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# Lower Envelopes near Zero and Infinity for Processes with Stable Components

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## Section 1. Introduction

The object of this paper is to investigate certain sample path properties of a type of Markov process in $R^d$ with stationary independent increments. To define the process, let $X_i(t)$ be a stable process of index $\alpha_i$ in Euclidean space of dimension $d_i$, for $i=1,2,\ldots,n$. Let $d=d_1+\cdots+d_n$. If the $X_i(t)$ are independent, the process $X(t)$ in $R^d$ defined by

$$X(t)\equiv(X_1(t),\ldots,X_n(t)), \tag{1.1}$$

where the $d_i$-dimensional subspaces in which the $X_i(t)$ take their values are orthogonal, is called a process with stable components. We may assume, with no loss of generality, that the indices $\alpha_i$ are distinct and that

$$\alpha_n<\alpha_{n-1}<\cdots<\alpha_2<\alpha_1.$$

Pruitt and Taylor have already studied processes of this type in [9], where they establish the asymptotic behavior of the first passage time out of a sphere and of the sojourn time within a sphere; they also find the correct Hausdorff measure function. We shall be interested in determining lower envelopes of functions with respect to $X(t)$ near zero and infinity. Integral tests are already known ([11], [12] and [13]) for monotone functions $h(t)$ which limit the behavior of $|X_{\alpha,n}(t)|$ for certain types of stable processes $X_{\alpha,n}(t)$ of index $\alpha<n$ in $R^n$ as $t$ approaches infinity (zero) in the sense that the event

$$[|X_{\alpha,n}(t)|\leqq h(t) \text{ i.o. as } t\to \text{infinity (zero)}]$$

has probability 0 or 1 according to whether or not the integral

$$\int^{\infty}\frac{\{h(t)\}^{n-\alpha}}{t^{n/\alpha}}\,dt\Big(\int_0 \text{——}\Big)$$

converges or diverges. For $X(t)$ as defined above we seek results in the form of integral tests near zero and infinity.

Our basic method of study is to use some potential theory to obtain estimates of delayed hitting probabilities of certain spheres with respect to two special classes of processes with stable components; this is done in Section 3. In Section 4 (Theorems 4.1 and 4.2) integral tests are developed for lower functions with respect

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to the processes considered in Section 3; as a corollary to each theorem we obtain an integral test for lower functions with respect to $X(t)$ as defined by (1.1).

An interesting special case of (1.1) is the process $X(t)$ defined by

$$X(t) \equiv (X_1(t), X_2(t)), \tag{1.2}$$

where $X_i(t)$ is a stable process of index $\alpha_i$ in $R^1$ for $i=1,2$ and $1<\alpha_2<\alpha_1 \leqq 2$. Pruitt and Taylor [9] noted that most of the possible kinds of behavior for the $X(t)$ of (1.1) are already obtainable for $X(t)$ as given in (1.2) and that the general proofs are not much harder. Moreover, each component of $X(t)$ in (1.2) is point recurrent but the process itself is transient and most information about the process cannot be obtained by analyzing its components.

For $X(t)$ as in (1.2) our results will show that the event

$$[|X(t)| \leqq h(t) \text{ i.o. as } t \to \text{infinity (zero)}]$$

has probability 0 or 1 according to whether the integral

$$\int^{\infty} \frac{\{h(t)\}^{\alpha_1[\lambda-1]}}{t^{\lambda}}\, dt \left( \int_0 \frac{\{h(t)\}^{\alpha_2[\lambda-1]}}{t^{\lambda}}\, dt \right)$$

converges or diverges, where $\lambda = \alpha_1^{-1} + \alpha_2^{-1}$ and $h(t)$ increases monotonely from zero to $+\infty$ as $t$ increases from zero to infinity. Thus, the event

$$[|X(t)|\, t^{-1/\alpha_2} \leqq 1 \text{ i.o. as } t \to 0]$$

has probability 1, and with probability one $|X(t)|\, t^{-1/\alpha_2}$ does not converge to $+\infty$ as $t$ approaches zero. On the other hand, it is not hard to prove that:

(i) $|X(t)|\, t^{-1/\alpha}$ converges to $+\infty$ in probability as $t \to 0$ if $\alpha < \alpha_1$;

(ii) $|X(t)|^{-1/\alpha}$ does not converge to $+\infty$ in probability as $t \to 0$ if $\alpha \geqq \alpha_1$.

We can also show that the parameters $\beta$, $\beta'$ and $\beta''$ defined by Blumenthal and Getoor [3] satisfy:

$$0 < \beta'' = \alpha_2 < \beta' = 1 + \alpha_2 - \alpha_2/\alpha_1 < \beta = \alpha_1.$$

Thus Corollary 5.1 in [3] can be strengthened in this case to give convergence of $|X(t)|\, t^{-1/\alpha}$ to $+\infty$ in probability as $t$ approaches zero whenever $\alpha < \beta$. Since the Hausdorff dimension of the range of the sample paths is almost surely equal to $\beta'$ (see [9]) we note in addition that

$$\sup\{\alpha : |X(t)|\, t^{-1/\alpha} \to +\infty \text{ almost surely as } t \to 0\} = \alpha_2$$

is in this case distinct from the Hausdorff dimension of the sample paths. It would also be interesting to characterize the asymptotic behavior of an arbitrary independent increment process (see [3]), but we have not yet been able to do this.

## Section 2. Preliminaries

The $n$-dimensional characteristic function of a stable process $X_{\alpha,n}(t)$ of index $\alpha+1$ in $R^n$ has the form $\exp[t\,\psi(y)]$, where

$$\psi(y) = i\langle a, y\rangle - \delta |y|^{\alpha} \int_{S_n} w_{\alpha}(y, \theta)\, \mu(d\theta),$$

with $a \in R^n$, $\delta > 0$, $\langle , \rangle$ the usual inner product in $R^n$,

$$w_\alpha(y, \theta) = [1 - i \operatorname{sgn} \langle y, \theta \rangle \tan \pi \alpha/2] \, |\langle \theta, y/|y| \rangle|^\alpha,$$

and $\mu$ a probability measure on the surface of the unit sphere $S_n$ in $R^n$ [8]. We assume $a=0$, $\delta=1$, and that $\mu$ is not supported by a proper linear subspace of $R^n$. If $\mu$ is uniform the process is said to be symmetric.

When speaking of a stable process $X_{\alpha,n}(t)$ we will always write the two subscripts to indicate index $\alpha$ and dimension $n$ except when the process is actually a component of a process with stable components. When the latter occurs we use a single subscript $i$ to indicate the $i$-th component. The symbol $X(t)$ will be understood to refer to a process with two or more stable components. Likewise, $p_{\alpha,n}(t, x)$, $p_i(t, x_i)$ and $p(t, x)$ will denote the densities of $X_{\alpha,n}(t)$, $X_i(t)$, and $X(t)$ respectively.

Taylor [13] classifies stable processes $X_{\alpha,n}$ as being of type $A$ or type $B$ according to whether or not $p_{\alpha,n}(1, 0) > 0$. When $\alpha > 1$, only processes of type $A$ can occur. If $\alpha \leqq 1$, we will assume that the process (or component) is type $A$. A key property of a stable density (except for some nonsymmetric processes of index 1, which we henceforth exclude) $p_{\alpha,n}(t, x)$ is the scaling property:

$$p_{\alpha,n}(t, x) = r^{n/\alpha} p_{\alpha,n}(r t, r^{1/\alpha} x) \tag{2.1}$$

for all $r > 0$, or in terms of the process itself, $X_{\alpha,n}(r t)$ and $r^{1/\alpha} X_{\alpha,n}(t)$ have the same distribution.

The stable density function $p_{\alpha,n}(t, x)$ is known to be positive, continuous, and bounded in $x$ for each fixed $t$ [10]. Since we are considering only processes or components of type $A$, $p_{\alpha,n}(1, x)$ is therefore bounded away from zero in any closed neighborhood of the origin.

All processes or components $X_{\alpha,n}(t)$ being considered will be regarded as being defined over some basic probability space $(\Omega, \mathscr{F}, P)$, with values $X_{\alpha,n}(t, \omega)$ in $R^n$; we take $t \geqq 0$, and $\omega \in \Omega$, although we usually suppress the $\omega$'s in our notation. We may assume that almost all sample functions $X_{\alpha,n}(\cdot, \omega)$ are right continuous and have left limits everywhere. Moreover, we assume that the strong Markov property holds and that $X_{\alpha,n}(0) = 0$ with probability one [2].

The letters $a$, $b$, and $c$ will be used to denote positive constants which can vary in size from statement to statement or line to line. Positive constants whose values remain fixed throughout the discussion will be introduced in order and denoted by $c_1, \ldots, c_{14}$.

### Section 3. Potential Theory and Hitting Probabilities

Taylor [13] gives a brief background of the potential theory which we need. Denoting the density of $X(t)$ as defined in (1.1) by $p(t, x)$ we have:

$$p(t, x) \equiv \prod_{i=1}^{n} p_i(t, x_i) \equiv \prod_{i=1}^{n} p_{\alpha_i, d_i}(t, x_i),$$

where $x = (x_1, \ldots, x_n) \in R^d$ and $d = d_1 + \cdots + d_n$. $U(y)$, the kernel of $X(t)$, is given by:

$$U(y) \equiv \int_0^\infty p(t, y) \, dt \tag{3.1}$$

and converges for all $y \neq 0$.

If we let $\mu$ be any measure defined on Borel subsets of compact sets $E$ in $R^d$, the potential at $x$ of the measure $\mu$ on $E$ is

$$W_\mu(x) \equiv \int_E U(y-x)\,\mu(dy).$$

The capacity of $E$ is zero iff $W_\mu$ is unbounded for every $\mu$ for which $\mu(E)>0$. If there are some $\mu$ such that $W_\mu$ is bounded, we define the capacity, $C(E)$, of $E$ by

$$C(E) = \sup\{\mu(E)\colon W_\mu(x) \leqq 1 \text{ for all } x\}.$$

When $E$ is compact this supremum is actually attained for a measure $\nu$, called the capacitory measure on $E$.

Finally, we denote the hitting probability of a Borel set $E$ starting from $x$ by:

$$\Phi(x, E) \equiv P^x[X(t) \in E \text{ for some } t>0].$$

Hitting probabilities are then given in terms of the kernel and the capacitory measure:

$$\Phi(x, E) = \int_E U(y-x)\,\nu(dy).$$

Our method will be to obtain bounds on the kernel, the capacity of closed spheres, and ultimately upon delayed hitting probabilities of certain spheres. The estimates for general $X(t)$ as defined above are not easily obtained, but fortunately are unnecessary. What will be needed will be estimates for large spheres in $R^{d_1+d_2}$ with respect to $(X_{\alpha_1,d_1}(t), X_{\alpha_2,d_2}(t))$ when $2 \geqq \alpha_1 > \alpha_2 > d_2 = 1$ and for small spheres in $R^{d_1+d_2}$ with respect to $(X_{\alpha_1,d_1}(t), X_{\alpha_2,d_2}(t))$ when $2 \geqq \alpha_1 > d_1 = 1$ and $\alpha_1 > \alpha_2$. Lemmas 3.1 and 3.2 deal with the first situation, while 3.3 and 3.4 deal with the latter.

**Lemma 3.1.** *Let $X(t)$ be defined by:*

$$X(t) \equiv (X_1(t), X_2(t)) \equiv (X_{\alpha_1,d_1}(t), X_{\alpha_2,d_2}(t)),$$

*where $2 \geqq \alpha_1 > \alpha_2 > d_2 = 1$. Let $S_r$ be the closed sphere in $R^{d_1+d_2}$ of radius $r \geqq 1$ which is centered at the origin, and denote the capacity of this sphere with respect to $X(t)$ by $C(S_r)$. Then positive constants $c_1$ and $c_2$, independent of $r$, can be found such that:*

$$c_1\, r^{\alpha_1[\lambda-1]} \leqq C(S_r) \leqq c_2\, r^{\alpha_1[\lambda-1]},$$

*where $\lambda = d_1 \alpha_1^{-1} + d_2 \alpha_2^{-1} > 1$.*

*Proof.* The kernel for $X(t)$ is given by

$$U(y) = \int_0^\infty p_1(t, y_1)\, p_2(t, y_2)\, dt$$

where $y = (y_1, y_2) \in R^{d_1+d_2}$ and $p_i(t, y_i)$ is the density of $X_i(t)$ for $i = 1, 2$.

To establish the upper bound for $C(S_r)$, let $\nu$ be the capacitory measure on $S_r$. Then:

$$1 \geqq \int_{S_r} U(y)\,\nu(dy) \geqq \min_{y\in S_r} U(y) \cdot \int_{S_r} \nu(dy) = C(S_r) \cdot \min_{y \in S_r} U(y). \tag{3.2}$$

Obtain lower bounds for the kernel inside $S_r$ by using the scaling property for the densities and the facts that $p_1(1,\cdot)$ and $p_2(1,\cdot)$ are bounded away from zero in closed neighborhoods about the origin and that $\lambda>1$.

$$U(y)=\int_0^\infty \frac{p_1(1, y_1 t^{-1/\alpha_1})\, p_2(1, y_2 t^{-1/\alpha_2})}{t^\lambda}\, dt$$

$$\geqq \begin{cases} \int_{|y_1|^{\alpha_1}}^\infty \text{——} \geqq a\,|y_1|^{\alpha_1[1-\lambda]} & \text{if} \quad |y_1|^{\alpha_1} \geqq |y_2|^{\alpha_2}; \\ \int_{|y_2|^{\alpha_2}}^\infty \text{——} \geqq b\,|y_2|^{\alpha_2[1-\lambda]} & \text{if} \quad |y_2|^{\alpha_2} \geqq |y_1|^{\alpha_1}. \end{cases}$$

Since $r\geqq 1$ we have:

$$\min_{y\in S_r} U(y) \geqq c \min_{i=1,2} \{r^{\alpha_i[1-\lambda]}\} = c\, r^{\alpha_1[1-\lambda]}.$$

Using this in (3.2), we obtain the desired upper bound:

$$C(S_r) \leqq [\min_{y\in S_r} U(y)]^{-1} = c_2\, r^{\alpha_1[\lambda-1]}.$$

The lower bound for $C(S_r)$ is established by first proving:

$$W_{\mu_L}(x) \equiv \int_{S_r} U(y-x)\, \mu_L(dy) \leqq c\, r^{d_2+\alpha_1-\alpha_1 d_2/\alpha_2}, \tag{3.3}$$

where $x=(x_1, x_2)\in R^{d_1+d_2}$, $\mu_L$ is Lebesgue measure in $R^{d_1+d_2}$ and $c>0$ is independent of $r$ and $x$. This is done by interchange of orders of integration and using Lebesgue measure $\mu_L(dy_1)$ and $\mu_L(dy_2)$ in $R^{d_1}$ and $R^{d_2}$ respectively to obtain:

$$\begin{aligned}
W_{\mu_L}(x) &= \int_{S_r} \int_0^\infty p_1(t, y_1-x_1)\, p_2(t, y_2-x_2)\, dt\, \mu_L(dy) \\
&\leqq \int_0^\infty \Big[\int_{|y_1|\leqq r} p_1(t, y_1-x_1)\, \mu_L(dy_1) \cdot \int_{|y_2|\leqq r} p_2(t, y_2-x_2)\, \mu_L(dy_2)\Big]\, dt \\
&= \int_0^\infty P[|X_1(t)+x_1|\leqq r] \cdot P[|X_2(t)+x_2|\leqq r]\, dt \\
&= \int_0^\infty P[|X_1(1)+x_1 t^{-1/\alpha_1}|\leqq r t^{-1/\alpha_1}] \cdot P[|X_2(1)+x_2 t^{-1/\alpha_2}|\leqq r t^{-1/\alpha_2}]\, dt \\
&\leqq \int_0^{r^{\alpha_2}} dt + \int_{r^{\alpha_2}}^{r^{\alpha_1}} a[r t^{-1/\alpha_2}]^{d_2}\, dt + \int_{r^{\alpha_1}}^\infty b[r t^{-1/\alpha_1}]^{d_1} [r t^{-1/\alpha_2}]^{d_2}\, dt \\
&\leqq r^{\alpha_2} + a\, r^{d_2+\alpha_1[1-d_2/\alpha_2]} + b\, r^{d_1+d_2+\alpha_1[1-\lambda]},
\end{aligned}$$

where the inequality preceding the sum is justified by the boundedness of $p_1(1, y_1)$ and $p_2(1, y_2)$. Now use the facts that $r\geqq 1$ and $\alpha_2 \leqq d_2+\alpha_1-\alpha_1 d_2/\alpha_2$ to obtain (3.3).

Having established (3.3) we define a measure $\mu(E)$ on Borel subsets $E$ of $S_r$ by:

$$\mu(E) = c^{-1} r^{-d_2-\alpha_1+\alpha_1 d_2/\alpha_2}\, \mu_L(E).$$

Then $W_\mu(x) \leqq 1$ for all $x$ and:

$$C(S_r) \geqq \mu(S_r) = c_1 r^{-d_2 - \alpha_1 + \alpha_1 d_2/\alpha_2} \cdot r^{d_1 + d_2},$$

which completes the proof of the lemma.

**Lemma 3.2.** *Let $X(t)$, $r$, and $S_r$ be as in Lemma 3.1 and define the delayed hitting probability of $S_r$ starting from $x$ by:*

$$Q(x, r, T) \equiv P^x[X(t) \in S_r \text{ for some } t \geqq T],$$

*where $T > 0$. Then positive constants $c_3$ and $c_4$, independent of $x, r$, and $T$, can be found such that:*

(i) $Q(x, r, T) \leqq c_3\, C(S_r)\, T^{1-\lambda}$;

(ii) $Q(x, r, T) \geqq c_4\, C(S_r)\, T^{1-\lambda}$ *if* $|x| \leqq r$ *and* $T \geqq (2r)^{\alpha_1}$.

*Proof.* Both parts use the following equality whose proof is patterned after by Jain and Pruitt [5] in the transient stable case:

$$\begin{aligned} Q(x, r, T) &= \int_{R^{d_1+d_2}} p(T, y-x)\, \Phi(y, S_r)\, dy \\ &= \int_{S_r} \int_0^\infty \int_{R^{d_1+d_2}} p(T, y-x)\, p(t, z-y)\, dy\, dt\, \nu(dz) \\ &= \int_{S_r} \int_0^\infty p(T+t, z-x)\, dt\, \nu(dz) \\ &= \int_{S_r} \int_T^\infty p_1\big(1, (y_1-x_1)\, t^{-1/\alpha_1}\big)\, p_2\big(1, (y_2-x_2)\, t^{-1/\alpha_2}\big)\, t^{-\lambda}\, dt\, \nu(dy), \end{aligned} \tag{3.4}$$

where $\nu$ is the capacitory measure on $S_r$. Using (3.4), (i) is an easy consequence of the boundedness of the $p_i(1, x)$ and the fact that $\lambda > 1$. The hypotheses in (ii) imply that for any $y = (y_1, y_2)$ in $S_r$:

$$|y_i - x_i| \leqq 2r \leqq T^{1/\alpha_i} \quad \text{for } i = 1, 2.$$

Now using the lower bounds for stable densities in closed neighborhoods of the origin, we obtain (ii) from (3.4) in the same way as (i).

We conclude this section with two lemmas whose proofs parallel those of Lemmas 3.1 and 3.2.

**Lemma 3.3.** *Let $X(t)$ be defined by:*

$$X(t) \equiv \big(X_1(t), X_2(t)\big) \equiv \big(X_{\alpha_1, d_1}(t), X_{\alpha_2, d_2}(t)\big),$$

*where $2 \geqq \alpha_1 > d_1 = 1$ and $\alpha_1 > \alpha_2$. Let $S_r$ be the closed sphere in $R^{d_1+d_2}$ of radius $r \leqq 1$ which is centered at the origin, and denote the capacity of this sphere with respect to $X(t)$ by $C(S_r)$. Then positive constants $c_5$ and $c_6$, independent of $r$, can be found such that:*

$$c_5\, r^{\alpha_2[\lambda-1]} \leqq C(S_r) \leqq c_6\, r^{\alpha_2[\lambda-1]},$$

*where $\lambda = d_1 \alpha_1^{-1} + d_2 \alpha_2^{-1} > 1$.*

*Proof.* The kernel for $X(t)$ is given by

$$U(y)=\int_0^\infty p_1(t, y_1)\, p_2(t, y_2)\, dt$$

where $y=(y_1, y_2)\in R^{d_1+d_2}$ and $p_i(t, y_i)$ is the density of $X_i(t)$ for $i=1, 2$. Let $v$ be the capacitory measure on $S_r$ and observe that:

$$1\geqq \int_{S_r} U(y)\, v(dy)\geqq C(S_r)\cdot \min_{y\in S_r} U(y). \tag{3.2'}$$

Obtain lower bounds for $U(y)$ inside $S_r$ in the same way as before:

$$\min_{y\in S_r} U(y)\geqq c \min_{i=1,2} \{r^{\alpha_i[1-\lambda]}\}=c\, r^{\alpha_2[1-\lambda]},$$

since $r\leqq 1$. Using this in (3.2'), we obtain the upper bound for $C(S_r)$.

The lower bound for $C(S_r)$ is established by first proving:

$$W_{\mu_L}(x)\equiv \int_{S_r} U(y-x)\, \mu_L(dy)\leqq c\, r^{d_1+\alpha_2-\alpha_2 d_1/\alpha_1}, \tag{3.3'}$$

where $x=(x_1, x_2)\in R^{d_1+d_2}$, $\mu$ is Lebesgue measure in $R^{d_1+d_2}$ and $c>0$ is independent of $r$ and $x$. Proceeding as before we have:

$$\begin{aligned} W_{\mu_L}(x) &\leqq \int_0^\infty P[|X_1(t)+x_1|\leqq r]\cdot P[|X_2(t)+x_2|\leqq r]\, dt \\ &\leqq \int_0^{r^{\alpha_1}} dt+\int_{r^{\alpha_1}}^{r^{\alpha_2}} a[r\, t^{-1/\alpha_1}]^{d_1}\, dt+\int_{r^{\alpha_2}}^{\infty} b[r\, t^{-1/\alpha_1}]^{d_1}[r\, t^{-1/\alpha_2}]^{d_1}\, dt \\ &\leqq r^{\alpha_1}+a\, r^{d_1+\alpha_2[1-d_1/\alpha_1]}+b\, r^{d_1+d_2}\, r^{\alpha_2[1-\lambda]}. \end{aligned}$$

Now use the facts that $r\leqq 1$ and $\alpha_1\geqq d_1+\alpha_2-\alpha_2 d_1/\alpha_1$ to obtain (3.3').

Next, define a measure $\mu(E)$ on Borel subsets $E$ of $S_r$ by:

$$\mu(E)=c^{-1}\, r^{-d_1-\alpha_2+\alpha_2 d_1/\alpha_1}\, \mu_L(E).$$

Then $W_\mu(x)\leqq 1$ for all $x$ and:

$$C(S_r)\geqq \mu(S_r)=c_5\, r^{-d_1-\alpha_2+\alpha_2 d_1/\alpha_1}\cdot r^{d_1+d_2},$$

which completes the proof of the lemma.

**Lemma 3.4.** *Let $X(t)$, $r$ and $S_r$ be as in Lemma 3.3 and define $Q(x, r, T)$ as before. Then positive constants $c_7$ and $c_8$, independent of $x$, $r$, and $T$, can be found such that:*

(i) $Q(x, r, T)\leqq c_7\, C(S_r)\, T^{1-\lambda}$;

(ii) $Q(x, r, T)\geqq c_8\, C(S_r)\, T^{1-\lambda}$ *if* $|x|\leqq r$ *and* $T\geqq (2r)^{\alpha_2}$.

*Proof.* Proceed as in the proof of Lemma 3.2.

## Section 4. Lower Envelopes near Zero and Infinity

We now consider the problem of finding monotone functions which limit the behavior of $|X(t)|$ as $t$ approaches zero or infinity, where $X(t)$ is a process with stable components. Takeuchi has done this in the transient symmetric stable case near zero in [11] and near infinity in [12], while Taylor [13] asserts that Takeuchi's results near infinity can be extended to include any stable process of type $A$. His results take the form of an integral test which has the same form at zero as at infinity. We too will establish an integral test, but the test near zero will differ significantly from that at infinity.

To consider $|X(t)|$ near infinity we suppose that $h(t)$ is a positive and non-decreasing function defined for all large real numbers $t$. Since $X(t)$ will prove to be transient we assume $h(t) \geqq 1$. By using the Hewitt-Savage zero-one law [4] we will show that the following event has probability zero or one:

$$[|X(t)| \leqq h(t) \text{ i.o. as } t \to \infty].$$

By this we shall always mean that with probability zero or one there is an increasing and unbounded sequence $\{t_j(\omega)\}_{j=1}^{\infty}$ of times such that $|X(t_j(\omega), \omega)| \leqq h(t_j(\omega))$ for all $j$.

$h(t)$ will be said to be in the lower class, $L_\infty$, with respect to the process $X(t)$ if the above event has probability zero and in the upper class, $U_\infty$, if the event has probability one. Hence, if $h \in L_\infty$ and $g(t) \leqq h(t)$ for all large $t$ we have $g \in L_\infty$, and similarly $g \in U_\infty$ if $h \in U_\infty$ and $g(t) \geqq h(t)$ whenever $t$ is large.

Our first integral test is given in Theorem 4.1, where we consider processes of the type defined in Lemmas 3.1 and 3.2. As a corollary, we then obtain the desired integral test near infinity for $X(t)$ as defined in (1.1).

**Theorem 4.1.** *Let $X(t)$ be defined as in Lemma* 3.1, *and let $L_\infty$ and $U_\infty$ be as above. Then a positive non-decreasing function $h(t)$ as defined above belongs to $L_\infty$ or $U_\infty$ according to whether the integral*

$$I_h \equiv \int^{\infty} \frac{\{h(t)\}^{\alpha_1[\lambda-1]}}{t^\lambda} \, dt$$

*converges or diverges, where $\lambda = d_1 \alpha_1^{-1} + d_2 \alpha_2^{-1}$.*

*Proof.* Suppose that the integral converges. Let $\lambda = d_1 \alpha_1^{-1} + d_2 d_2^{-1}$ and define events $M_n$ by

$$M_n = [\omega\colon\ |X(t, \omega)| \leqq h(t) \text{ for some } t \text{ in } [2^n, 2^{n+1}]],$$

for $n = 1, 2, 3, \ldots$. Then, since $h(t) \geqq 1$, Lemmas 3.1 and 3.2 give:

$$\begin{aligned} P[M_n] &\leqq Q(0, h(2^{n+1}), 2^n) \\ &\leqq c_2 c_3 \frac{\{h(2^{n+1})\}^{\alpha_1[\lambda-1]}}{2^{n[\lambda-1]}} \leqq c \int_{2^{n+1}}^{2^{n+2}} \frac{\{h(t)\}^{\alpha_1[\lambda-1]}}{t^\lambda} \, dt. \end{aligned}$$

Thus, $\sum_n P[M_n] \leqq c \cdot I_h < \infty$. By the Borel-Cantelli lemma, $P[M_n \text{ i.o.}] = 0$ and $h$ is in $L_\infty$.

Conversely, suppose that the integral diverges. We accomplish the proof by means of several lemmas.

**Lemma 4.1.** *Without loss of generality, we may assume that $h(t) \leqq t^{1/\alpha_1}$ when $X(t)$ is as above.*

*Proof.* We must show that assuming the theorem valid for functions which never exceed $t^{1/\alpha_1}$, we can show $h$ to be in $U_\infty$ whenever $I_h = \infty$. So let $h$ be any function for which the integral diverges and define $h^*(t) = \min\{h(t), t^{1/\alpha_1}\}$. Then $h^* \leqq h$, $h^*(t) \leqq t^{1/\alpha_1}$, and $h^*$ is non-decreasing. Therefore:

$$\int_{2^n}^{2^{n+1}} \frac{\{h^*(t)\}^{\alpha_1[\lambda-1]}}{t^\lambda} dt \geqq \frac{2^n \cdot \{h^*(2^n)\}^{\alpha_1[\lambda-1]}}{\{2^{n+1}\}^\lambda} = \frac{c\, 2^{n-1} \cdot \{h^*(2^n)\}^{\alpha_1[\lambda-1]}}{\{2^{n-1}\}^\lambda} \tag{4.1}$$

for some $c > 0$ independent of $n$.

If $h^*(2^n) = h(2^n)$ for all large $n$, the above expression is:

$$\frac{c\, 2^{n-1} \cdot \{h(2^n)\}^{\alpha_1[\lambda-1]}}{\{2^{n-1}\}^\lambda} \geqq c \int_{2^{n-1}}^{2^n} \frac{\{h(t)\}^{\alpha_1[\lambda-1]}}{t^\lambda} dt,$$

so that $I_{h^*} = \infty$. Since $h^* \in U_\infty$ by hypothesis and $h^* \leqq h$, we have $h$ in $U_\infty$ also.

On the other hand, if $h^*(2^n) = 2^{n/\alpha_1}$ for infinitely many values of $n$, for such $n$ we have (see (4.1)):

$$\frac{2^n \cdot \{h^*(2^n)\}^{\alpha_1[\lambda-1]}}{\{2^{n+1}\}^\lambda} = 2^{-\lambda}.$$

Therefore $I_{h^*} = \infty$ in this case also. Once again we conclude that $h$ is in $U_\infty$ and the lemma is proved. We note in passing that Lemma 4.1 remains valid if we replace $t^{1/\alpha_1}$ by $c\, t^{1/\alpha_1}$, where $c$ is any positive constant.

Next we obtain a lower bound on the probability of returning to a large sphere in $[T_1, T_2]$ which is of the same order as returning in $[T_1, \infty]$. Using both parts of Lemma 3.2 we prove:

**Lemma 4.2.** *Let $X(t)$ and $S_r$ be as in Lemma* 3.1*; also let $T_1 \geqq (2r)^{\alpha_1}$ and $x \in S_r$. Then positive constants $c_9 > 1$ and $c_{10}$, independent of $r$ and $x$, can be found such that when $T_2 \geqq c_9 T_1$:*

$$P^x[X(t) \in S_r \text{ for some } t \in [T_1, T_2]] \geqq c_{10}\, C(S_r)\, T_1^{1-\lambda}.$$

*Proof.* Choose $c_9 > (2c_3/c_4)^{1/(\lambda-1)} > 1$. Let $T_2 \geqq c_9 T_1$. The probability of interest is at least:

$$Q(x, r, T_1) - Q(x, r, T_2) \geqq Q(x, r, T_1) - Q(x, r, c_9 T_1) \geqq C(S_r)\,[c_4 T_1^{1-\lambda} - c_3 (c_9 T_1)^{1-\lambda}],$$

from which the desired results follows.

The next step in the proof of the theorem is to make use of the fundamental lemma along the lines of Borel-Cantelli [11]:

**Lemma 4.3.** *Let $\{Q_n, n = 1, 2, 3, \ldots\}$ be a sequence of events of a common probability space. Suppose that* (i), (ii) *and* (iii) *are true.*

(i) $\sum_n P[Q_n] = \infty$.

(ii) $P[Q_n \text{ i.o.}] = 0$ *or* 1.

(iii) *There exist two positive constants $c_{11}$ and $c_{12}$ with the property that to each integer $j$ there corresponds a finite subset $S_j$ of positive integers such that:*

$$\sum_{i \in S_j} P[Q_j \cap Q_i] \leqq c_{11} P[Q_j] \quad \text{and} \quad P[Q_j \cap Q_k] \leqq c_{12} P[Q_j] \cdot P[Q_k] \quad \text{when } k \notin S_j, \quad k > j.$$

*Then* $P[Q_n \text{ i.o.}] = 1$.

We now apply Lemma 4.3 to events $Q_n$ defined by:

$$Q_n = \left[|X(t)| \leqq h(c_9^n) \text{ for some } t \text{ in } [c_9^n, c_9^{n+1}]\right],$$

where $c_9 > 1$ is the constant which appears in Lemma 4.2. Once the three conditions of the lemma are verified for any $h$ whose integral diverges, we will have shown that such $h$ are in $U_\infty$ and the proof of the theorem will be complete.

To verify (i) note that by Lemma 4.1 $c_9^n \geqq [2h(c_9^n)]^{\alpha_1}$, so that we can use Lemmas 4.1 and 3.1 to obtain (for some $c > 0$ independent of $n$):

$$P[Q_n] \geqq c_{10} c_1 \frac{\{h(c_9^n)\}^{\alpha_1[\lambda-1]}}{c_9^{n[\lambda-1]}} = \frac{c[c_9^n - c_9^{n-1}]\{h(c_9^n)\}^{\alpha_1[\lambda-1]}}{\{c_9^{n-1}\}^\lambda} \geqq c \int_{c_9^{n-1}}^{c_9^n} \frac{\{h(t)\}^{\alpha_1[\lambda-1]}}{t^\lambda} dt.$$

Thus, $\sum_n P[Q_n] = \infty$.

For (iii) we first show that if $k > j$,

$$P[Q_j \cap Q_k] \leqq P[|X(t)| \leqq h(c_9^j) + h(c_9^k) \text{ for some } t \geqq c_9^k - c_9^{j+1}] \cdot P[Q_j]. \tag{4.2}$$

This is done by defining r.v.'s $\sigma_n(\omega)$:

$$\sigma_n(\omega) = \begin{cases} \inf\{t \geqq c_9^n \colon |X(t, \omega)| \leqq h(c_9^n) \text{ for some } t \leqq c_9^{n+1}\} \\ c_9^{n+1} + 1 \text{ otherwise.} \end{cases}$$

Then

$$\begin{aligned} P[Q_j \cap Q_k] &= P[\sigma_j \leqq c_9^{j+1}; \sigma_k \leqq c_9^{k+1}] \\ &= \int_{c_9^j}^{c_9^{j+1}} P\left[|X(t)| \leqq h(c_9^k) \text{ for some } t \in [c_9^k, c_9^{k+1}] | \sigma_j = s\right] \cdot P[\sigma_j \in ds] \\ &\leqq P[|X(t)| \leqq h(c_9^k) + h(c_9^j) \text{ for some } t \geqq c_9^k - c_9^{j+1}] \cdot P[\sigma_j \leqq c_9^{j+1}]. \end{aligned}$$

The final inequality, justified by the strong Markov property, establishes (4.2).

Letting $k > j+1$ and using Lemma 3.2, we obtain an estimate on one of the factors in (4.2):

$$\begin{aligned} &P[|X(t)| \leqq h(c_9^j) + h(c_9^k) \text{ for some } t \geqq c_9^k - c_9^{j+1}] \\ &\quad \leqq Q\left(0, 2h(c_9^k), b\, c_9^k\right) \quad \text{for some } b > 0 \text{ independent of } k \text{ and } j \\ &\quad \leqq c_2 c_3 \frac{\{2h(c_9^k)\}^{\alpha_1[\lambda-1]}}{(b\, c_9^k)^{\lambda-1}} \leqq c_{12} P[Q_k] \quad \text{by Lemma 4.2.} \end{aligned}$$

Use of this result and (4.2) gives (when $k > j+1$):

$$P[Q_j \cap Q_k] \leqq c_{12} P[Q_j] \cdot P[Q_k],$$

where $c_{12}>0$ is independent of $k$ and $j$. Now we take $S_j=\{j+1\}$ and $c_{11}=1$ to establish (iii).

(ii) of the fundamental lemma is difficult to verify directly, so we define $Q_n^* \subset Q_n$, $n=1, 2, 3, \ldots$, by:

$$Q_n^* = [|X(j)| \leqq h(c_9^n) \text{ for some integer } j \text{ in } [c_9^n, c_9^{n+1}]].$$

(ii) can easily be verified for the events $Q_n^*$ using the Hewitt-Savage zero-one law [4] and the fact that the process has independent increments. We omit the details. Suppose that we could show that, for all large $n$, $P[Q_n] \leqq c\, P[Q_n^*]$ for some $c>0$ which does not depend upon $n$. Then (i) and (iii) of the fundamental lemma would hold for the $Q_n^*$ since they hold for the $Q_n$, and we could conclude that $P[Q_n^* \text{ i.o.}]=1$. Thus we would obtain $P[Q_n \text{ i.o.}] \geqq P[Q_n^* \text{ i.o.}]=1$ and the proof of the theorem would be complete.

We now show that $P[Q_n] \leqq c\, P[Q_n^*]$ for all large $n$ and some $c>0$ independent of $n$. First observe that

$$P[Q_n^*] \geqq P[|X(t)| \leqq h(c_9^n)/2 \text{ for some } t \in [c_9^n, c_9^{n+1}-1]] \cdot a$$

where $a=P[M(1) \leqq \frac{1}{2}]>0$ and

$$M(t) \equiv \sup_{0 \leqq s \leqq t} |X(s)|.$$

This follows from the fact that if

$$|X(t, \omega)| \leqq h(c_9^n)/2 \qquad \text{for some } \; t=\sigma(\omega) \text{ in } [c_9^n, c_9^{n+1}-1]$$

and if

$$\sup_{\sigma \leqq t \leqq \sigma+1} |X(t, \omega) - X(\sigma, \omega)| \leqq \tfrac{1}{2},$$

then

$$|X(j, \omega)| \leqq 2^{-1}[1+h(c_9^n)] \leqq h(c_9^n) \qquad \text{for some integer } j \text{ in } [c_9^n, c_9^{n+1}].$$

Estimate the first factor, $a_n$, on the right in the above estimate of $P[Q_n^*]$ by Lemmas 4.2 and 3.2(i) to obtain (for all large $n$):

$$\begin{aligned} a_n \geqq c_{10}\, c_1 \frac{\{h(c_9^n)/2\}^{\alpha_1[\lambda-1]}}{c_9^{n[\lambda-1]}} &\geqq b\, Q(0, h(c_9^n), c_9^n) \\ &\geqq b\, P[|X(t)| \leqq h(c_9^n) \text{ for some } t \in [c_9^n, c_9^{n+1}]]. \end{aligned}$$

Therefore $P[Q_n^*] \geqq a\, b\, P[Q_n] = c\, P[Q_n]$ for all large $n$, where $c=a\,b>0$ does not depend upon $n$. This completes the proof.

Before generalizing the above theorem, we state and prove a lemma whose proof depends upon Khinchine's [6] results about upper envelopes near infinity in the stable case.

**Lemma 4.4.** *Let $X_{\alpha,n}(t)$ be a stable process of index $\alpha \leqq 2$ in $R^n$, and let $0<\theta<\alpha$. Then:*

$$P[|X_{\alpha,n}(t)| \leqq t^{1/\theta} \textit{ for all large } t]=1.$$

*Proof.* For $\alpha$ in (0, 2), Khinchine's results imply that the event

$$[|X_{\alpha,1}(t)| \leqq h(t) \text{ for all large } t]$$

has probability 1 or 0 according to whether the integral

$$\int^{\infty} \{h(t)\}^{-\alpha}\, dt$$

converges or diverges, where $h(t)$ is defined for all large $t$ and $t^{-1/\alpha} h(t)$ increases monotonely to infinity as $t$ approaches infinity. Thus:

$$P[|X_{\alpha,1}(t)| \leqq t^{1/\theta} \text{ for all large } t] = 1.$$

Applying this to each of the $n$ projections of $X_{\alpha,n}(t)$ onto the coordinate axes and using the triangle inequality, we obtain the desired result. If $\alpha = 2$, the conclusion follows from the law of the iterated logarithm.

We now obtain the desired generalized integral test near infinity in

**Corollary 4.1.** *Let $X(t)$ be defined as in* (1.1). *Let $h(t)$ be as defined in Theorem* 4.1, *and $L_\infty$ and $U_\infty$ be the lower and upper classes (near $\infty$) respectively with respect to $X(t)$. Then:*

(i) *If $\alpha_n < d_n$, $h$ is in $L_\infty$ or $U_\infty$ according to whether the integral*

$$\int^{\infty} \frac{\{h(t)\}^{d_n - \alpha_n}}{t^{d_n/\alpha_n}}\, dt$$

*converges or diverges.*

(ii) *If $\alpha_n > d_n$, $h$ is in $L_\infty$ or $U_\infty$ according to whether the integral*

$$\int^{\infty} \frac{\{h(t)\}^{\alpha_{n-1}[\lambda-1]}}{t^{\lambda}}\, dt$$

*converges or diverges, when $\lambda = d_n \alpha_n^{-1} + d_{n-1} \alpha_{n-1}^{-1}$.*

*Proof of* (i). Suppose that the integral, $I_h$, converges. Then, according to Theorem 2, Taylor [13], with probability 1

$$h(t) < |X_{\alpha_n, d_n}(t)| \equiv |X_n(t)| \leqq |X(t)|$$

for all large $t$, so that $h \in L_\infty$.

Conversely, let $I_h = \infty$. Again by Taylor [13]:

$$P[|X_n(t)| \leqq h(t)/n \text{ i.o. as } t \to \infty] = 1. \tag{4.3}$$

Choose $\theta$ so that $\alpha_n < \theta < \alpha_{n-1}$. Then $\theta^{-1} = \alpha_n^{-1} - \varepsilon$ for some $\varepsilon > 0$ and:

$$\int^{\infty} \frac{\{t^{1/\theta}\}^{d_n - \alpha_n}}{t^{d_n/\alpha_n}}\, dt = \int^{\infty} t^{-1-\varepsilon[d_n - \alpha_n]}\, dt < \infty.$$

By Taylor's theorem we therefore have:

$$t^{1/\theta} < |X_n(t)| \qquad \text{for all large } t \text{ with probability } 1.$$

According to Lemma 4.4 for $1 \leqq i \leqq n-1$:

$$|X_i(t)| \leqq t^{1/\theta} \qquad \text{for all large } t \text{ with probability } 1.$$

Therefore, for all large $t$ with probability 1, we have:

$$|X(t)| \leqq |X_1(t)| + \cdots + |X_n(t)| \leqq (n-1)\, t^{1/\theta} + |X_n(t)| \leqq n\, |X_n(t)|.$$

We use this in (4.3) to complete the proof of (i).

*Proof of* (ii). Denote the integral in (ii) by $I_h$, and first suppose that $I_h < \infty$. Then:

$$h(t) < |(X_{n-1}(t), X_n(t))| \leqq |X(t)|$$

with probability 1 for all large $t$ by Theorem 4.1.

Conversely, let $I_h = \infty$. Again by Theorem 4.1:

$$P\left[|(X_{n-1}(t), X_n(t))| \leqq h(t)/(n-1) \text{ i.o. as } t \to \infty\right] = 1. \tag{4.4}$$

Choose $\theta$ so that $\alpha_{n-1} < \theta < \alpha_{n-2}$. Then $\theta^{-1} = \alpha_{n-1}^{-1} - \varepsilon$ for some $\varepsilon > 0$ and:

$$\int^{\infty} \frac{\{t^{1/\theta}\}^{\alpha_{n-1}[\lambda-1]}}{t^{\lambda}}\, dt = \int^{\infty} t^{-1-\varepsilon[\lambda-1]}\, dt < \infty.$$

By Theorem 4.1 we therefore have:

$$t^{1/\theta} < |(X_{n-1}(t), X_n(t))| \qquad \text{for all large } t \text{ with probability } 1.$$

According to Lemma 4.4, for $1 \leqq i \leqq n-2$:

$$|X_i(t)| \leqq t^{1/\theta} \qquad \text{for all large } t \text{ with probability } 1.$$

Therefore for all large $t$ with probability 1 we have:

$$\begin{aligned} |X(t)| &\leqq |X_1(t)| + \cdots + |X_{n-2}(t)| + |(X_{n-1}(t), X_n(t))| \\ &\leqq (n-2)\, t^{1/\theta} + |(X_{n-1}(t), X_n(t))| < (n-1)\, |(X_{n-1}(t), X_n(t))|. \end{aligned}$$

We use this in (4.4) to complete the proof of (ii).

We turn now to the situation when $t$ is near zero. Since $X(0, \omega) \equiv 0$, we assume $h(t)$ to be a non-decreasing function which vanishes at 0 and is less than 1 for small $t$. Blumenthal's [1] zero-one law implies that the event

$$[|X(t)| \leqq h(t) \text{ i.o. as } t \to 0]$$

has probability zero or one. By this we shall always mean that with probability zero or one, there is a sequence $\{t_j(\omega)\}_{j=1}^{\infty}$ of times which decrease monotonely to zero and for which

$$|X(t_j(\omega), \omega)| \leqq h(t_j(\omega))$$

for all $j$.

The function $h(t)$ will be said to be in the lower class, $L_0$, with respect to the process $X(t)$ if the above event has probability zero and in the upper class, $U_0$, if the event has probability one. The properties mentioned immediately after the definition of $L_\infty$ and $U_\infty$ now hold for small $t$.

Our first integral test near zero is given in Theorem 4.2, where we consider processes of the type defined in Lemmas 3.3 and 3.4. As a corollary, we then obtain the desired integral test near zero for $X(t)$ as defined in (1.1).

**Theorem 4.2.** *Let $X(t)$ be defined as in Lemma 3.3, and let $L_0$ and $U_0$ be as above. Then a positive non-decreasing function $h(t)$ as defined above belongs to $L_0$ or $U_0$ according to whether the integral*

$$I_h \equiv \int_0 \frac{\{h(t)\}^{\alpha_2[\lambda-1]}}{t^\lambda} dt$$

*converges or diverges, where $\lambda = d_1 \alpha_1^{-1} + d_2 \alpha_2^{-1}$.*

*Proof.* The proof closely parallels that of Theorem 4.1 and is sketched below. First suppose that $I_h < \infty$ and for $n = 1, 2, 3, \ldots$ define events $E_n$:

$$E_n = [|X(t)| \leqq h(2^{-n}) \text{ for some } t \text{ in } [2^{-n-1}, 2^{-n}]].$$

$$P[E_n] \leqq Q(0, h(2^{-n}), 2^{-n-1}) \leqq c \int_{2^{-n}}^{2^{-n+1}} \frac{\{h(t)\}^{\alpha_2[\lambda-1]}}{t^\lambda} dt.$$

Therefore, $\sum_n P[E_n] < \infty$ and $h$ is in $L_0$.

Conversely, let $I_h = \infty$. We first obtain

**Lemma 4.5.** *Without loss of generality we may assume that $h(t) \leqq t^{1/\alpha_2}$ when $X(t)$ is as above.*

*Proof.* Define $h^*(t) = \min\{h(t), t^{1/\alpha_2}\}$ and proceed as in the proof of Lemma 4.1.

The counterpart of Lemma 4.2 is given below in Lemma 4.6; the proof, which we omit, uses both parts of Lemma 3.4.

**Lemma 4.6.** *Let $X(t)$ and $S_r$ be as in Lemma 3.3; also let $T_1 \geqq (2r)^{\alpha_2}$ and $x \in S_r$. Then positive constants $c_{13} > 1$ and $c_{14}$, independent or $r$ and $x$, can be found such that when $T_2 \geqq c_{13} T_1$:*

$$P^x[X(t) \in S_r \text{ for some } t \text{ in } [T_1, T_2]] \geqq c_{14}\, C(S_r)\, T_1^{1-\lambda}.$$

Now define events $Q_n$ by:

$$Q_n = [|X(t)| \leqq h(c_{13}^{-n}) \text{ for some } t \text{ in } [c_{13}^{-n}, c_{13}^{-n+1}]],$$

and proceed as before using Lemmas 3.3, 4.5 and 4.6 to verify (i) of the fundamental lemma, while (ii) follows easily from the Blumenthal zero-one law [1].

For (iii) of the fundamental lemma we first show that if $k > j$,

$$P[Q_j \cap Q_k] \leqq P[|X(t)| \leqq h(c_{13}^{-j}) + h(c_{13}^{-k}) \text{ for some } t \geqq c_{13}^{-j} - c_{13}^{-k+1}] \cdot P[Q_k]. \quad (4.5)$$

Letting $k > j+1$ and using Lemmas 3.3 and 3.4 we have (for some positive constants $b$ and $c$ independent of $j$ and $k$):

$$P[|X(t)| \leqq h(c_{13}^{-j}) + h(c_{13}^{-k}) \text{ for some } t \geqq c_{13}^{-j} - c_{13}^{-k+1}]$$
$$\leqq Q(0, 2h(c_{13}^{-j}), b\, c_{13}^{-j}) \leqq c_6\, c_7 \frac{\{2h(c_{13}^{-j})\}^{\alpha_2[\lambda-1]}}{(b\, c_{13}^{-j})^{\lambda-1}} \leqq c\, P[Q_j],$$

where the final inequality is justified by Lemma 4.6. Use of this estimate in (4.5) and the choice $S_j = \{j+1\}$ suffices to prove (iii) of Lemma 4.3, so that the proof of the theorem is complete.

Before generalizing Theorem 4.2 we prove a lemma whose proof depends upon Khinchine's [6] results about upper envelopes near zero in the stable case.

**Lemma 4.7.** *Let $X_{\alpha,n}(t)$ be a stable process of index $\alpha<2$ in $R^n$, and let $\theta>\alpha$. Then*

$$P[|X_{\alpha,n}(t)|\leqq t^{1/\theta} \text{ for all small } t]=1.$$

*Proof.* For $\alpha$ in (0, 2), Khinchine's results imply that the event

$$[|X_{\alpha,1}(t)|\leqq h(t) \text{ for all small } t]$$

has probability 1 or 0 according to whether the integral $\int_0 h(t)^{-\alpha}\,dt$ converges or diverges, where $h(t)$ is defined for all small positive $t$, $h(t)$ decreases monotonely to 0 as $t$ approaches zero, and $t^{-1/\alpha}h(t)$ approaches infinity as $t$ approaches zero. Reasoning as in Lemma 4.4 completes the proof when $\alpha\in(0,2)$.

The following corollary gives the generalized integral test near zero.

**Corollary 4.2.** *Let $X(t)$ be defined as in* (1.1). *Let $h(t)$ be defined as in Theorem* 4.2, *and $L_0$ and $U_0$ the lower and upper classes (near* 0*) respectively with respect to $X(t)$. Then:*

(i) *If $\alpha_1<d_1$, $h$ is in $L_0$ or $U_0$ according to whether the integral*

$$\int_0 \frac{\{h(t)\}^{\alpha_1-d_1}}{t^{d_1/\alpha_1}}\,dt$$

*converges or diverges.*

(ii) *If $\alpha_1>d_1$, $h$ is in $L_0$ or $U_0$ according to whether the integral*

$$\int_0 \frac{\{h(t)\}^{\alpha_2[\lambda-1]}}{t^{\lambda}}\,dt$$

*converges or diverges, where $\lambda=d_1\alpha_1^{-1}+d_2\alpha_2^{-1}$.*

*Proof of* (i). Using Takeuchi's [11] method near zero, Theorem 2 of Taylor [13] can be proven for lower envelopes near zero with respect to type $A$ stable processes $X_{\alpha,n}(t)$ of stable index $\alpha<n$ in $R^n$ (provided that the scaling property holds if $\alpha=1$). Therefore, if we assume that the integral in (i) converges, with probability 1

$$h(t)<|X_{\alpha_1,d_1}(t)|\equiv|X_1(t)|\leqq|X(t)|$$

for all small $t$, so that $h\in L_0$.

To prove the converse let the integral in (i) diverge and use Taylor's theorem near zero to obtain

$$P[|X_1(t)|\leqq h(t)/n \text{ i.o. as } t\to 0]=1. \tag{4.6}$$

Choose $\theta$ so that $\alpha_2<\theta<\alpha_1$. Then $\theta^{-1}=\alpha_1^{-1}+\varepsilon$ for some $\varepsilon>0$ and:

$$\int_0 \frac{\{t^{1/\theta}\}^{d_1-\alpha_1}}{t^{d_1/\alpha_1}}\,dt=\int_0 t^{-1+\varepsilon[d_1-\alpha_1]}\,dt<\infty.$$

By Taylor's theorem we therefore have:

$$t^{1/\theta}<|X_1(t)| \quad \text{for all small } t \text{ with probability 1.}$$

According to Lemma 4.7, for $2 \leqq i \leqq n$:

$$|X_i(t)| \leqq t^{1/\theta} \quad \text{for all small } t \text{ with probability } 1.$$

Therefore for all small $t$ with probability 1 we have:

$$|X(t)| \leqq |X_1(t)| + \cdots + |X_n(t)| \leqq |X_1(t)| + (n-1)\, t^{1/\theta} < n\, |X_1(t)|.$$

Use of this in (4.6) completes the proof of (i).

*Proof of* (ii). Denote the integral in (ii) by $I_h$ and first suppose that $I_h < \infty$. Then by Theorem 4.2:

$$h(t) < |(X_1(t), X_2(t))| \leqq |X(t)|$$

with probability 1 for all small $t$.

Conversely, let $I_h = \infty$. Again by Theorem 4.2:

$$P\big[|(X_1(t), X_2(t))| \leqq h(t)/(n-1) \text{ i.o. as } t \to 0\big] = 1. \tag{4.7}$$

Choose $\theta$ so that $\alpha_3 < \theta < \alpha_2$. Then $\theta^{-1} = \alpha_2^{-1} + \varepsilon$ for some $\varepsilon > 0$ and:

$$\int_0 \frac{\{t^{1/\theta}\}^{\alpha_2[\lambda-1]}}{t^\lambda} = \int_0 t^{-1+\varepsilon[\lambda-1]}\, dt < \infty.$$

By Theorem 4.2 we therefore have:

$$t^{1/\theta} < |(X_1(t), X_2(t))| \quad \text{for all small } t \text{ with probability } 1.$$

According to Lemma 4.7, for $3 \leqq i \leqq n$:

$$|X_i(t)| \leqq t^{1/\theta} \quad \text{for all small } t \text{ with probability } 1.$$

Therefore for all small $t$ with probability 1, we have:

$$\begin{aligned} |X(t)| &\leqq |(X_1(t), X_2(t))| + |X_3(t)| + \cdots + |X_n(t)| \\ &\leqq |(X_1(t), X_2(t))| + (n-2)\, t^{1/\theta} < (n-1)\, |(X_1(t), X_2(t))|. \end{aligned}$$

Use of this in (4.7) completes the proof of (ii).

*Remark* 4.1. In the above proofs we have not actually needed to assume that all of the components of $X(t)$ as defined by (1.1) are type $A$. It suffices to know that those components whose indices appear in a given integral test are type $A$. Thus in Corollary 4.1, if $\alpha_n < d_n$, we need only assume $X_n(t)$ is type $A$; and if $\alpha_n > d_n$, we need only assume $X_{n-1}(t)$ and $X_n(t)$ are type $A$; similarly for Corollary 4.2.

*Remark* 4.2. For $h(t)$ of the type indicated in the proof of Lemma 4.4 and $X(t)$ as defined by (1.1), the event

$$[|X(t)| \leqq h(t) \text{ for all large } t]$$

has probability 1 or 0 according to whether the integral

$$\int^\infty \{h(t)\}^{-\alpha_n}\, dt$$

converges or diverges.

*Proof.* If the integral converges, so does $\int^{\infty} h(t)^{-\alpha_j}\,dt$ for $1 \leqq j \leqq (n-1)$. Thus $|X_j(t)| \leqq h(t)/n$ for all large $t$ with probability one by Khinchine's theorem for $1 \leqq j \leqq n$. By the triangle inequality $|X(t)| \leqq h(t)$ for all large $t$ with probability one. If the integral diverges, the event

$$[|X_n(t)| \leqq h(t) \text{ for all large } t]$$

has probability zero; since this event contains the event of interest, the proof is complete.

*Remark* 4.3. For $h(t)$ of the type indicated in the proof of Lemma 4.7 and $X(t)$ as defined by (1.1), the event

$$[|X(t)| \leqq h(t) \text{ for all small } t]$$

has probability 1 or 0 according to whether the integral

$$\int_0 \{h(t)\}^{-\alpha_1}\,dt$$

converges or diverges.

*Proof.* The proof is analogous to that of Remark 4.2.

*Remark* 4.4. In (ii) of Corollary 4.1, the critical power of $t$ in passing from convergence to divergence of the integral is $t^{1/\alpha_{n-1}}$, and in Remark 4.2 the critical power is $t^{1/\alpha_n}$. Therefore, for any $\varepsilon > 0$ it can be shown that if $\alpha_n > d_n$ for $X(t)$ as in (1.1):

$$P[t^{-\varepsilon+1/\alpha_{n-1}} < |X(t)| \leqq t^{\varepsilon+1/\alpha_n} \text{ for all large } t] = 1.$$

In the stable case (type $A$) of index $\alpha < d$ in $R^d$ it can be shown that:

$$P[t^{-\varepsilon+1/\alpha} < |X_{\alpha,d}(t)| \leqq t^{\varepsilon+1/\alpha} \text{ for all large } t] = 1,$$

so that there is less room for fluctuation of $|X_{\alpha,d}(t)|$ than for $|X(t)|$ (when $\alpha_n > d_n$) as $t$ approaches infinity. Similar observations using (ii) of Corollary 4.2 and Remark 4.3 can be made near zero.

*Remark* 4.5. We have been unable to establish complete results if $\alpha_n = d_n = 1$ in Corollary 4.1 or if $\alpha_1 = d_1 = 1$ or 2 in Corollary 4.2.

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