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Multivariate Characteristic Functions and Tail Behaviour*

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1. The Result

Let $d \geqq 1$ be an integer and $F(x)$, $x=(x_1, \ldots, x_d) \in R^d$, be a d-variate probability distribution function with characteristic function

$$C(t)=\int_{R^d} \exp(i\langle t,x\rangle)\, dF(x), \qquad t=(t_1, \ldots, t_d) \in R^d,$$

where $\langle t,x\rangle = \sum_{k=1}^{d} t_k x_k$ is the inner product. Denote by $(R^d, \mathscr{A}_F^d, \mu_F)$ the measure space induced by F, and let $\|t\| = \max(|t_1|, \ldots, |t_d|)$ be the maximum-norm on R^d. Finally, for a positive u, introduce

$$B_u=\{x=(x_1, \ldots, x_d)\colon \max(|x_1|, \ldots, |x_d|)>u\}.$$

Let $0<\alpha<2$. This note investigates the relationship between the two conditions

$$1-\operatorname{Re} C(t)=O(\|t\|^\alpha), \qquad t \to (0, \ldots, 0), \tag{1.1}$$

and

$$\mu_F(B_u)=\int_{B_u} dF(x)=O(u^{-\alpha}), \qquad u \to \infty. \tag{1.2}$$

Theorem 1. *If* $0<\alpha<2$, *then* (1.1) *implies* (1.2). *Conversely, if* $0<\alpha<1$, *then* (1.2) *also implies* (1.1), *and if* $1<\alpha<2$, *then* (1.2) *implies*

$$1-\operatorname{Re} C(t)=O(\|t\|), \qquad t \to (0, \ldots, 0).$$

Consider also the condition

$$1-C(t)=O(\|t\|^\alpha), \qquad t \to (0, \ldots, 0). \tag{1.3}$$

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Theorem 2. *If* $0<\alpha<2$, *then* (1.3) *implies* (1.1). *Conversely, if* $0<\alpha<1$, *then* (1.1) *also implies* (1.3), *and if* $1<\alpha<2$, *then* (1.1) *implies*

$$1-C(t)=O(\|t\|), \qquad t\to(0,\dots,0). \tag{1.4}$$

Both theorems remain valid if O is replaced by o.

If $d=1$, then Theorem 1 is due to Binmore and Stratton [1] and is also presented as Theorem 11.3.2 by Kawata [5]. But in this unvariate case (1.2) implies (1.1) also for $1\leqq\alpha<2$, i.e., (1.1) and (1.2) are equivalent for all $0<\alpha<2$. This univariate result was applied in [3] when strongly approximating the empirical characteristic function. While working on a multivariate analogue of this approximation in [4], the above multivariate extension became a need. Although (1.2) $\Rightarrow$ (1.1) for $0<\alpha<1$ is sufficient in this application, it remained an interesting open question to the author whether (1.2) implies (1.1) for $1\leqq\alpha<2$ if $d\geqq2$.

If $d=1$, then Theorem 2 is due to Boas [2] and is also presented as Theorem 11.3.3 by Kawata [5]. It is known that (1.1) (or (1.2)) fails to imply (1.3) for $\alpha=1$ in the univariate case, and this fact implies that the same is true for $d\geqq2$. Kawata [5] also states (1.2) $\Rightarrow$ (1.3) for $1<\alpha<2$, $d=1$. But the referee of the present note has pointed out that Kawata's argument on p. 422 of [5] is incorrect if $1<\alpha<2$, and it gives only (1.4) ($d=1$). Therefore Theorem 2 is a complete generalisation of what is known at present in the case $d=1$. I am very grateful to the referee for this observation which also saved me from copying the univariate error here.

Just as in the special case $d=1$, the problem of the equivalence of (1.1) and (1.2) or of (1.3) and (1.2) is meaningless if $\alpha\geqq2$. This remark and the open problem mentioned above are more conveniently discussed after the proofs in Sect. 3.

2. Proofs

The following formula will frequently be used. It is probably well known, and proved by the usual extension procedure from the easily checked case when f is an indicator function. It can also be extended such that instead of the special $g(x_1,\dots,x_d)=|x_1|+\dots+|x_d|$ one has a general $\mathscr{A}_F^d$-mesurable g: $R^d\to R^1$.

Lemma 1. *If* f: $[0,\infty)\to[0,\infty)$ *is Borel measurable, then*

$$\int_{R^d} f\left(\sum_{k=1}^{d}|x_k|\right)dF(x)=-\int_0^\infty f(u)\,d\mu_F(A_u),$$

where

$$\mu_F(A_u)=\int_{A_u} dF(x)$$

and

$$A_u=\left\{x=(x_1,\dots,x_d)\colon \sum_{k=1}^{d}|x_k|\geqq u\right\}. \tag{2.1}$$

It can be noted right away that the tail condition (1.2) will be used in the form

$$\mu_F(A_u)=O(u^{-\alpha}), \qquad u\to\infty. \tag{2.2}$$

They are equivalent since

$$A_{ud}\subset B_u\subset A_u, \qquad u>0.$$

Consider the cube

$$I^d=\{x=(x_1,\dots,x_d):\ -1\leqq x_1\leqq 1,\dots,\ -1\leqq x_d\leqq 1\},$$

and let $\varepsilon^{(j)}=(\varepsilon_1^{(j)},\dots,\varepsilon_d^{(j)})$, $j=1,\dots,2^d$, be the vertices of I^d numbered so that $\varepsilon^{(j)}=-\varepsilon^{(j+2^{d-1})}$, $j=1,\dots,2^{d-1}$. The following lemma is a d-variate generalisation of the "truncation inequality" of Loève [6, p. 209], which is Theorem 3.7.3 in Kawata [5]. In its proof, as in further proofs, the ideas of the univariate proofs are, naturally, used.

Lemma 2. *For all* $u>0$

$$\int\limits_{\sum_{k=1}^{d}|x_k|>\frac{1}{u}} dF(x_1,\dots,x_d)\leqq\frac{7}{u}\sum_{j=1}^{2^{d-1}}\int_0^u[1-\operatorname{Re}C(\varepsilon_1^{(j)}v,\dots,\varepsilon_d^{(j)}v)]\,dv.$$

Proof. Let $E_1,\dots,E_{2^d}$ denote the "$(1/2^d)^{\text{th}}$ spaces" of R^d labelled by $\varepsilon^{(1)},\dots,\varepsilon^{(2^d)}$ respectively, i.e., $\varepsilon^{(j)}\in E_j$, $j=1,\dots,2^d$. This means that if $x=(x_1,\dots,x_d)\in E_j$, $j=1,\dots,2^{d-1}$, then $\varepsilon_k^{(j)}x_k=|x_k|$, $k=1,\dots,d$, and if $x=(x_1,\dots,x_d)\in E_{j+2^{d-1}}$, $j=1,\dots,2^{d-1}$, then $\varepsilon_k^{(j)}x_k=-|x_k|$, $k=1,\dots,d$. With $F_j=E_j\cup E_{j+2^{d-1}}$ we get $\left(\sum|x_k|=\sum_{k=1}^{d}|x_k|\right)$

$$\begin{aligned}
&\sum_{j=1}^{2^{d-1}}\frac{1}{u}\int_0^u[1-\operatorname{Re}C(\varepsilon_1^{(j)}v,\dots,\varepsilon_d^{(j)}v)\,dv\\
&=\sum_{j=1}^{2^{d-1}}\int_{R^d}\left\{\frac{1}{u}\int_0^u[1-\cos(v\Sigma\,\varepsilon_k^{(j)}x_k)]\,dv\right\}dF(x)\\
&=\sum_{j=1}^{2^{d-1}}\int_{R^d}\left\{1-\frac{\sin(u\Sigma\,\varepsilon_k^{(j)}x_k)}{u\Sigma\,\varepsilon_k^{(j)}x_k}\right\}dF(x)\\
&\geqq\sum_{j=1}^{2^{d-1}}\int_{F_j\cap A_{1/u}}\left\{1-\frac{\sin(u\Sigma|x_k|)}{u\Sigma|x_k|}\right\}dF(x)\\
&\geqq\frac{1}{7}\sum_{j=1}^{2^{d-1}}\int_{F_j\cap A_{1/u}}dF(x)\\
&=\tfrac{1}{7}\int_{A_{1/u}}dF(x),
\end{aligned}$$

since $1-(\sin v)/v\geqq 1/7$ for $v\geqq 1$.

The following lemma is in fact the main body in the proof of Theorem 1, but it deserves to be separated since it is valid also if $1\leqq\alpha<2$.

Lemma 3. *Let* $0<\alpha<2$. *The necessary and sufficient condition that*

$$\sum_{j=1}^{2^d} [1-\operatorname{Re} C(\varepsilon_1^{(j)}h, \ldots, \varepsilon_d^{(j)}h)] = O(|h|^\alpha), \qquad h\to 0, \tag{2.3}$$

is the tail condition (1.2).

Proof. Denoting by $\phi(h)$ the left hand side of (2.3), for $u\geqq 1$ (say) we get from Lemma 2 that

$$\begin{aligned}\mu_F(A_u) &\leqq \tfrac{7}{2}u \int_0^{1/u} \phi(h)\,dh \\ &\leqq \tfrac{7}{2} \sup_{0\leqq h\leqq 1/u} \phi(\mathrm{h}) \\ &\leqq K u^{-\alpha},\end{aligned}$$

with some constant K, proving the necessity of (1.2).

To show sufficiency, let us suppose now (2.2). It is enough to show only that for all $j=1, \ldots, 2^d$

$$1-\operatorname{Re} C(\varepsilon_1^{(j)}h, \ldots, \varepsilon_d^{(j)}h) = O(h^\alpha), \qquad h\to 0+. \tag{2.4}$$

Following Binmore and Stratton [1] or Kawata [5, p. 420], let

$$\limsup_{u\to\infty} Q_\alpha(u) = L < \infty \tag{2.5}$$

with

$$Q_\alpha(u) = u^\alpha \mu_F(A_u), \qquad u>0.$$

Using that for any $u>0$ one has $\sin^2 u \leqq 2\int_0^{2u} (\sin^2 v)/v\,dv$, and then Lemma 1, we get

$$\begin{aligned}1-\operatorname{Re} C(\varepsilon_1^{(j)}h, \ldots, \varepsilon_d^{(j)}h) &= 2\int_{R^d} \sin^2(\tfrac{1}{2}h|\Sigma\varepsilon_k^{(j)}x_k|)\,dF(x) \\ &\leqq 4\int_{R^d}\left\{\int_0^{h|\Sigma\varepsilon_k^{(j)}x_k|}\frac{\sin^2 v}{v}dv\right\}dF(x) \\ &\leqq 4\int_{R^d}\left\{\int_0^{h\Sigma|x_k|}\frac{\sin^2 v}{v}dv\right\}dF(x) \\ &= -4\int_0^\infty\left\{\int_0^{hu}\frac{\sin^2 v}{v}dv\right\}d\frac{Q_\alpha(u)}{u^\alpha} \\ &= -4\int_0^\infty\left\{\int_0^{w}\frac{\sin^2 v}{v}dv\right\}d\left[\left(\frac{w}{h}\right)^{-\alpha}Q_\alpha\left(\frac{w}{h}\right)\right] \\ &= 4h^\alpha\int_0^\infty Q_\alpha\left(\frac{w}{h}\right)\frac{\sin^2 w}{w^{1+\alpha}}dw,\end{aligned}$$

the last equality obtained via integration by parts. Hence

$$\limsup_{h\to 0+}\frac{1}{h^\alpha}[1-\operatorname{Re} C(\varepsilon_1^{(j)}h, \ldots, \varepsilon_d^{(j)}h)] \leqq 4L\int_0^\infty \frac{\sin^2 w}{w^{1+\alpha}}dw,$$

for all $j=1,\dots,2^d$, showing (2.4). The last integral is finite since $0<\alpha<2$, and this fact was also used to show that the integrated out term disappears when integrating by parts before.

Proof of Theorem 1. Suppose first (1.1). Then we have (2.3), and by Lemma 3 (1.2) follows.

Suppose now (1.2) with $0<\alpha<2$, $\alpha\neq 1$. Given a $t\neq(0,\dots,0)$, there is a $j=j_t$, $1\leqq j\leqq 2^d$, such that $t\in E_j$. We associate then with this t its orthogonal projection $t^*=(\varepsilon_1^{(j)}u_t,\dots,\varepsilon_d^{(j)}u_t)$ on the ray $(\varepsilon_1^{(j)}u,\dots,\varepsilon_d^{(j)}u)$, where $0<u_t\leqq\|t\|$. Now

$$1-\operatorname{Re}C(t)\leqq[1-\operatorname{Re}C(t^*)]+|\operatorname{Re}C(t)-\operatorname{Re}C(t^*)|,$$

where the first term is $O(\|t\|^\alpha)$ as $t\to(0,\dots,0)$ by Lemma 3, while the second is majorized by

$$2\int_{R^d}|\sin\tfrac{1}{2}\langle t-t^*,x\rangle|\,dF(x)\leqq\|t-t^*\|\int_{\Sigma|x_k|<1/\|t\|}\sum|x_k|\,dF(x)+\int_{A_{1/\|t\|}}dF(x).$$

Here the second term is $O(\|t\|^\alpha)$ by hypothesis, while the first, using again Lemma 1, is not greater than

$$\begin{aligned}\|t\|\int_0^{1/\|t\|}u\,d\mu_F(A_u)&=-\|t\|\,[u\mu_F(A_u)]_{u=0}^{1/\|t\|}+\|t\|\int_0^{1/\|t\|}\mu_F(A_u)\,du\\&=-\mu_F(A_{1/\|t\|})+\|t\|\,O\left(\int_{\varepsilon(\alpha)}^{1/\|t\|}u^{-\alpha}\,du\right)\\&=\begin{cases}O(\|t\|^\alpha)+\|t\|\,O(\|t\|^{\alpha-1}), & 0<\alpha<1,\\ O(\|t\|^\alpha)+\|t\|\,O(1), & 1<\alpha<2,\end{cases}\\&=\begin{cases}O(\|t\|^\alpha), & 0<\alpha<1,\\ O(\|t\|), & 1<\alpha<2,\end{cases}\end{aligned}$$

as $t\to(0,\dots,0)$. $\varepsilon(\alpha)$ above is 0 if $0<\alpha<1$ and $\varepsilon(\alpha)>0$ if $1<\alpha<2$.

Proof of Theorem 2. Obviously (1.3) implies (1.1) for all $\alpha>0$.

Suppose (1.1) with $0<\alpha<2$, $\alpha\neq 1$. What we have to show is

$$\operatorname{Im}C(t)=\int_{R^d}\sin\langle t,x\rangle\,dF(x)=\begin{cases}O(\|t\|^\alpha), & 0<\alpha<1,\\ O(\|t\|), & 1<\alpha<2,\end{cases}$$

as $t\to(0,\dots,0)$. Since by the first part of Theorem 1 we have (2.2), this can be done exactly the same way as in the proof of the converse part of Theorem 1.

If in all the proofs of this section O is replaced by o, and $L=0$ in (2.5), then we obtain both theorems with o instead of O.

3. Discussion

Proposition. *If* $\operatorname{Re} C(t)=1+o(\|t\|^2)$ *as* $t\to(0,\ldots,0)$, *then* F *is the degenerate distribution with unit mass at* $(0,\ldots,0)$.

Proof. The proofs of the following inequalities are the same as in the univariate case (Kawata [5, p. 96]).

$$|C(t+s)-C(t)|^2 \leqq 2(1-\operatorname{Re} C(s)), \qquad t,s\in R^d,$$
$$1-\operatorname{Re} C(2t)\leqq 4(1-\operatorname{Re} C(t)), \qquad t\in R^d.$$

Using these inequalities the Proposition is proved the same way as in the univariate case (Kawata [5, p. 98]).

So the theorems are meaningless for $\alpha\geqq 2$.

After Sect. 2 the question whether (1.2) $\Rightarrow$ (1.1) holds or not for $1\leqq\alpha<2$ can be asked the following way. Let $d\geqq 2$, and assume that

$$1-\operatorname{Re} C(\varepsilon_1^{(j)}h,\ldots,\varepsilon_d^{(j)}h)=O(|h|^\alpha), \qquad h\to 0,\ j=1,\ldots,2^d. \tag{3.1}$$

The question is whether there exist a $C(t)$ for which (3.1) holds but

$$\limsup_{t\to(0,\ldots,0)} \|t\|^{-\alpha}[1-\operatorname{Re} C(t)]=\infty,$$

or from (3.1) it always follows that the latter limsup is finite. For $\alpha>1$ this is basically the same question whether (1.2) implies $\operatorname{Im} C(t)=O(\|t\|^\alpha)$ as $t\to(0,\ldots,0)$ or not, and the answer is unknown even if $d=1$.

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