denotes convergence in distribution and Y is nondegenerate. A central limit theorem for discounted sums has already been proved in [2]. See also [1]. In the next section we will prove

Theorem. *Suppose X_0 is in the domain of attraction of a stable distribution with index α, where $0<\alpha\leqq 2$ but $\alpha\neq 1$ unless X_0 has a symmetric d.f. In case $E|X_0|<\infty$ assume $EX_0=0$. Then there exists a function $A(r)$: $[0,1[\to\mathbb{R}^+$ such that $A(r)\,Y(r)\overset{\mathcal{D}}{\Rightarrow} Y$ $(r\to 1-)$ where Y is stable with index α. We can choose $A(r)$ such that*

$$A^2(r)\,EX_0^2\,1_{\left\{|X_0|\leqq\frac{1}{A(r)}\right\}}\sim\alpha(1-r) \qquad (r\to 1-).$$

In case the variance of X_0 is finite we obtain

Corollary [2]. *If* $EX_0=0$ *and* $EX_0^2=\sigma^2<\infty$, *then*

$$\sqrt{1-r^2}\,Y(r)\xrightarrow{\mathcal{D}} Y$$

where $Y\sim N(0,\sigma^2)$.

2. Proofs

The conditions on X_0 imply that for small t, the characteristic function $\varphi(t)$ of X_0 can be written as

$$\varphi(t)=\exp-c|t|^\alpha h\left(\frac{1}{|t|}\right)(1+a\,\mathrm{sgn}(t))(1+o(1))$$

where $c>0$, $a\in\mathbb{C}$, and $h(x)$ is slowly varying. Furthermore $h(x)$ satisfies

$$x^{2-\alpha}h(x)\sim EX_0^2 1_{\{|X_0|\leqq x\}} \qquad (x\to\infty).$$

This follows e.g. from the results of [5, Ch. 5.1] or [3, Ch. 2.6]. Now let $G(x)=\dfrac{\alpha x^\alpha}{h(x)}$ and $B(x)$ its inverse in the sense of Seneta [6, p. 21]. Then $A(r)=\dfrac{1}{B\left(\frac{1}{1-r}\right)}$ satisfies $A(r)\to 0$ $(r\to 1-)$ and

$$A^\alpha(r)\,h\left(\frac{1}{A(r)}\right)\sim\alpha(1-r) \qquad (r\to 1-). \tag{1}$$

Now observe that

$$E(e^{isA(r)Y(r)})=\exp-c|s|^\alpha\sum_{n=0}^\infty r^{n\alpha}A^\alpha(r)\,h\left(\frac{1}{|s|\,r^n A(r)}\right)(1+a\,\mathrm{sgn}(s))(1+o(1)).$$

Since h is slowly varying we have

$$(1-\varepsilon)(|s|\,r^n)^\varepsilon h\left(\frac{1}{A(r)}\right)\leqq h\left(\frac{1}{|s|\,r^n A(r)}\right)\leqq(1+\varepsilon)(|s|\,r^n)^{-\varepsilon}h\left(\frac{1}{A(r)}\right)$$

for $0<\varepsilon<\alpha$ and $r_0(\varepsilon)<r<1$. Hence

$$\frac{(1-\varepsilon)|s|^\varepsilon A^\alpha(r)\,h\left(\frac{1}{A(r)}\right)}{1-r^{\alpha+\varepsilon}}\leqq\sum_{n=0}^\infty r^{n\alpha}A^\alpha(r)\,h\left(\frac{1}{|s|\,r^n A(r)}\right)\leqq\frac{(1+\varepsilon)|s|^{-\varepsilon}A^\alpha(r)\,h\left(\frac{1}{A(r)}\right)}{1-r^{\alpha-\varepsilon}}$$

and using (1),

$$\frac{\alpha(1-\varepsilon)|s|^\varepsilon}{\alpha+\varepsilon}\leqq\lim_{r\to 1-}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\sum_{n=0}^\infty r^{n\alpha}A^\alpha(r)\,h\left(\frac{1}{|s|\,r^n A(r)}\right)\leqq\frac{\alpha(1+\varepsilon)|s|^{-\varepsilon}}{\alpha-\varepsilon}.$$

Now let $\varepsilon\downarrow 0$ to see that

$$\lim_{r\to 1-}E(e^{isA(r)Y(r)})=\exp-c|s|^\alpha(1+a\,\mathrm{sgn}(s)).$$

This proves the theorem.

To prove the corollary, observe that in this case $\alpha=2$, $c=\frac{1}{2}$, $a=0$ and $h(x)\to\sigma^2$ $(x\to\infty)$. With $A(r)$ as in (1) we have

$$A^2(r)\sim\frac{2(1-r)}{\sigma^2}\sim\frac{1-r^2}{\sigma^2}$$

from which the corollary follows. □

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