Z. Wahrscheinlichkeitstheorie verw. Geb. 21, 77–80 (1972)

On the Converse of Hopf's Ergodic Theorem

G. HELMBERG

Let $(E, \mathscr{F}, \pi)$ be a probability space and let T be a sub-Markovian endomorphism of $\mathscr{L}_1(E, \mathscr{F}, \pi)$ with conservative part C and dissipative part D (cf. [1] V.4 and V.5; we shall also further adhere to the terminology of [1]). The purpose of this note is to show the following theorem:

Theorem 1. *The following two conditions are equivalent:*

a) $\lim_{k \to \infty} T^k 1_D = 0$ *and there exists a function* $f \in \mathscr{L}_1^+$ *satisfying* $fT = f$ *and* $\{f > 0\} = C$.

b) *For every* $h \in \mathscr{L}_\infty$ *the sequence*

$$\left\{\frac{1}{n+1} \sum_{k \leqq n} T^k h\right\}$$

converges a.s. on E.

In order to avoid confusion it should be pointed out that this theorem stems from a correction of problem V.6.6 in [1] which should read as follows:

Theorem 1′. *The following two conditions are equivalent:*

a) *There exists a function* $f \in \mathscr{L}_1^+$ *satisfying* $fT = f$ *and* $\{f > 0\} = C$.

b) *For every* $h \in \mathscr{L}_\infty$ *the sequence*

$$\left\{\frac{1}{n+1} \sum_{k \leqq n} T^k h\right\}$$

converges a.s. on C.

We shall further rely on Theorem 1′, the proof of which proceeds exactly along the lines indicated by the hints given in [1], and we shall use the following lemma:

Lemma. *Suppose* $\|h_n\|_\infty \leqq c$ *for all* $n \geqq 1$ *and*

$$\lim_{n \to \infty} \text{a.s. } h_n = h.$$

Then

$$\lim_{n \to \infty} \text{a.s. } T h_n = T h.$$

Proof. Let $\underline{h}_n = \inf_{m \geqq n} h_m$ and $\bar{h}_n = \sup_{m \geqq n} h_m$. Then $\underline{h}_n \in \mathscr{L}_\infty$, $\bar{h}_n \in \mathscr{L}_\infty$ and

$$\underline{h}_n \leqq h_m \leqq \bar{h}_n \quad \text{for } m \geqq n,$$

$$\lim_{n \to \infty} \uparrow \underline{h}_n = h = \lim_{n \to \infty} \downarrow \bar{h}_n.$$

We conclude

$$\lim_{n\to\infty}\uparrow T\underline{h}_n = T\lim_{n\to\infty}\uparrow \underline{h}_n = Th = T\lim_{n\to\infty}\downarrow \bar{h}_n = \lim_{n\to\infty}\downarrow T\bar{h}_n,$$

$$T\underline{h}_n \leqq Th_m \leqq T\bar{h}_n \quad \text{for } m \geqq n,$$

$$T\underline{h}_n \leqq \inf_{m\geqq n} Th_m \leqq \sup_{m\geqq n} Th_m \leqq T\bar{h}_n,$$

$$\lim_{n\to\infty} \text{a.s. } T\underline{h}_n \leqq \lim_{n\to\infty} \inf_{m\geqq n} Th_m \leqq \lim_{n\to\infty} \sup_{m\geqq n} Th_m \leqq \lim_{n\to\infty} \text{a.s. } T\bar{h}_n,$$

$$Th = \lim_{m\to\infty} \text{a.s. } Th_m.$$

Proof of Theorem 1. We may assume $\pi(C)>0$. Furthermore we note that $1_C(T1_D)=(1_C T)\,1_D\equiv 0$ (cf. Corollary 1 of Proposition V. 5.2 [1]). Thus we have $1_D \geqq T1_D \geqq T^2 1_D \geqq \cdots$ and therefore in general there exists the non-negative limit

$$\lim_{k\to\infty} \text{a.s. } T^k 1_D = \lim_{n\to\infty} \text{a.s. } \frac{1}{n+1} \sum_{k=0}^{n} T^k 1_D.$$

a) $\Rightarrow$ b) Let $h \in \mathscr{L}_\infty^+(E, \mathscr{F}, \pi)$. By the implication a) $\Rightarrow$ b) of Theorem 1′, there exists the limit

$$h' = \lim_{n\to\infty} \text{a.s. } 1_C \frac{1}{n+1} \sum_{k\leqq n} T^k h. \tag{$*$}$$

Starting from the decomposition $T = TI_D + TI_C$ we show by induction that

$$T^k h = \sum_{l=0}^{k} (TI_D)^{k-l} (TI_C)^l h$$

and therefore

$$\begin{aligned}
\sum_{k=0}^{n} T^k h &= \sum_{k=0}^{n} \sum_{l=0}^{k} (TI_D)^{k-l}(TI_C)^l h \\
&= \sum_{l=0}^{n} \sum_{m=0}^{n-l} (TI_D)^m (TI_C)^l h \\
&= \sum_{m=0}^{n} (TI_D)^m h + \sum_{l=1}^{n} \sum_{m=0}^{n-l} (TI_D)^m (TI_C)(TI_C)^{l-1} h \\
&= \sum_{m=0}^{n} (TI_D)^m h + \sum_{l=1}^{n} {}^C H (TI_C)^{l-1} h - \sum_{l=1}^{n} \sum_{m=n-l+1}^{\infty} (TI_D)^m (TI_C)(TI_C)^{l-1} h.
\end{aligned}$$

We now consider separately each of the terms on the right-hand side.

$$\frac{1}{n+1} \sum_{m=0}^{n} (TI_D)^m h \leqq \frac{\|h\|_\infty}{n+1} \left(1 + \sum_{m=1}^{n} T^m 1_D\right). \tag{1}$$

For $n\to\infty$ this term vanishes a.s. on E by the remark preceding this proof.

$$\frac{1}{n+1} \sum_{l=1}^{n} {}^C H (TI_C)^{l-1} h = {}^C H \left(\frac{n}{n+1} \frac{1_C}{n} \sum_{l=0}^{n-1} (TI_C)^l h\right). \tag{2}$$

For $n\to\infty$ the expression between parentheses converges a.s. on E to h' by $(*)$ (cf. also Corollary 1 of Proposition V.5.2 [1]). Furthermore it is absolutely bounded by $\|h\|_\infty$. By the lemma this term converges a.s. on E to ${}^C H\,h'$.

Note that outside of a suitably chosen fixed π-null-set we have

$$\varepsilon_m(x)=\sum_{k=m+1}^{\infty}(TI_D)^k\,T1_C(x)\downarrow 0 \qquad \text{for } m\to\infty,$$

$$\sum_{k=m+1}^{\infty}(TI_D)^k(TI_C)(TI_C)^{l-1}h(x)\leqq\|h\|_\infty\sum_{k=m+1}^{\infty}(TI_D)^k\,T1_C(x)=\|h\|_\infty\,\varepsilon_m(x),$$

$$\begin{aligned}&\frac{1}{n+1}\sum_{l=1}^{n}\left[\sum_{k=n-l+1}^{\infty}(TI_D)^k(TI_C)^l h\right](x)\\&\leqq\frac{1}{n+1}\sum_{l=n-m+1}^{n}\|h\|_\infty+\|h_\infty\|\frac{1}{n+1}\sum_{l=1}^{n-m}\left[\sum_{k=n-l+1}^{\infty}(TI_D)^k\,T1_C\right](x)\\&\leqq\frac{m}{n+1}\|h\|_\infty+\frac{n-m}{n+1}\|h\|_\infty\,\varepsilon_m(x)\\&\leqq 2\,\|h\|_\infty\,\varepsilon_m(x)\qquad\text{if } n\geqq n(x),\end{aligned}$$

and therefore

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{l=1}^{n}\left[\sum_{k=n-l+1}^{\infty}(TI_D)^k(TI_C)^l h\right](x)=0. \tag{3}$$

From (1), (2), and (3) we conclude

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}T^k h={}^C H\,h'.$$

b) $\Rightarrow$ a) Define

$$f_n=\frac{1}{n+1}\sum_{k\leqq n}1\,T^k.$$

Precisely as in the proof of Theorem 1′ it is shown that the sequence $\{f_n\}_{n=1}^{\infty}$ converges (in the topology $\sigma(\mathscr{L}_1,\mathscr{L}_\infty)$) to a function $f\in\mathscr{L}_1$ which satisfies $fT=f$ and $\{x: f(x)>0\}=C$. The last assertion of a) follows from

$$\begin{aligned}0=\int_D f=\lim_{n\to\infty}\int_D f_n&=\lim_{n\to\infty}\int\frac{1}{n+1}\sum_{k\leqq n}T^k 1_D\\&=\int\lim_{n\to\infty}\frac{1}{n+1}\sum_{k\leqq n}T^k 1_D=\int\lim_{k\to\infty}T^k 1_D\end{aligned}$$

by the remark preceding this proof.

The limit h' in $(*)$ may be identified by applying Hopf's ergodic theorem ([1] Proposition V.6.3) to the Markov operator T' induced by T in C ([1] Proposition V.5.2, Corollary 1). Denoting by $\mathscr{C}$ the σ-algebra of invariant sets in C and taking account of the last line in the proof of the implication a) $\Rightarrow$ b) we obtain:

Corollary. *If any of the equivalent assertions* a) *or* b) *of Theorem* 1 *holds, then for any h*

$$\lim_{n\to\infty} \text{a.s.} \frac{1}{n+1} \sum_{k\leq n} T^k h = {}^C H \frac{E^{\mathscr{C}}(1_C f h)}{E^{\mathscr{C}}(1_C f)} = \sum_{k=0}^{\infty} (T 1_D)^k T \frac{E^{\mathscr{C}}(1_C f h)}{E^{\mathscr{C}}(1_C f)}.$$

The following example shows that the equivalent assertion a) and b) of Theorem 1′ do not imply the stronger assertions a) and b) of Theorem 1. Let $(E, \mathscr{F})$ be the space of integers in which all singleton sets are measurable and let the probability π be chosen equivalent to counting measure on $(E, \mathscr{F})$ (e.g. $\pi(x) = (3.2^{|x|})^{-1}$). Let τ be the left shift transformation in E given by $\tau x = x - 1$ and define the Markov operator T on $\mathscr{L}_\infty(E, \mathscr{F}, \pi)$ by

$$Th(x) = h(\tau x) \quad \text{for all } x \in E.$$

Then C is empty and condition b) of Theorem 1′ is trivially satisfied. On the other hand, it is easy to construct a function h assuming only the values 0 and 1 for which

$$\frac{1}{n+1} \sum_{k=0}^{n} T^k h(x)$$

oscillates between zero and 1 for all $x \in E$. Thus assertion b) of Theorem 1 fails. Indeed, we also have $\lim_{k\to\infty} T^k 1_D = 1 \neq 0$ on E in this example.

Reference

1. Neveu, J.: Mathematical foundations of the calculus of probability. San Francisco: Holden-Day 1965.

G. Helmberg
Technische Fakultät der Universität
Technikerstraße 13
A-6020 Innsbruck
Austria

(Received December 3, 1970)