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An Invariance Principle for Conditioned Recurrent Random Walk Attracted to a Stable Law*

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1. Introduction

In this paper we will be concerned with one-dimensional, aperiodic, recurrent, lattice random walk. We let $S_n = x_0 + \sum_{i=1}^{n} X_i$ denote the n-th partial sum and assume that the common distribution of the increments X_i lies in the domain of attraction of a stable law. Thus there exists normalizing constants $B_n > 0$ and A_n and a stable distribution G_α (α is the exponent or index of the stable law) such that

$$\lim_{n\to\infty} P\left[\frac{S_n}{B_n} - A_n \leqq x\right] = G_\alpha(x). \tag{1.1}$$

Let $\mathbf{A} = \{x_1, x_2, \ldots, x_M\}$ be a finite set of integers and let $T_\mathbf{A}$ denote the hitting time of $\mathbf{A}$, i.e.,

$$T_\mathbf{A} = \begin{cases} \min\{n>0\colon S_n \in \mathbf{A}\} & \text{if such an } n \text{ exists} \\ \infty & \text{otherwise.} \end{cases}$$

In [1] the author considered the effect of the conditioning $T_\mathbf{A} > n$ on the limiting behavior of the distribution of S_n. In particular, the following result was proved.

Theorem (1.1). *If* $\tilde{g}_\mathbf{A}(0) \neq 0$ *and*

(i) $1 < \alpha \leqq 2$

or

(ii) $\alpha = 1$, *the attraction is normal* ($B_n = n^{1/\alpha}$), *and*

$$\lim_{x\to\infty} \int_{-x}^{x} \xi \, dF(\xi) = \mu < \infty,$$

then

$$\lim_{n\to\infty} P\left[\frac{S_n}{B_n} \leqq x \mid T_\mathbf{A} > n\right] = H_{\alpha,\mathbf{A}}(x). \tag{1.2}$$

Here $H_{\alpha,\mathbf{A}}$ *is a probability distribution with characteristic function* $\Psi_{\alpha,\mathbf{A}}$ *with the following properties:*

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(i) *If* $1 \leqq \alpha < 2$ *or if* $\alpha = 2$ *and F has infinite variance, then*

$$\Psi_{\alpha, \mathbf{A}}(t) = \Psi_\alpha(t) = 1 - b\,|t|^\alpha \int_0^1 x^{(1/\alpha)-1}\, \Phi_\alpha[t(1-x)^{1/\alpha}]\, dx,$$

where Φ_α *is the characteristic function of* G_α *and consequently*

$$\ln \Phi_\alpha(t) = \begin{cases} -c\,|t|^\alpha (1 + i \operatorname{sgn}(t)\, \beta \tan \pi\alpha/2) & \text{for } 1 < \alpha \leqq 2 \\ -|t|\,(c + i \operatorname{sgn}(t)\, \mu) & \text{for } \alpha = 1 \end{cases}$$
$$= b\,|t|^\alpha.$$

In particular, the limit does not depend on the choice of the set **A**. *Furthermore, if* $\mathfrak{h}_\alpha$ *is the density corresponding to* Ψ_α, *then*

$$\mathfrak{h}_2(x) = \frac{|x|}{2\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

is two-sided Rayleigh with $\sigma^2 = 2c$, *and*

$$\mathfrak{h}_1(x) = \frac{1}{\pi} \frac{c}{c^2 + (x-\mu)^2}$$

is Cauchy (i.e., the limit is unaffected by the conditioning on $T_{\mathbf{A}} > n$*).*

(ii) *If* $\alpha = 2$ *and F has finite variance* σ,

$$\Psi_{2,\mathbf{A}}(t) = \Psi_2(t) - i \sqrt{\frac{\pi}{2}}\, \frac{E[S_{T_{\mathbf{A}}}]}{\tilde{g}_{\mathbf{A}}(0)}\, \frac{t}{\sigma} \exp\left(-\frac{t^2\sigma^2}{2}\right).$$

In this paper, using this theorem as a basis, we propose to extend the results of [1] to prove an invariance principle for the conditioned random walk. In the particular cases $\alpha = 1, 2$ we shall actually identify the respective limit processes as the Cauchy process and the Brownian excursion process studied by Ito and McKean [6].

We remark that at several points it will be necessary to make heavy use of the results in [1]. Although some of the necessary definitions are not repeated here, an effort has been made to keep the notation consistent.

2. The Invariance Principle for Stable Processes

Suppose the underlying random walk has mean zero and finite variance σ. We construct a sequence P_n of probability measures on $C[0,1]$, the family of continuous functions on the unit interval, in the following manner: Assign the probability

$$P\left[\frac{S_1}{\sigma\sqrt{n}} = x_1, \ldots, \frac{S_n}{\sigma\sqrt{n}} = x_n\right] \tag{2.1}$$

to the polygonal line segment $\xi \in C[0,1]$ with vertices at k/n, $k = 0, 1, \ldots, n$ such that $\xi(0) = 0$ and $\xi(k/n) = x_k$ otherwise. Define P_n to be the resulting probability measure. Now if $C[0,1]$ is given the topology of uniform convergence, then it is known (see e.g. Donsker's original proof in [4] or a thorough treatment in [3]) that the sequence P_n converges weakly to Wiener measure W on $C[0,1]$. Stated in the form

$$P_n[f(\xi)\leqq x]\to W[f(\xi)\leqq x],$$

for any continuous functional on $C[0,1]$, this fact is generally known as the invariance principle.

Suppose now that one substitutes

$$P\left[\frac{S_1}{\sigma\sqrt{n}}=x_1,\ldots,\frac{S_n}{\sigma\sqrt{n}}=x_n\middle|\, T_{\mathbf{A}}>n\right], \tag{2.2}$$

with $\mathbf{A}$ a finite set of integers for (2.1) in the above definition of P_n. The question naturally arises whether weak convergence to some limit measure on $C[0,1]$ still holds. We will show that not only does a limit measure exist, but that it corresponds to a Markov process whose (non-stationary) transition density may be determined explicitly. As stated earlier we actually identify the process as a modification of the Brownian excursions discussed by Ito and McKean in [6].

As an extension of this result allow the distribution F of X_1 to belong to the domain of attraction of a stable law for the cases mentioned above:

(i) $\quad 1<\alpha\leqq 2 \quad \text{and} \quad E[X_1]=0,$

or

(ii) $\alpha=1$, the attraction is normal and

$$\lim_{x\to\infty}\int_{-x}^{x}\zeta\,dF(\zeta)=\mu<\infty.$$

One then assigns the probability

$$P\left[\frac{S_1}{B_n}=x_1,\ldots,\frac{S_n}{B_n}=x_n\right] \tag{2.3}$$

to the appropriate polygonal line.

The weak limit is then a stable process having the following characterization.

Definition. The process $\xi(t)$ for $t\geqq 0$ is *stable* of index α, $0<\alpha\leqq 2$, provided the characteristic function $\Phi_\alpha(t,u)$ of the distribution of $\xi(t)$ has the form

$$\ln\Phi_\alpha(t,u)=-t\,|u|^\alpha[1+i\beta\operatorname{sgn}(u)\,\omega(u,\alpha)], \tag{2.4}$$

where $-1\leqq\beta\leqq 1$, and

$$\omega(u,\alpha)=\begin{cases}\tan\dfrac{\pi\alpha}{2} & \text{for } \alpha\neq 1\\[2ex] \dfrac{2}{\pi}\ln u & \text{for } \alpha=1.\end{cases}$$

Now it is a consequence of (1.1) that the finite dimensional distributions of the measures defined analogously to P_n, but assigning the probability in (2.3) to the appropriate polygonal path, must converge to those of the appropriate stable process. However, the situation is complicated by the fact that the sample paths of the stable processes for $0<\alpha<2$ almost surely are not continuous. What can be shown however is that they belong with probability one to $D[0,1]$, the family of real-valued functions on the unit interval which are right continuous with left limits, provided separable versions are chosen. Therefore our first difficulty is the choice of an appropriate topology for $D[0,1]$. This problem has been studied extensively by Skorokhod in [12] and Prokhorov in [11]. We now describe Skorokhod's J_1-topology, the strongest among those which he studied for which our methods apply.

The following characterization of J_1 convergence (which we take as a definition) is based on the conditions for J_1 compactness given by Billingsley [3]. It should be noted that this is not quite the characterization originally given by Skorokhod who assumes the elements of $D[0,1]$ to be left continuous at 1. Define the following (in the terminology of Billingsley) *moduli of continuity* for ξ in $D[0,1]$

$$\Delta(T_0, \xi) = \sup\{|\xi(s)-\xi(t)|: s, t \in T_0\}$$

for $T_0 \subset [0,1]$ and

$$\Delta_{J_1}(\rho, \xi) = \sup_{t-\rho<t_1 \leqq t \leqq t_2 < t+\rho} \min\{|\xi(t)-\xi(t_1)|, |\xi(t_2)-\xi(t)|\}$$

for $\rho>0$. Then a sequence ξ_n in $D[0,1]$ *converges* in the J_1 topology to ξ in $D[0,1]$ provided

(i) $\xi_n(t) \to \xi(t)$ as $n\to\infty$ on a dense set containing 0 and 1

and

(ii)
$$\lim_{\rho\to 0} \varlimsup_{n\to\infty} \Delta_{J_1}(\rho, \xi_n) = 0$$
$$\lim_{\rho\to 0} \varlimsup_{n\to\infty} \Delta([0,\rho), \xi_n) = 0$$
$$\lim_{\rho\to 0} \varlimsup_{n\to\infty} \Delta([1-\rho, 1), \xi_n) = 0.$$

Suppose P_n is a sequence of probability measures on $D[0,1]$ defined on the usual Borel field $\mathscr{B}$ generated by the cylinder sets $\{\xi(t_1)\leqq x_1, \ldots, \xi(t_N)\leqq x_N\}$ for finite N. In what follows it will be important to have criteria for the weak convergence of such a sequence. Following [3] in this regard we define a sequence P_n of probability measures (on a general metric space S) to be *tight* if for every $\varepsilon>0$ there exists a compact set K such that $P_n(K)>1-\varepsilon$ for every n. Then the desired conditions for weak convergence are contained in the following results proven in [3]. (We write $P_n \Rightarrow P$ to denote the weak convergence of P_n to P.)

Theorem (2.1). *Suppose*

(i) *The finite dimensional distributions of P_n converge to those of P.*

(ii) *The sequence P_n is tight.*

Then $P_n \Rightarrow P$.

Theorem (2.2). *The sequence P_n is tight if and only if*

(i) *For every $\varepsilon>0$, there exists a constant A such that*

$$P_n[\sup_t |\xi(t)|>A]<\varepsilon$$

(ii)
$$\lim_{\rho\to 0}\overline{\lim_{n\to\infty}} P_n[\Delta_{J_1}(\rho,\xi)>\varepsilon]=0$$

$$\lim_{\rho\to 0}\overline{\lim_{n\to\infty}} P_n[\Delta([0,\rho),\xi)>\varepsilon]=0 \tag{2.5}$$

$$\lim_{\rho\to 0}\overline{\lim_{n\to\infty}} P_n[\Delta([1-\rho,1),\xi)]=0.$$

We remark that the above results remain valid if in the definition of weak convergence one assumes only that the real-valued functional on $D[0,1]$ is J_1-continuous almost surely $[P]$. Also an examination of the proof shows that if P_n and P are concentrated on $C[0,1]$, then the conclusion of the theorem remains valid for functionals on $C[0,1]$ continuous in the subspace topology induced by J_1. This, however, is easily seen to be the topology of uniform convergence.

Lemma (2.1). *Assume the conditions of Theorem 1.1. For $1\leqq\alpha\leqq 2$ define two sequences $P_{\alpha,n}$ and $\hat{P}_{\alpha,n}$ of probability measures on $D[0,1]$ by assigning the probability*

$$P\left[\frac{S_1}{B_n}=x_1,\dots,\frac{S_n}{B_n}=x_n\right] \tag{2.6}$$

in the case of $P_{\alpha,n}$, and

$$P\left[\frac{S_1}{B_n}=x_1,\dots,\frac{S_n}{B_n}=x_n \,\Big|\, T_{\mathbf{A}}>n\right] \tag{2.7}$$

in the case of $\hat{P}_{\alpha,n}$, to the right continuous step function ξ with jumps at k/n, $k=1,2,\dots,n$, and $\xi(0)=0$, $\xi(k/n)=x_k$. Then

$$\lim_{\rho\to 0}\overline{\lim_{n\to\infty}} Q_n[\Delta_{J_1}(\rho,\xi)>\varepsilon]=0 \tag{2.8}$$

for $\varepsilon>0$, where either $\{Q_n\}=\{P_{\alpha,n}\}$ or $\{Q_n\}=\{\hat{P}_{\alpha,n}\}$.

Proof. As pointed out to the author by the referee, the validity of (2.8) for $P_{\alpha,n}$ is a special case of a theorem of Skorokhod (Theorem 2.7 of [13]).

The key to the proof for the $\hat{P}_{\alpha,n}$ is the demonstration of

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}} \frac{1}{r_n(\mathbf{A})} P\left[\max_{t\leqq\delta}\left|\frac{S_{[nt]}}{B_n}\right|\geqq\varepsilon;\ T_{\mathbf{A}}>[n\delta]\right]=0 \tag{2.9}$$

for every fixed $\varepsilon>0$.

It is easily seen that if $x_+=\max\{x\in\mathbf{A}\}$, then for $x>\varepsilon$

$$\begin{aligned}
P^{xB_n}&[S_{[n\delta]-k}\geqq xB_n;\ T_{\mathbf{A}}>[n\delta]-k]\\
&\geqq P\left[S_{[n\delta]-k}\geqq 0;\ \min_{j\leqq[n\delta]-k}\frac{S_j}{B_n}>\frac{x_+}{B_n}-x\right]\\
&\geqq P\left[S_{[n\delta]-k}\geqq 0;\ \min_{j\leqq[n\delta]-k}\frac{S_j}{B_{[n\delta]-k}}\geqq-\frac{\varepsilon B_n}{B_{[n\delta]}}\right],
\end{aligned} \tag{2.10}$$

provided n is sufficiently large and $B_{[n\delta]-k}\to\infty$ as $n\to\infty$.

Now

$$\lim_{n\to\infty} P\left[\frac{S_n}{B_n}\leqq x_1;\ \min_{j\leqq n}\frac{S_j}{B_n}\geqq -x_2\right]=F_\alpha(x_1,x_2) \tag{2.11}$$

where

$$F_\alpha(x_1,x_2)=P[\xi(1)\leqq x_1;\ \min_{0\leqq t\leqq 1}\xi(t)\geqq -x_2]$$

and $\xi(t)$ is the appropriate limit stable process. A proof of this statement can be patterned after the proof that in the finite variance case $\max_{j\leqq n}\frac{S_j}{\sigma\sqrt{n}}$ converges in distribution to $\sup_{0\leqq t\leqq 1}\xi(t)$, where $\xi(t)$ is a Brownian motion (see Theorem (1) of Section 26 in [8]) or one can simply appeal to the invariance principal for the $P_{\alpha,n}$ which is already known. In particular then,

$$\lim_{n\to\infty} P\left[S_n\geqq 0;\ \min_{j\leqq n}\frac{S_j}{B_n}\geqq -\frac{\varepsilon}{\delta^{1/\alpha}}\right]=\lambda(\delta,\varepsilon)$$

with $\lambda(\delta,\varepsilon)>0$.

Therefore, for $[n\delta]-k\geqq M=M(\delta,\varepsilon)$ we have

$$P\left[S_{[n\delta]-k}\geqq 0;\ \min_{j\leqq[n\delta]-k}\frac{S_j}{B_{[n\delta]-k}}\geqq -\frac{\varepsilon B_n}{B_{[n\delta]}}\right]\geqq\frac{\lambda(\delta,\varepsilon)}{2}.$$

Define $\tau_n=\tau_n(\delta,\varepsilon)$ by

$$\tau_n=\min\left\{j\colon 1\leqq j\leqq[n\delta];\ \frac{S_j}{B_n}\geqq\varepsilon\right\}.$$

Then, in view of (2.10), we have the following estimates

$$\begin{aligned}
&P\left[\max_{t\leqq\delta}\frac{S_{[nt]}}{B_n}\geqq\varepsilon;\ \tau_n<[n\delta]-M;\ T_{\mathbf{A}}>[n\delta]\right]\\
&\quad\leqq\frac{2}{\lambda(\delta,\varepsilon)}\sum_{x>\varepsilon}\sum_{k=0}^{[n\delta]-M}P\left[\tau_n=k;\ \frac{S_k}{B_n}=x;\ T_{\mathbf{A}}>k\right]\\
&\qquad\qquad P^{xB_n}[S_{[n\delta]-k}\geqq xB_n;\ T_{\mathbf{A}}>[n\delta]-k]\\
&\quad\leqq\frac{2}{\lambda(\delta,\varepsilon)}P\left[\frac{S_{[n\delta]}}{B_n}\geqq\varepsilon;\ \tau_n<[n\delta]-M;\ T_{\mathbf{A}}>[n\delta]\right].
\end{aligned} \tag{2.12}$$

In addition,

$$\begin{aligned}
&P\left[\max_{t\leqq\delta}\frac{S_{[nt]}}{B_n}\geqq\varepsilon;\ \tau_n\geqq[n\delta]-M;\ T_{\mathbf{A}}>[n\delta]\right]\\
&\quad\leqq P\left[\frac{S_{[n\delta]}}{B_n}\geqq\frac{\varepsilon}{2};\ \tau_n\geqq[n\delta]-M;\ T_{\mathbf{A}}>[n\delta]\right]\\
&\qquad+P\left[\max_{j\leqq M}\frac{S_j}{B_n}>\frac{\varepsilon}{2}\right]P[T_{\mathbf{A}}>[n\delta]-M].
\end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13) we obtain

$$P\left[\max_{t\leqq\delta}\frac{S_{[nt]}}{B_n}\geqq\varepsilon;\ T_{\mathbf{A}}>[n\delta]\right]\leqq\left(\frac{2}{\lambda(\delta,\varepsilon)}+1\right)P\left[\frac{S_{[n\delta]}}{B_n}\geqq\frac{\varepsilon}{2};\ T_{\mathbf{A}}>[n\delta]\right]$$
$$+P\left[\max_{j\leqq M}\frac{S_j}{B_n}\geqq\frac{\varepsilon}{2}\right]r_{[n\delta]-M}(\mathbf{A}).$$

Now

(i) $$\lim_{n\to\infty}\frac{r_{[n\delta]}(\mathbf{A})}{r_n(\mathbf{A})}=\begin{cases}\delta^{(1/\alpha)-1} & \text{for } 1<\alpha\leqq 2 \text{ (see Lemma (2.1) of [1])}\\ 1 & \text{for } \alpha=1,\end{cases}$$

(ii) $$\lim_{n\to\infty}P\left[\max_{j\leqq M}\frac{S_j}{B_n}\geqq\frac{\varepsilon}{2}\right]=0,$$

and

(iii) $$\lim_{n\to\infty}P\left[\frac{S_{[n\delta]}}{B_n}\geqq\frac{\varepsilon}{2}\,\middle|\,T_{\mathbf{A}}>[n\delta]\right]=P\left[X\geqq\frac{\varepsilon}{2\delta^{1/\alpha}}\right]$$

with (iii) following from Theorem (1.1) $\left(\text{recall }\lim_{n\to\infty}\frac{B_n}{B_{[n\delta]}}=\delta^{-(1/\alpha)}\right)$, where X has the distribution with characteristic function

$$\Psi_\alpha(t)=1-b\,|t|^\alpha\int_0^1 x^{(1/\alpha)-1}\,\Phi_\alpha\big(t(1-x)^{1/\alpha}\big)\,dx.$$

We obtain finally, therefore, that

$$\overline{\lim_{n\to\infty}}\frac{P\left[\max_{t\leqq\delta}\frac{S_{[nt]}}{B_n}\geqq\varepsilon;\ T_{\mathbf{A}}>[n\delta]\right]}{r_n(\mathbf{A})}\leqq\left(\frac{2}{\lambda(\delta,\varepsilon)}+1\right)\delta^{(1/\alpha)-1}P\left[X>\frac{\varepsilon}{2\delta^{1/\alpha}}\right].\quad(2.14)$$

By (1) of Section 13 in [5] we have

$$P[|X|>2x]\leqq x\int_{-1/x}^{1/x}|1-\Psi_\alpha(t)|\,dt,$$

and thus

$$P\left[X>\frac{\varepsilon}{2\delta^{1/\alpha}}\right]\leqq\frac{\varepsilon}{4\delta^{1/\alpha}}\int_{-4\delta^{1/\alpha}/\varepsilon}^{4\delta^{1/\alpha}/\varepsilon}|t|^\alpha\int_0^1 x^{(1/\alpha)-1}\,dx\,dt\leqq K(\varepsilon)\,\delta$$

for some constant $K(\varepsilon)$ not depending on δ.

Observing that for fixed ε, $\lambda(\delta,\varepsilon)$ is non-decreasing as $\delta\to 0$, we have by (2.14)

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\frac{P\left[\max_{t\leqq\delta}\frac{S_{[nt]}}{B_n}\geqq\varepsilon;\ T_{\mathbf{A}}>[n\delta]\right]}{r_n(\mathbf{A})}\leqq\lim_{\delta\to 0}\frac{K(\varepsilon)}{\lambda(\delta,\varepsilon)}\delta^{1/\alpha}=0.$$

This proves (2.9).

Define a random time $\tau=\tau(\rho,\varepsilon)$ by

$$\tau=\begin{cases}\inf\{t: \sup\limits_{t-\rho<t_1<t\leqq t_2<t+\rho} \min(|\xi(t)-\xi(t_1)|, |\xi(t_2)-\xi(t)|)>\varepsilon\} & \text{if such } t \text{ exists}\\ \infty & \text{otherwise.}\end{cases}$$

Then for $\delta>0$ we have

$$\hat{P}_{\alpha,n}[\Delta_{J_1}(\rho,\xi)>\varepsilon]=\hat{P}_{\alpha,n}[\tau\leqq\delta]+\hat{P}_{\alpha,n}[\delta<\tau\leqq 1];$$

this implies

$$\begin{aligned}\overline{\lim_{n\to\infty}}\,\hat{P}_{\alpha,n}[\Delta_{J_1}(\rho,\xi)>\varepsilon]\leqq \overline{\lim_{n\to\infty}}\,\frac{r_{[n\delta]}(\mathbf{A})}{r_n(\mathbf{A})}\,P\left[\max_{j\leqq[n\delta]}\left|\frac{S_j}{B_n}\right|>\varepsilon\,|\,T_{\mathbf{A}}>[n\delta]\right]\\ +\overline{\lim_{n\to\infty}}\,\frac{r_{[n\delta]}(\mathbf{A})}{r_n(\mathbf{A})}\,P_{\alpha,n}[\Delta_{J_1}(\rho,\xi)>\varepsilon].\end{aligned} \tag{2.15}$$

Applying (2.9), the fact that the lemma is already known to hold for the $P_{\alpha,n}$, and (i) of the three facts listed above, we get

$$\lim_{\rho\to 0}\,\overline{\lim_{n\to\infty}}\,\hat{P}_{\alpha,n}[\Delta_{J_1}(\rho,\xi)>\varepsilon]=0.$$

This completes the proof of Lemma (2.1).

3. The Main Theorem

We have already indicated that in the sense of Skorokhod's J_1 convergence

$$\lim_{n\to\infty} P_{\alpha,n}\Rightarrow P_\alpha.$$

We now state our main result giving the analogue for the measures $\hat{P}_{\alpha,n}$.

Theorem (3.1). *Suppose that the distribution F of X_1 belongs to the domain of attraction of a stable law of index α, $1\leqq\alpha\leqq 2$. When $\alpha=1$ we require that the attraction be normal and*

$$\lim_{x\to\infty}\int_{-x}^{x}\zeta\,dF(\zeta) \quad \textit{exists and is finite.}$$

Then the sequence $\hat{P}_{\alpha,n}$ of probability measures on $D[0,1]$ defined in (2.7) converges weakly relative to Skorokhod's J_1 topology to a probability measure $\hat{P}_\alpha$ on $D[0,1]$ which for $\alpha\neq 1$ corresponds to a Markov process with non-stationary transition measure.

The finite dimensional distributions of $\hat{P}_\alpha$ are given by

$$\begin{aligned}\hat{P}_\alpha[\xi(t_1)\leqq x_1,\dots,\xi(t_k)\leqq x_k]=&\int_{-\infty}^{x_1}\frac{1}{t_1}\mathfrak{h}_\alpha(t_1^{-1/\alpha}y_1)\,P_\alpha^{y_1}[T>1-t_1]\,dy_1\\ &\cdot\int_{-\infty}^{x_2}\frac{P_\alpha^{y_2}[T>1-t_2]}{P_\alpha^{y_1}[T>1-t_1]}P_\alpha^{y_1}[\xi(t_2-t_1)\in dy_2;\,T>t_2-t_1]\cdots\\ &\cdot\int_{-\infty}^{x_k}\frac{P_\alpha^{y_k}[T>1-t_k]}{P_\alpha^{y_{k-1}}[T>1-t_{k-1}]}P_\alpha^{y_{k-1}}[\xi(t_k-t_{k-1})\in dy_k;\,T>t_k-t_{k-1}];\end{aligned} \tag{3.1}$$

where $\mathfrak{h}_\alpha$ is the density corresponding to the characteristic function Ψ_α, and P_α is the weak limit of the sequence $P_{\alpha,n}$.

When $\alpha=1,2$ substantially more can be said. For $\alpha=1$

$$\lim_{n\to\infty} \hat{P}_{1,n} = \lim_{n\to\infty} P_{1,n} = P_1, \tag{3.2}$$

i.e. the weak limits of $P_{1,n}$ and $\hat{P}_{1,n}$ both correspond to the same Cauchy process. For $\alpha=2$, $\hat{P}_2$ is concentrated on $C[0,1]$ and corresponds to the process with transition density p given by

$$p(0,0;t,x) = \frac{|x|}{t^{\frac{3}{2}}} \exp\left(-\frac{x^2}{2t}\right) N\left(\frac{|x|}{\sqrt{1-t}}\right),$$

$$p(t_1,x_1;t_2,x_2) = \frac{1}{\sqrt{2\pi(t_2-t_1)}}\left(\exp\left[-\frac{(x_2-x_1)^2}{2(t_2-t_1)}\right] - \exp\left[-\frac{(x_2+x_1)^2}{2(t_2-t_1)}\right]\right)$$

$$\cdot \frac{N\left(|x_2|/\sqrt{1-t_2}\right)}{N\left(|x_1|/\sqrt{1-t_1}\right)} \quad \text{for } 0 \leqq t_1 < t_2 \leqq 1,\ x_1 x_2 > 0,$$

$$p(t_1,x_1;t_2,x_2) = 0 \qquad \text{otherwise};$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-y^2/2}\, dy.$

Moreover, let $X(t)$ for $t \geqq 0$ be a standard Brownian motion, and for an arbitrary but fixed time $\tau_0 > 0$ define

(i) $\quad \tau = \sup[t: t<\tau_0 \text{ and } X(t)=0] \quad \text{and} \quad \Delta = \tau_0 - \tau,$

(ii) $\quad \tilde{\mathfrak{e}}(t) = \frac{|X(t\Delta+\tau)|}{\sqrt{\Delta}} \quad \text{for } 0 \leqq t \leqq 1,$

(iii) $\quad \mathfrak{s} = -1, +1 \quad \text{for } X(\tau_0)<0, X(\tau_0)>0 \quad \text{respectively}.$

Then $\mathfrak{e}(t) = \mathfrak{s}\,\tilde{\mathfrak{e}}(t)$ is a Markov process with transition density p.

Proof. By Theorem (2.1) we have only to show that $\hat{P}_{\alpha,n}$ is tight and then prove the convergence of the finite dimensional distributions of $\hat{P}_{\alpha,n}$ to those of $\hat{P}_\alpha$.

We use the criteria in Theorem (2.2) to show the tightness of the sequence $\hat{P}_{\alpha,n}$. The first of the three conditions in (ii) of the theorem is the content of Lemma 2.1 and the second follows immediately from (2.9). The third condition as well as (i) of the theorem follow from the obvious analogues of (2.15) and the tightness of the $P_{\alpha,n}$.

We now turn to the proof that the finite dimensional distributions of $\hat{P}_{\alpha,n}$ converge to those of $\hat{P}_\alpha$. We begin with some definitions.

Let

$$\mu_n(0,0;t,dy) = \frac{1}{r_n(\mathbf{A})} P\left[\frac{S_{[nt]}}{B_n} \in dy;\ T_{\mathbf{A}} > [nt]\right]$$

$$\mu_n(t_1,y_1;t_2,dy_2) = P^{B_n y_1}\left[\frac{S_{[nt_2]-[nt_1]}}{B_n} \in dy_2;\ T_{\mathbf{A}} > [nt_2]-[nt_1]\right]$$

$$\text{for } 0 \leqq t_1 \leqq t_2 \leqq 1,\ (t_1,y_1) \neq (0,0).$$

We then have

$$P\left[\frac{S_{[nt_1]}}{B_n} \leqq x_1, \ldots, \frac{S_{[nt_k]}}{B_n} \leqq x_k \,\middle|\, T_{\mathbf{A}} > n\right] \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_1} \mu_n(0,0; t_1, dy_1) \ldots \mu_n(t_{k-1}, y_{k-1}; t_k, dy_k)\, \mu_n(t_k, y_k; 1, dy). \tag{3.3}$$

Now

$$\hat{P}_{\alpha,n}[\xi(t_1) \leqq x_1, \ldots, \xi(t_k) \leqq x_k] = P\left[\frac{S_{[nt_1]}}{B_n} \leqq x_1, \ldots, \frac{S_{[nt_k]}}{B_n} \leqq x_k \,\middle|\, T_{\mathbf{A}} > n\right]. \tag{3.4}$$

The convergence of the finite dimensional distributions of the $\hat{P}_{\alpha,n}$ to those of $\hat{P}_\alpha$ will follow from

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_1} \mu_n(0,0; t_1, dy_1) \ldots \mu_n(t_{k-1}, y_{k-1}; t_k, dy_k)\, \mu_n(t_k, y_k; 1, dy) \\ = \hat{P}_\alpha[\xi(t_1) \leqq x_1, \ldots, \xi(t_k) \leqq x_k]. \tag{3.5}$$

We now state without proof the following elementary result which will be a valuable analytic tool in the proof of (3.5).

Lemma (3.1). *Let v_n, $n=1,2,\ldots$ be a sequence of finite (positive) measures on the σ-field $\mathcal{B}$ of Borel sets in* $\mathbf{R}$ *converging weakly to a finite measure v which is absolutely continuous with respect to Lebesgue measure. If f_n is a sequence of uniformly bounded $\mathcal{B}$-measurable functions converging uniformly on compact sets to an everywhere bounded continuous limit f, then*

$$\int_B f_n \, dv_n \to \int_B f \, dv \tag{3.6}$$

as $n \to \infty$ for $B \in \mathcal{B}$ provided the Lebesgue measure of the boundary of B is zero.

For the *Proof* of (3.5) we will want to make recursive use of the lemma. For $0 \leqq t_1 < t_2 \leqq 1$ set

$$h_n(t_1, y_1; t_2, x) = \int_{-\infty}^{x} \mu_n(t_1, y_1; t_2, dy_2)$$

$$h_n(t_1, y_1; t_2) = \lim_{x\to\infty} h_n(t_1, y_1; t_2, x).$$

Suppose that we have shown first that

$$\lim_{n\to\infty} \int_{-\infty}^{x_{i-1}} \cdots \int_{-\infty}^{x_1} \mu_n(0,0; t_1, dy_1) \ldots \mu_n(t_{i-1}, y_{i-1}; t_i, dy_i) \\ = \int_{-\infty}^{x_{i-1}} \cdots \int_{-\infty}^{x_1} \frac{1}{t_1} \mathfrak{h}_\alpha(t_1^{-1/\alpha} y_1)\, dy_1 \ldots P_\alpha^{y_{i-1}}[\xi(t_i - t_{i-1}) \in dy_i;\ T > t_i - t_{i-1}]\, dy_{i-1}; \tag{3.7}$$

and second that as a function of y_i, $h_n(t_i, y_i; t_{i+1}, x)$ satisfies the appropriate conditions of Lemma (1.2). Then, by the existence of densities for stable processes, the limit measure is absolutely continuous with respect to Lebesgue measure, and hence by the lemma we obtain (3.7) with i replaced by $i+1$.

We proceed then to verify (3.7) for $1<\alpha\leqq 2$. Let

$$R(\xi;\, t_1, t_2)=\{\xi(t)\colon t\in[t_1, t_2]\}$$

and

$$R_t(\xi)=R(\xi;\, 0, t);$$

then except when $y\in\mathbf{A}$ and $y=0$ we have

$$P^{B_n y}\left[\frac{S_{[nt]}}{B_n}\leqq x;\ T_{\mathbf{A}}>[nt]\right]=h_n(0, y;\, t, x)$$
$$=P_{\alpha,n}[R_t(\xi)\cap\mathbf{A}_n=\Phi;\ \xi(t)\leqq x-y]$$

for n sufficiently large, where the set $\mathbf{A}_n$ is given by

$$\mathbf{A}_n=\frac{\mathbf{A}}{B_n}-y.$$

The required result in (3.8) below in the exceptional case can be handled by noting that, on the one hand,

$$P\left[\frac{S_{[nt]}}{B_n}\leqq x;\ T_{\mathbf{A}}>[nt]\right]\leqq r_{[nt]}(\mathbf{A})\to 0 \quad \text{as} \quad n\to\infty,$$

while for $1<\alpha\leqq 2$ for all points x, x is regular for $\{x\}$ and thus

$$P_\alpha[\xi(t)\leqq x;\ T>t]\leqq P_\alpha[T>t]=0.$$

For a Borel set $C\subset\mathbf{R}$, let $\chi_C(t, x, \cdot)$ be the functional on $D[0,1]$ given by

$$\chi_C(t, x, \xi)=\begin{cases}1 & \text{if} \quad \mathbf{R}_t(\xi)\cap C=\Phi \quad \text{and} \quad \xi(t)\leqq x\\ 0 & \text{otherwise}.\end{cases}$$

Then we must first show

$$\begin{aligned}\lim_{n\to\infty} P_{\alpha,n}[\chi_{\mathbf{A}_n}(t, x-y, \xi)=1]&=P_\alpha[\chi_{\{-y\}}(t, x-y, \xi)=1]\\ &=P_\alpha^y[\xi(t)\leqq x;\ T>t],\end{aligned} \tag{3.8}$$

and then demonstrate the uniformity of the convergence for y on compact sets as well as the continuity of the limit. A start in this direction is the following.

Lemma (3.2). *Let $I=I(z,\varepsilon)$ be the open interval $(z-\varepsilon, z+\varepsilon)$, then $\chi_I(t, x, \cdot)$ is J_1-continuous almost surely $[P_\alpha]$.*

Proof. Assume $\xi_n \xrightarrow{J_1} \xi$. We consider several possible cases.

Case (i):

$$\chi_I(t, x, \xi)=0.$$

Either $R_t(\xi)\cap I\neq\Phi$ or $\xi(t)>x$. We may assume ξ is right continuous on all of $[0,1]$ and continuous at t since both of these events occur almost surely P_α. The criterion for J_1 convergence then implies $\chi_I(t, x, \xi_n)\to\chi_I(t, x, \xi)$.

Case (ii):

$$\chi_I(t,x,\xi)=1 \quad \text{and} \quad \bar{R}_t(\xi)\cap\bar{I}=\Phi.$$

Let $d=d(\bar{R}(\xi),\bar{I})$, with the usual definition of the distance between two closed sets in a metric space. Then, given any $c>0$, choose from the set where $\xi_n(t)\to\xi(t)$ a partition, $0=t_0<t_1<\cdots<t_k=1$, such that

$$\max_{1\leq i\leq k}(t_i-t_{i-1})\leq\frac{c}{2}.$$

Then choose N so large that $n\geq N$ implies

$$\max_{0\leq i\leq k}|\xi_n(t_i)-\xi(t_i)|\leq\frac{d}{2}.$$

For any such n it is clear that $\Delta_{J_1}(c,\xi_n)\geq\frac{d}{2}$ if $R_t(\xi_n)\cap I\neq\Phi$. This is enough to guarantee the continuity of χ_I at ξ.

To complete the proof of the lemma we show that either Case (i) or Case (ii) must hold with probability 1 $[P_\alpha]$. Thus it must be shown that with probability 1 sample paths coming arbitrarily close to a given open interval during $[0,t]$, must in fact have hit it. This result can be gotten from a general theorem about stopping times for Hunt processes, but we give a direct proof.

Let $\tau_N=T_{I(x,1/N)}$. Then τ_N increases to some limit τ. By the existence of left limits along sample paths, $\tau-$ exists almost surely and clearly then $\xi(\tau-)=x$ almost surely. The assertion of the last paragraph will follow if we can show that the probability of a jump at the random time τ is zero. But

$$|\xi(\tau)-\xi(\tau-)|>\delta$$

implies

$$\liminf_{N\to\infty}\sup_{t\leq\varepsilon}|\xi(\tau_N+t)-\xi(\tau_N)|>\frac{\delta}{2}$$

for $\varepsilon>0$ by the right continuity of sample paths. Consequently, applying the strong Markov property we obtain

$$P_\alpha[|\xi(\tau)-\xi(\tau-)|>\delta]\leq P_\alpha\left[\sup_{t\leq\varepsilon}|\xi(t)|>\frac{\delta}{2}\right]\to 0 \quad \text{as} \quad \varepsilon\to 0,$$

again by the right continuity of sample paths. Therefore it indeed must be that

$$P_\alpha[x\in\bar{R}_t(\xi)]=P_\alpha[x\in R_t(\xi)].$$

For $\alpha=1$, $P_1^x[T>t]=1$ (see [10] for an elegant proof of this somewhat remarkable property of the Cauchy process). It follows that $\chi_I=1$ implies $\bar{R}_t\cap\bar{I}=\Phi$.

If $1<\alpha\leq 2$, we still must prove that once an end point of an interval is hit the interior must also be hit. This, however, follows immediately from another result of Port [10] which states that for $1<\alpha\leq 2$

$$\lim_{y\to x}P_\alpha^y[T_{\{x\}}\leq\varepsilon]=1 \qquad \text{for } \varepsilon>0. \tag{3.9}$$

This completes the proof of the lemma.

Denote by $\Lambda_t(y, x, \varepsilon)$ the event

$$[R_t(\xi) \cap I(-y, \varepsilon) \neq \Phi;\ \xi(t) \leqq x - y],$$

and by $\Lambda_t(y, x)$ the event

$$[R_t(\xi) \cap \{-y\} \neq \Phi;\ \xi(t) \leqq x - y].$$

Then Lemma (3.2) and Skorokhod's theorem imply that

$$\lim_{n \to \infty} P_{\alpha, n}[\Lambda_t(y, x, \varepsilon)] = P_\alpha[\Lambda_t(y, x, \varepsilon)]. \tag{3.10}$$

Also by (3.9) we have

$$\lim_{\varepsilon \to 0} P_\alpha[\Lambda_t(y, x, \varepsilon)] = P_\alpha[\Lambda_t(y, x)]. \tag{3.11}$$

Now define

$$\Lambda_t^{(n)}(y, x) = [R_t(\xi) \cap \mathbf{A}_n \neq \Phi;\ \xi(t) \leqq x - y].$$

We now show that

$$\lim_{n \to \infty} P_{\alpha, n}[\Lambda_t^{(n)}(y, x)] = P_\alpha[\Lambda_t(y, x)]. \tag{3.12}$$

In view of (3.10) and (3.11) it is enough to show

$$\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \{P_{\alpha, n}[\Lambda_t(y, x, \varepsilon)] - P_{\alpha, n}[\Lambda_t^{(n)}(y, x)]\} = 0. \tag{3.13}$$

(Note that $\Lambda_t^{(n)}(y, x) \subset \Lambda_t(y, x, \varepsilon)$ for large n.)

Suppose that we are able to show that

$$\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \sup_{|z| < \varepsilon B_n} P^z(T > \varepsilon n] = 0, \tag{3.14}$$

then

$$\begin{aligned}
&\overline{\lim_{n \to \infty}} P_{\alpha, n}\left[\Lambda_t\left(y, x, \frac{\varepsilon}{2}\right) - \Lambda_t^{(n)}(y, x)\right] \\
&\quad \leqq \overline{\lim_{n \to \infty}} P_{\alpha, n}[\Lambda_t(y, x, \varepsilon) - \Lambda_t^{(n)}(y, x);\ T_{I(-y, \frac{\varepsilon}{2})} < n(1-\varepsilon)] \\
&\quad\quad + \overline{\lim_{n \to \infty}} P_{\alpha, n}\left[R(\xi;\ 1-\varepsilon, 1) \cap I\left(-y, \frac{\varepsilon}{2}\right) \neq \Phi\right].
\end{aligned}$$

Now if n is so large that

$$\frac{\max\{|x|: x \in \mathbf{A}\}}{B_n} \leqq \frac{\varepsilon}{2},$$

then

$$P_{\alpha, n}\left[\Lambda_t\left(y, x, \frac{\varepsilon}{2}\right) - \Lambda_t^{(n)}(y, x);\ T_{I(-y, \frac{\varepsilon}{2})} < n(1-\varepsilon)\right] \leqq \sup_{|z| \leqq \varepsilon B_n} P^z(T > \varepsilon n].$$

Consequently, by the assumed validity of (3.14) and by Lemma (3.2)

$$\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} P_{\alpha, n}[\Lambda_t(y, x, \varepsilon) - \Lambda_t^{(n)}(y, x)] \leqq \lim_{\varepsilon \to 0} P_\alpha\left[R(\xi;\ 1-\varepsilon, 1) \cap I\left(-y, \frac{\varepsilon}{2}\right) \neq \Phi\right].$$

Now an easy estimate using the existence for stable processes of a continuous transition density p_α satisfying the scalling property $p_\alpha(t, x) = t^{-1/\alpha} p_\alpha(1, x t^{-1/\alpha})$, implies

$$\lim_{\varepsilon\to 0} P_\alpha[R(\xi;\, 1-\varepsilon, 1)\cap I(-y,\varepsilon)\neq\Phi]=0.$$

It follows then that (3.13) is true. We have therefore reduced the proof of (3.12) to showing (3.14). We require the following lemma.

Lemma (3.3). *Let* $a(x)$ *be the potential kernel of a recurrent random walk belonging to the domain of attraction of a stable law of index* α, $1<\alpha\leqq 2$, *then*

(i) *for* $1<\alpha<2$

$$\lim_{x\to\pm\infty}\frac{a(x)\,L(|x|)}{|x|^{\alpha-1}}=-\frac{\tan\frac{\pi\alpha}{2}}{\pi(1+h^2)}(1\pm\beta),$$

where $\chi(x)=1-F(x)+F(-x)=x^{-\alpha}L(x)$ *for* $x>0$ *with* L *slowly varying, and* $h=\beta\tan\frac{\pi\alpha}{2}$;

(ii) *for* $\alpha=2$, $\sigma^2<\infty$
$$\lim_{x\to\infty}\frac{a(x)}{|x|}=\frac{1}{\sigma^2};$$

(iii) *for* $\alpha=2$, $\sigma^2=\infty$
$$\lim_{x\to\infty}\frac{a(x)+a(-x)}{2x}\int_{-x}^{x}\zeta^2\,dF(\zeta)=1.$$

Proof. For $1<\alpha<2$ it is not difficult to show that

$$a(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-e^{ixt}}{1-\Phi(t)}\,dt.$$

By Theorem (1) of the Appendix of [2]

$$\lim_{t\to 0\pm}\frac{1-\Phi(t)}{\chi\left(\frac{1}{|t|}\right)}=\frac{c}{c_1+c_2}(1\pm i\,h),$$

and so the proof is of an analytic nature much the same as the proof of 32.3 in [14] and the details will be omitted.

For $\alpha=2$, $\sigma^2<\infty$ we simply quote the results of Section 29 in [14].

Finally, for $\alpha=2$, $\sigma^2=\infty$ we have

$$\frac{a(x)+a(-x)}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-\cos x\,t}{1-\Phi(t)}\,dt.$$

Again by Theorem (1) of the Appendix of [2]

$$\lim_{t\to 0}\frac{1-\Phi(t)}{t^2\int_{-1/t}^{1/t}\zeta^2\,dF(\zeta)}=\frac{1}{2},$$

and we omit the detailed estimates.

It is shown in [7] that

$$\frac{P^x[T>n]}{P[T>n]}=\frac{r_{n+1}}{r_n}\sum_{t\neq 0} g(x,t)\, v_n(t) \tag{3.15}$$

where $\sum_{t\neq 0} v_n(t)=1$ for $n\geqq 1$, and $\lim_{n\to\infty} v_n(t)=0$ for $t\neq 0$.

Proposition 29.4 in [14] states

(i) $$g(x,t)=a(x)+a(-t)-a(x-t)$$

(ii) $$a(-t)-a(x-t)\leqq a(-x).$$

From the representation in (3.15) and (ii) it follows that

$$P^x[T>n]\leqq (a(x)+a(-x))\, r_n;$$

and in particular

$$P^{[\varepsilon B_n]}[T>\varepsilon n]\leqq (a([\varepsilon B_n])+a(-[\varepsilon B_n]))\, r_{[\varepsilon n]} \overset{\text{def}}{=} K_n(\varepsilon). \tag{3.16}$$

Combining Lemma (3.3) above and Lemma (2.1) of [1], we have for $1<\alpha<2$ and $n\geqq N(\varepsilon)$

$$K_n(\varepsilon)\leqq K\,\frac{B(\varepsilon n)}{\varepsilon}\,\frac{1}{\varepsilon B(n)}\,\frac{\chi\left(\chi^{-1}\left(\frac{c_1+c_2}{n}\right)\right)}{\chi[\varepsilon B(n)]}\to \varepsilon^{(\alpha-1)^2/\alpha} \quad \text{as} \quad n\to\infty,$$

where K is some absolute constant.

For $\alpha=2$ we have the following estimate

$$K_n(\varepsilon)\leqq K'\,\frac{\varepsilon B_n}{\int_{-\varepsilon B_n}^{\varepsilon B_n} x^2\, dF(x)}\,\frac{B_{[\varepsilon n]}}{[\varepsilon n]}\to K'\sqrt{\varepsilon} \quad \text{as} \quad n\to\infty,$$

since

$$\lim_{n\to\infty}\frac{n}{B_n^2}\int_{|x|\leqq \varepsilon B_n} x^2\, dF(x)=1$$

(see e.g. (1.22) and the subsequent comments in [1]). Thus, in each case we conclude that

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{n\to\infty}}\, K_n(\varepsilon)=0,$$

completing the verification of (3.14) and hence of (3.12) as well as the existence and identification of the limit in (3.8). The uniformity of the convergence on compact sets is proved by contradiction. If $y_1, y_2, \ldots$ were a bounded sequence such that $y_n\to y$ and

$$|P_{\alpha,n}[A_t^{(n)}(y_n,x)]-P_\alpha[A_t(y_n,x)]|>\delta>0,$$

then, either

$$\overline{\lim_{n\to\infty}}\, P_{\alpha,n}[A_t^{(n)}(y_n,x)\,\triangledown\, A_t(y,x)]>\frac{\delta}{2}, \tag{3.17}$$

or

$$\overline{\lim_{n'\to\infty}} P_\alpha[A_t(y_n, x) \triangledown A_t(y, x)] > \frac{\delta}{2}. \tag{3.18}$$

Defining

$$A_t(y) = [R_t(\xi) \cap \{-y\} \neq \Phi]$$

$$A_t^{(n)}(y) = [R_t(\xi) \cap \mathbf{A}_n \neq \Phi],$$

a straightforward argument gives

$$P_{\alpha,n}[A_t^{(n)}(y_n, x) \triangledown A_t(y, x)] \leqq P_{\alpha,n}[A_t^{(n)}(y_n) \triangledown A_t(y)] + P_{\alpha,n}[|\xi(t) - (x-y)| \leqq |y - y_n|].$$

Now by (3.16) and the estimates following it

$$\overline{\lim_{n'\to\infty}} P_{\alpha,n}[A_t^{(n)}(y_n) \triangledown A_t(y)] = 0;$$

while

$$P\left[\frac{S_{[nt]}}{B_n} \leqq x\right] \to P_\alpha[\xi(t) \leqq x]$$

as $n \to \infty$ uniformly in x guarantees that

$$\overline{\lim_{n'\to\infty}} P_{\alpha,n}[|\xi(t) - (x+y)| \leqq |y - y_n|] = 0.$$

Thus (3.17) is impossible. A simple argument using (3.9) shows that (3.18) is impossible as well.

Finally the continuity in y of $P_\alpha[A(y, x)]$ is an immediate consequence of the impossibility of (3.18).

This completes the proof of (3.5). Thus we have that for $1 < \alpha < 2$, $\hat{P}_\alpha$ is a probability measure and we have identified its finite dimensional distributions. We now dispense with the remaining cases.

Our assertions for $\alpha = 1$ are largely a consequence of the fact that $P_1^x[T > t] = 1$ for every x. The assertion in (3.2) thus reduces to showing

$$\lim_{n\to\infty} \hat{P}_{1,n} = P_1.$$

We proceed as in the case $\alpha > 1$.

First we note that

$$\chi_y(t, x+y, \xi) = \begin{cases} 1 & \text{if } \xi(t) \leqq x+y \\ 0 & \text{otherwise} \end{cases}$$

is almost surely J_1-continuous, so Skorokhod's theorem applies and

$$\lim_{n\to\infty} h_n(t_1, y; t_2, x) = P_1^y[\xi(t_2 - t_1) \leqq x]. \tag{3.19}$$

Finally $P_1^y[\xi(t) \leqq x]$ is continuous in y, so all that remains to be shown in order to justify the use of Lemma (3.1) is the uniformity of the convergence in (3.19) for y on compact sets. For this it suffices to show that if $y_n \to y$ as $n \to \infty$, then

$$\overline{\lim_{n'\to\infty}} P_{1,n}[T_{\{y_n\}} \leqq t] = 0. \tag{3.20}$$

However, by Lemma (3.2)

$$\overline{\lim_{n\to\infty}} P_{1,n}[T_{\{y_n\}} \leqq t] \leqq \overline{\lim_{n\to\infty}} P_{1,n}[T_{I(y,\varepsilon)} \leqq t] = P_1[T_{I(y,\varepsilon)} \leqq t] \to 0 \quad \text{as} \quad \varepsilon \to 0$$

since

$$P_1[y \in \bar{R}_t] = P_1[y \in R_t] = 0.$$

This proves (3.20) and therefore that $\lim_{n\to\infty} \hat{P}_{1,n} = P_1$. We have already remarked that $\lim_{n\to\infty} P_{1,n} = P_1$.

For $\alpha = 2$ and $y > 0$ we have by the sample path continuity of Brownian motion,

$$\begin{aligned} P_2^y[\xi(t_1) \in dx;\ T > t_1] &= P_2^y[\xi(t_1) \in dx;\ \min_{0 \leqq t \leqq t_1} \xi(t) > 0] \\ &= \frac{1}{\sqrt{2\pi t_1}} \left(\exp\left[-\frac{(x-y)^2}{2t_1}\right] - \exp\left[-\frac{(x+y)^2}{2t_1}\right]\right) dx \\ &\quad \text{for } x > 0. \end{aligned}$$

This last evaluation is a well-known result for Brownian motion which we simply quote (see e.g. 1.7 in Ito-McKean [6]). The expression given in the statement of the theorem for p, the transition density of $\hat{P}_2$, follows easily.

For the identification of the limit process we follow an approach of Ito and McKean in their treatment of Brownian excursions (see 2.9 in [6]). Let $X(t)$ for $t > 0$ be a standard Brownian motion and define τ_0, τ, Δ, $\tilde{\mathfrak{e}}(t)$, $\mathfrak{s}$, and $\mathfrak{e}(t)$ as in the statement of Theorem (3.1). Letting $C_0(\mathbf{R}_+^N)$ denote the continuous real-valued functions on $\prod_{i=1}^{N} \mathbf{R}_+$ which vanish at infinity (are zero outside some compact set), we choose

(1) $f_1 \in C_0(\mathbf{R}_+^N)$ and define $f_1(\tilde{\mathfrak{e}}) = f_1(\tilde{\mathfrak{e}}(t_1), \ldots, \tilde{\mathfrak{e}}(t_N))$ for some fixed choice of $0 < t_1 < \cdots < t_N \leqq 1$;

(2) $f_2 \in C_0(\mathbf{R}_+)$, with f_2 vanishing in some neighborhood of the origin.

(3) f_3 an arbitrary real-valued function on $\{-1, 1\}$.

Also let

$$g(t,x,y) = \frac{1}{\sqrt{2\pi t}} \left(\exp\left[-\frac{(x-y)^2}{2t}\right] - \exp\left[-\frac{(x+y)^2}{2t}\right]\right) \quad \text{for } xy > 0,\ 0 < t \leqq 1$$

$$g(t,x,y) = 0 \quad \text{otherwise;}$$

and

$$\tilde{p}(0,0;t,x) = \frac{2x}{t^{\frac{3}{2}}} \exp\left(-\frac{x^2}{2t}\right) N\left(\frac{x}{\sqrt{1-t}}\right) \quad \text{for } x > 0$$

$$\tilde{p}(t_1, x_1; t_2, x_2) = \begin{cases} g(t_2 - t_1, x_1, x_2)\, N\left(\dfrac{x_2}{\sqrt{1-t_2}}\right) \Big/ N\left(\dfrac{x_1}{\sqrt{1-t_1}}\right) \\ \qquad\qquad \text{for } x_1, x_2 > 0,\ 0 < t_1 < t_2 \leqq 1 \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

We compute $E[f_1(\tilde{\mathfrak{e}})f_2(\Delta)f_3(\mathfrak{s})]$ (the functions $\tilde{\mathfrak{e}}(t_1),\dots,\tilde{\mathfrak{e}}(t_N)$, Δ, and $\mathfrak{s}$ are Borel measurable on the Brownian paths).

$$\begin{aligned}
&E[f_1(\tilde{\mathfrak{e}})f_2(\Delta)f_3(\mathfrak{s})]\\
&=\lim_{n\to\infty}\sum_{i=0}^{\infty}E\left[f_1\left(\frac{|X(t_1 i 2^{-n}+(\tau_0-i2^{-n}))|}{\sqrt{i2^{-n}}},\dots,\frac{|X(t_N i 2^{-n}+(\tau_0-i2^{-n}))|}{\sqrt{i2^{-n}}}\right)\right.\\
&\quad\left.\cdot f_2\left(\frac{i}{2^n}\right)f_3(\mathfrak{s}X(\tau_0));\ \frac{i}{2^n}<\Delta\leqq\frac{i+1}{2^n}\right]\\
&=\tfrac{1}{2}[f_3(-1)+f_3(+1)]\\
&\quad\cdot\lim_{n\to\infty}\sum_{i=0}^{\infty}\int_0^{\infty}P\left[\left|X\left(\tau_0-\frac{i}{2^n}\right)\right|\in da;\ X(t)=0 \text{ for some } \frac{i}{2^n}<\tau_0-t\leqq\frac{i+1}{2^n}\right]f_2\left(\frac{i}{2^n}\right)\\
&\quad\cdot\int_{\mathbf{R}_+^N}g\left(t_1\frac{i}{2^n},a,x_1\right)g\left((t_2-t_1)\frac{i}{2^n},x_1,x_2\right)\dots g\left((t_N-t_{N-1})\frac{i}{2^n},x_{N-1},x_N\right)\\
&\quad\cdot f_1\left(\frac{x_1}{\sqrt{i2^{-n}}},\dots,\frac{x_N}{\sqrt{i2^{-n}}}\right)dx_1\,dx_2\dots dx_N\int_0^{\infty}g\left((1-t_N)\frac{i}{2^n},x_N,b\right)db.
\end{aligned}$$

Replacing x_k by $\sqrt{\frac{i}{2^n}}\,x_k$ for $1\leqq k\leqq N$, we obtain

$$\begin{aligned}
&\tfrac{1}{2}[f_3(-1)+f_3(+1)]\\
&\quad\cdot\lim_{n\to\infty}\sum_{i=0}^{\infty}\int_0^{\infty}P\left[\left|X\left(\tau_0-\frac{i}{2^n}\right)\right|\in da;\ X(t)=0 \text{ for some } \frac{i}{2^n}<\tau_0-t\leqq\frac{i+1}{2^n}\right]f_2\left(\frac{i}{2^n}\right)\\
&\quad\cdot\int_0^{\infty}g\left(\frac{i}{2^n},a,b\right)db\int_{\mathbf{R}_+^N}\tilde{p}(0,0;t_1,x_1)\,\tilde{p}(t_1,x_1;t_2,x_2)\dots\tilde{p}(t_{N-1},x_{N-1};t_N,x_N)\\
&\quad\cdot f_1(x_1,\dots,x_N)\,\gamma\left(\frac{i}{2^n},a,x_1\right)dx_1\,dx_2\dots dx_N,
\end{aligned}$$

where

$$\begin{aligned}
\gamma\left(\frac{i}{2^n},a,x_1\right)&=\frac{\sqrt{\frac{i}{2^n}}\,g\left(t_1\frac{i}{2^n},a,\sqrt{\frac{i}{2^n}}\,x_1\right)N\left(\frac{x_1}{\sqrt{1-t_1}}\right)}{\tilde{p}(0,0;t_1,x_1)\int_0^{\infty}g\left(\frac{i}{2^n},a,b\right)db}\\
&=\frac{1}{\sqrt{2\pi}}R\left(\frac{a x_1}{t_1\sqrt{i2^{-n}}}\right)\frac{a}{\sqrt{i2^{-n}}}\,\frac{1}{N\left(\frac{a}{\sqrt{i2^{-n}}}\right)}\exp\left(-\frac{a^2}{2t_1 i 2^{-n}}\right)
\end{aligned}$$

with $R(x)=\dfrac{\sinh x}{x}$.

It is easily verified that

$$\lim_{a\to 0}\gamma\left(\frac{i}{2^n},a,x_1\right)=1\quad\text{uniformly in}\quad\frac{i}{2^n}>\varepsilon>0\quad\text{and}\quad x_1\leqq K<\infty.$$

Thus, since f_1 was assumed to vanish outside some compact set in $\mathbf{R}_+^N$, and f_2 outside a compact in $\mathbf{R}_+$, we conclude:

$$\begin{aligned}
&E[f_1(\tilde{\mathfrak{e}})\, f_2(\Delta)\, f_3(\mathfrak{s})]\\
&\quad=\frac{1}{2}[f_3(-1)+f_3(+1)] \lim_{n\to\infty} \sum_{i=0}^{\infty} \int_0^{\infty} P\left[\left|X\left(\tau_0-\frac{i}{2^n}\right)\right|\in da;\right.\\
&\qquad\qquad\qquad\left. X(t)=0 \text{ for some } \frac{i}{2^n}<\tau_0-t\leqq\frac{i+1}{2^n}\right]\\
&\qquad\cdot\int_0^{\infty} g\left(\frac{i}{2^n}, a, b\right) db\, f_2\left(\frac{i}{2^n}\right) \int_{\mathbf{R}_+^N} \tilde{p}(0,0;t_1,x_1)\,\tilde{p}(t_1,x_1;t_2,x_2)\cdots\\
&\qquad\cdot\tilde{p}(t_{N-1},x_{N-1};t_N,x_N)\, f_1(x_1,\dots,x_N)\, dx_1\dots dx_N\\
&\quad=\tfrac{1}{2}[f_3(-1)+f_3(+1)]\, E[f_2(\Delta)] \int_{\mathbf{R}_+^N} \tilde{p}(0,0;t_1,x_1)\,\tilde{p}(t_1,x_1;t_2,x_2)\cdots\\
&\qquad\cdot\tilde{p}(t_{N-1},x_{N-1};t_N,x_N)\, f_1(x_1,\dots,x_N)\, dx_1\dots dx_N.
\end{aligned}\tag{3.21}$$

This shows that $\mathfrak{s}$ is a standard coin tossing game which is independent of $\tilde{\mathfrak{e}}(t)$, and that the latter is a Markov process with transition density $\tilde{p}$. From this it follows that $\mathfrak{e}(t)$ is a Markov process with the transition density p given in the statement of Theorem (3.1).

This completes the proof of the convergence of the finite dimensional distributions of $\hat{P}_{\alpha,n}$ to the correct limits for $1\leqq\alpha\leqq 2$ and thus the proof of Theorem (3.1).

We conclude with several remarks.

4. Remarks

The question arises with regard to Theorem (3.1) whether the choice of the family of right continuous step functions as the support of the measures $\hat{P}_{\alpha,n}$ was really essential to the proof. In particular suppose we had defined sequences of probability measures $\Pi_{\alpha,n}$ and $\hat{\Pi}_{\alpha,n}$ by assigning the probabilities in (2.6) and (2.7) respectively, to the polygonal line segment ξ with vertices at k/n, $k=0,1,\dots,n$ such that $\xi(0)=0$ and $\xi(k/n)=x_k$ for $1\leqq k\leqq n$. Then, having already demonstrated the convergence of the finite dimensional distributions of the $\hat{P}_{\alpha,n}$ to those of the $\hat{P}_\alpha$, it is an easy matter to show that the finite dimensional distributions of the $\hat{\Pi}_{\alpha,n}$ must have the same limits. However, one finds that for $1\leqq\alpha<2$ condition (ii) of Skorokhod's theorem must fail. One way to see this is to note that the almost surely $[\hat{P}_\alpha]$ J_1-continuous functional K_ε for $\varepsilon>0$ defined by

$$K_\varepsilon(\xi)=\text{the number of jumps of } \xi \text{ of magnitude exceeding } \varepsilon,$$

fails to satisfy the conclusion of Skorokhod's theorem since the sample paths of the $\hat{P}_\alpha$ processes almost surely have jumps greater than ε.

For $\alpha=2$, however, condition (ii) is satisfied, and in view of the remark immediately following the statement of Skorokhod's theorem, we see that with respect to the $\hat{\Pi}_{2,n}$ and $\hat{P}_2$ the conclusion of Theorem (3.1) holds for functionals on $C[0,1]$ continuous almost surely in the topology of uniform convergence.

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