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Characterizing the Gaussian and Exponential Laws via Mappings onto the Unit Interval*

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1. Introduction

Let $(\mathscr{X}, \mathscr{A})$ be a measurable space and let $\mathscr{P}$ be a set of probability distributions defined on $\mathscr{A}$. Let $(\mathscr{Y}, \mathscr{B})$ be another measurable space and let $Y = T(X)$, $X \in \mathscr{X}$, be a measurable mapping of $(\mathscr{X}, \mathscr{A})$ onto $(\mathscr{Y}, \mathscr{B})$. With this mapping every distribution $P \in \mathscr{P}$ induces on $\mathscr{B}$ a corresponding distribution which will be denoted by Q_P^Y.

In general we will be interested in the mappings Y which satisfy the following two properties:

$$Q_P^Y \text{ is the same for all } P \in \mathscr{P}\text{; in this case we write } Q_{\mathscr{P}}^Y. \tag{1.1}$$

$$\text{If for some } P' \text{ on } \mathscr{A} \text{ one has } Q_{P'}^Y = Q_{\mathscr{P}}^Y, \text{ then } P' \in \mathscr{P}. \tag{1.2}$$

In this paper $(\mathscr{X}, \mathscr{A})$ is an n-dimensional Euclidean space of points $X = (X_1, \ldots, X_n)$ with the σ-algebra of Borel sets and the distributions belonging to $\mathscr{P}$ have a product probability density

$$f(x_1, \theta) f(x_2, \theta) \ldots f(x_n, \theta), \tag{1.3}$$

where f is a one-dimensional density function and θ is a parameter taking values in an appropriate parameter space. Given some density function of the form as in (1.3) with $f(.,.)$ specified, in particular we will be interested in mappings Y satisfying (1.1) and (1.2) in such a way that the induced distribution $Q_{\mathscr{P}}^Y$ will be that of k, $k < n$, ordered random variables of k independent uniformly distributed random variables on $[0, 1]$. In Section 2 $f(.,.)$ of (1.3) is the exponential density with various spaces for θ. In Section 3 the product density of (1.3) is the finite-dimensional family of density functions of the Poisson Process. In Section 4 we deal with gamma densities of order $1/n$ and in Section 5 $f(.,.)$ of (1.3) is the normal family.

From the point of view of statistics our statements can be used to replace composite statistical hypotheses by equivalent simple ones in the spirit of Prohorov's paper [4]. As regards to this point of view we will shortly publish another paper, in the Review of the International Statistical Institute, which will be based on the results of this paper and will also contain some further results satisfying the properties (1.1) and (1.2).

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2. Preliminaries. Some Characteristic Properties of the Exponential Law

We start with two theorems, which will play a fundamental role in the sequel, and which the two of us first proved in [6].

We consider the exponential families of density functions

$$f(x)=B^{-1}\exp(-x/B), \quad x>0,\ B>0, \tag{2.1}$$

or more generally

$$f(x)=B^{-1}\exp\big(-(x-A)/B\big), \quad x>A,\ B>0, \tag{2.2}$$

respectively written as Exp(0, B) and Exp(A, B) from now on.

Also, the uniform distribution on [0, 1] will be denoted by $U(0, 1)$ from now on.

Theorem 1. *Let $X_1, X_2, \ldots, X_n$, $n\geqq 3$, be independent identically distributed positive random variables with continuous density function and mean $B>0$. Let $S=\Sigma_1^n X_i$ and define $Z_r=\Sigma_1^r X_i/S$, $r=1, 2, \ldots, n-1$. Then the Z_r act like $(n-1)$ order statistics of $(n-1)$ independent random variables from $U(0, 1)$ if and only if the X_i are* Exp(0, B).

Later on, and also to handle the exponential family of distribution functions Exp(A, B) in a similar manner, we will need the following well known lemma, whose proof can also be found in Section 7 of [1].

Lemma 1. *Let $X_1, X_2, \ldots, X_n$ be independent identically distributed positive random variables. Define $\delta_i=(n+1-i)(X_{(i)}-X_{(i-1)})$, $i=1, 2, \ldots, n$, where $X_{(0)}=0$ and $X_{(1)}<X_{(2)}<\cdots<X_{(n)}$ are the order statistics of our corresponding random sample. Then the δ_i are independent* Exp(0, B) *random variables if and only if the X_i are* Exp(0, B) *random variables.*

The extension to X_i when they are Exp(A, B), and the construction from δ_i of a set Z_i^* which acts like an ordered set of $U(0, 1)$ random variables is given in Theorem 2.

Theorem 2. *Let $X_1, X_2, \ldots, X_n$, $n\geqq 3$, be independent identically distributed random variables with $X_i>A$ for all i. Define*

$$\delta_i=(n+1-i)(X_{(i)}-X_{(i-1)}), \quad i=1, 2, \ldots, n, \quad \textit{where } X_{(0)}=A$$

and $X_{(i)}$ are the order statistics of X_i. Further let $A_i=\delta_i/S(\delta)$, $i=1, 2, \ldots, n$, where $S(\delta)=\Sigma_1^n \delta_i$, and let $Z_r^*=\Sigma_1^r A_i$, $r=1, 2, \ldots, n-1$. Then the Z_r^* act like $(n-1)$ order statistics of $(n-1)$ independent $U(0, 1)$ random variables if and only if the X_i are Exp(A, B).

3. A Characterization of the Poisson Process

As an immediate consequence of the results of Section 2 we have the following two characterizations of the Poisson process.

Let $\{X(t), t\in T\}$ be a real, stationary (strict sense), independent increment stochastic process. Let $T=[0, +\infty)$, and define $X(0)=0$ with probability 1.

Theorem 3. *Let $X(t)$ represent the number of events occuring in the interval $[0, t)$ and suppose that n events have occured at $t_1<t_2<\cdots<t_n$, where we take $t_n=t$. Let s_1 be the time to the first event and for $i>1$ let s_i be the time between the $(i-1)$-th* *and the i-th event. Let* $t = s_1 + s_2 + \cdots + s_n$ *and define* $\tau_r = \Sigma_1^r s_i/t$, $r = 1, 2, \ldots, n-1$, $n \geqq 3$. *Then* $X(t)$ *is a Poisson process with mean* λt *if and only if the* τ_r *act like* $(n-1)$ *order statistics of* $(n-1)$ *independent* $U(0, 1)$ *random variables.*

Proof. $\{X(t), t \in [0, +\infty)\}$ is a Poisson process with mean λt if and only if the s_i are mutually independent $\operatorname{Exp}(0, 1/\lambda)$ random variables and Theorem 1 gives the s_i mutually independent $\operatorname{Exp}(0, 1/\lambda)$ random variables if and only if τ_r act like $(n-1)$ ordered independent $U(0, 1)$ random variables.

A Remark. Theorem 3 is a special case of *K.* Nawrotzki's characterization theorem of the mixed Poisson process (see Satz 4 of his paper, Ein Grenzwertsatz für homogene zufällige Punktfolgen (Verallgemeinerung eines Satzes von A. Rényi), Math. Nachrichten **24**, 201–217 (1962)).

Theorem 4. *Let* $X(t)$ *represent the number of events occuring in the interval* $[0, t)$ *and suppose that n events occured at* $t_1 < t_2 < \cdots < t_n = t$. *Let* s_i, $i = 1, \ldots, n$, *be as in Theorem* 3. *Define* $T_i = (n+1-i)(s_{(i)} - s_{(i-1)})$, $i = 1, 2, \ldots, n$, *where* $s_{(0)} = 0$ *and* $s_{(i)}$ *are the order statistics of* s_i. *Further let* $\tau_r^* = \Sigma_1^r T_i/S(T)$, *where* $S(T) = \Sigma_1^n T_i$ *and* $r = 1, 2, \ldots, n-1$, $n \geqq 3$. *Then the* τ_r^* *act like* $(n-1)$ *order statistics of* $(n-1)$ *independent* $U(0, 1)$ *random variables if and only if* $X(t)$ *is a Poisson process with mean* λt.

Proof. $\{X(t), t \in [0, +\infty)\}$ is a Poisson process with mean λt if and only if the s_i are mutually independent $\operatorname{Exp}(0, 1/\lambda)$ random variables. Lemma 1 gives the s_i mutually independent $\operatorname{Exp}(0, 1/\lambda)$ if and only if the T_i are mutually independent $\operatorname{Exp}(0, 1/\lambda)$. Theorem 1 gives the T_i mutually independent $\operatorname{Exp}(0, 1/\lambda)$ if and only if the τ_r^* act like $(n-1)$ ordered independent $U(0, 1)$ random variables.

4. Characterizations of the Gamma Distribution of Order $1/n$

First consider

$$f(x) = (B\pi)^{-\frac{1}{2}} x^{\frac{1}{2}-1} \exp(-x/B), \qquad x > 0, B > 0, \tag{4.1}$$

or more generally

$$f(x) = (B\pi)^{-\frac{1}{2}} (x-A)^{\frac{1}{2}-1} \exp(-(x-A)/B), \qquad x > A,\ B > 0, \tag{4.2}$$

respectively written as $\Gamma(\frac{1}{2}, 0, B)$ and $\Gamma(\frac{1}{2}, A, B)$ from now on.

The following lemma will be needed here in addition to the theory of Section 2.

Lemma 2. *Let* $X_1, X_2, \ldots, X_n$, *n even* $\geqq 2$, *be independent identically distributed positive random variables. Let* $Y_1, Y_2, \ldots, Y_k$ $k = n/2$, *be defined as* $Y_i = X_{2i-1} + X_{2i}$, $i = 1, 2, \ldots, k$. *Then the* Y_i *are* $\operatorname{Exp}(0, B)$ *random variables if and only if the* X_i *are* $\Gamma(\frac{1}{2}, 0, B)$ *random variables.*

The *Proof* of this statement is trivial.

Consequently, Theorem 1, Lemma 1 and Theorem 2 of Section 2 hold for the Y_i of Lemma 2. Putting Lemma 2 and Theorem 1 together we get

Theorem 5. *Let* $X_1, X_2, \ldots, X_n$, *n even* $\geqq 6$, *independent identically distributed positive random variables. Let* $Y_i = X_{2i-1} + X_{2i}$, $i = 1, 2, \ldots, k$, *with* $k = n/2$. *Further let* $S_k = \Sigma_1^k Y_i$ *and define* $\zeta_r = \Sigma_1^r Y_i/S_k$, $r = 1, 2, \ldots, k-1$. *Then the* ζ_r *act like* $(k-1)$ *order statistics of* $(k-1)$ *independent random variables from* $U(0, 1)$ *if and only if the* X_i *are* $\Gamma(\frac{1}{2}, 0, B)$.

Putting Lemmas 1, 2 and Theorem 1 together we have

Theorem 6. *Let $X_1, X_2, \ldots, X_n$, n even $\geqq 6$, be independent identically distributed random variables with $X_i > A$ for all i. Let $Y_i = Y_{2i-1} + X_{2i}$, $i = 1, 2, \ldots, k$, with $k = n/2$. Define*

$$d_i = (k+1-i)(Y_{(i)} - Y_{(i-1)}), \qquad i = 1, 2, \ldots, k,$$

where $Y_{(0)} = 2A$ and $Y_{(i)}$ are the order statistics of Y_i. Further define $K_i = d_i/S(d)$, $i = 1, 2, \ldots, k$, where $S(d) = \Sigma_1^k d_i$, and let $\zeta_r^* = \Sigma_1^r K_i$, $r = 1, 2, \ldots, k-1$. Then the ζ_r^* act like $(k-1)$ order statistics of $(k-1)$ independent $U(0,1)$ random variables if and only if the S_i are $\Gamma(\frac{1}{2}, A, B)$.

Proof. Let $W_i = X_i - A$, $i = 1, 2, \ldots, n$, and let $M_i = W_{2i-1} + W_{2i}$, $i = 1, 2, \ldots, k$, with $k = n/2$. Then $M_i > 0$, all i, and $d_i = (k+1-i)\,(M_{(i)} - M_{(i-1)})$, with $M_{(0)} = 0$, $i = 1, 2, \ldots, k$. Let $N_i = d_i/B$, $i = 1, 2, \ldots, k$; then the values of K_i and ζ_r^*, defined in terms of N_i instead of d_i, are not changed. Now Lemma 2 implies that the M_i are independent $\mathrm{Exp}(0, B)$ if and only if the W_i are $\Gamma(\frac{1}{2}, 0, B)$, that is if and only if X_i are $\Gamma(\frac{1}{2}, A, B)$. Lemma 1 gives the d_i independent $\mathrm{Exp}(0, B)$ and, consequently, the N_i independent $\mathrm{Exp}(0, 1)$ if and only if the M_i are independent $\mathrm{Exp}(0, B)$. Applying now Theorem 1 we get that the ζ_r^* act like $(k-1)$ ordered independent $U(0,1)$ random variables if and only if the d_i are $\mathrm{Exp}(0, B)$. This completes the proof of Theorem 6.

These results can be immediately generalized to hold for the $\Gamma(1/n, 0, B)$ and $\Gamma(1/n, A, B,)$ families, with $n \geqq 2$. Namely we have the following corollaries.

Corollary 1. *Let $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$, $i = 1, 2, \ldots, k$, $k \geqq 3$, $n \geqq 2$, be k independent sets of n independent positive random variables. Let $Y_i = X_{i,1} + X_{i,2} + \cdots + X_{i,n}$, $i = 1, \ldots, k$. Further let $S_k = \Sigma_1^k Y_i$ and define $\zeta_r = \Sigma_1^r Y_i/S_k$, $r = 1, 2, \ldots, k-1$. Then the ζ_r act like $(k-1)$ order statistics of $(k-1)$ independent $U(0,1)$ random variables if and only if the $X_{i,j}$, $j = 1, \ldots, n$; $i = 1, \ldots, k$ are $\Gamma(1/n, 0, B)$.*

Corollary 2. *Let $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$, $i = 1, 2, \ldots, k$, $k \geqq 3$, $n \geqq 2$, be k independent sets of n independent random variables with $X_{i,j} > A$, $j = 1, 2, \ldots, n$; $i = 1, 2, \ldots, k$. Let Y_i be as in Corollary 1 and define*

$$d_i = (k+1-i)(Y_{(i)} - Y_{(i-1)}), \qquad i = 1, 2, \ldots, k,$$

where $Y_{(0)} = nA$ and $Y_{(i)}$ are order statistics of Y_i. Further define $k_i = d_i/S(d)$, $i = 1, 2, \ldots, k$, where $S(d) = \Sigma_1^k d_i$, and let $\zeta_r^* = \Sigma_1^r K_i$, $r = 1, 2, \ldots, k-1$. Then the ζ_r^* act like $(k-1)$ order statistics of $(k-1)$ independent $U(0,1)$ random variables if and only if the $X_{i,j}$, $j = 1, 2, \ldots, n$; $i = 1, 2, \ldots, k$, are $\Gamma(1/n, A, B)$.

All one needs to prove these theorems is a restatement of Lemma 2 for k independent sets of n independent positive random variables, $n \geqq 2$, for the $\Gamma(1/n, 0, B)$ case.

5. Characterizations of the Normal Law

In order to characterize the normal distribution via measurable mappings onto the unit interval we will only need the following simple lemma in addition to the statements of Sections 2 and 4.

Lemma 3. *Let X be a random variable. If X has the same density as $-X$, and if X^2 is a chi-square one (written as χ_1^2 from now on) random variable, then X is normal with zero mean and unit variance.*

The *Proof* of this lemma follows immediately from the assumed equality of the densities $f(x)$ and $f(-x)$ and the transformation $y=x^2$.

Theorem 7. *Let $X_1, X_2, \ldots, X_n$, $n=2k+3$, $k \geqq 2$, be independent identically distributed random variables with mean μ and variance σ^2, $-\infty<\mu<+\infty$, $\sigma>0$. Let*

$$Z_1=(X_1-X_2)/\sqrt{2},\ Z_2=(X_1+X_2-2X_3)/\sqrt{6},\ \ldots,$$
$$Z_{n-1}=(X_1+X_2+\cdots+X_{n-1}-(n-1)X_n)/\sqrt{n(n-1)},\ Z_n=(X_1+\cdots+X_n)/\sqrt{n},$$

and define

$$Y_1=Z_1^2+Z_2^2,\ Y_2=Z_3^2+Z_4^2,\ \ldots,\ Y_{k+1}=Z_{n-2}^2+Z_{n-1}^2, \quad k+1=\frac{n-1}{2}.$$

Further let $S_{k+1}=\Sigma_1^{k+1} Y_i$ and define $\eta_r^*=\Sigma_1^r Y_i/S_{k+1}$, $r=1, 2, \ldots, k$. Then the η_r^* act like k order statistics of k independent $U(0, 1)$ random variables if and only if the X_i are $N(\mu, \sigma^2)$.

Proof. The random variables η_r^* are independent of σ^2, and we can therefore assume $\sigma^2=1$, without loss of generality. From the theory of normal distribution it follows that $Z_i, i=1, 2, \ldots, n$, and therefore $Z_i, i=1, 2, \ldots, n-1$, are independent normal random variables if and only if the X_i are normal. Next we show that the Z_i, $i=1, 2, \ldots, n-1$, are independent $N(0, 1)$ if and only if the Z_i^2, $i, 1, 2, \ldots, n-1$, are χ_1^2. To see this, we consider Z_1. We know that Z_1 has the same distribution as $-Z_1$ by construction. By assumption Z_1^2 is χ_1^2 and Lemma 3 gives $(X_1-X_2)/\sqrt{2}$ as $N(0, 1)$ and Cramér's theorem implies that X_1 and X_2 are $N(\mu, 1)$. Consequently our basic assumption on the X_i implies that all of them are $N(\mu, 1)$ and therefore all the Z_i are independent $N(0, 1)$, where independence of Z_i is implied by the orthogonal transformation of X_i to Z_i. An application of Lemma 2 gives the Y_i Exp$(0, 2)$ if and only if the Z_i^2 are χ_1^2, and it follows from Theorem 1 that the η_r^* act like k ordered independent $U(0, 1)$ random variables if and only if the Y_i are Exp$(0, 2)$. This completes the proof of Theorem 7.

In case we can assume the mean to be known, the first transformation of Theorem 7 is not necessary and in such situations the following theorem is applicable.

Theorem 8. *Let $X_1, X_2, \ldots, X_n$, $n=2k$, $k \geqq 3$, be independent identically distributed random variables with known mean μ and unknown variance σ^2, $-\infty<\mu<+\infty$, $\sigma>0$ and assume also that the X_i have symmetric distribution about μ. Let*

$$Z_1=X_1-\mu,\ Z_2=X_2-\mu,\ \ldots,\ Z_n=X_n-\mu$$

and define

$$Y_1=Z_1^2+Z_2^2,\ Y_2=Z_3^2+Z_4^2,\ \ldots,\ Y_k=Z_{n-1}^2+Z_n^2, \quad k=n/2.$$

Further let $S_k=\Sigma_1^k Y_i$ and define $\eta_r=\Sigma_1^r Y_i/S_k$, $r=1, 2, \ldots, k-1$. Then the η_r act like $(k-1)$ order statistics of $(k-1)$ independent $U(0, 1)$ random variables if and only if the X_i are $N(\mu, \sigma^2)$.

Proof. The η_r being independent of σ^2, we can again assume $\sigma^2=1$, without loss of generality. As a result of our symmetry assumption on the X_i, the Z_i have symmetric distribution about zero, and Lemma 3 gives the Z_i^2 as independent χ_1^2 random variables if and only if the Z_i are $N(0,1)$, that is if and only if the X_i are $N(\mu, 1)$. The rest of the proof is the same as that of Theorem 7.

The method of this paper, that is characterisation of families of distributions via measurable mappings onto the unit interval has a wide scope of application. The next two statements are given here to further demonstrate this fact.

Corollary 3. *Let $X_1, X_2, \ldots, X_n$, $n \geqq 3$ be independent identically distributed positive random variables with finite mean and an absolutely continuous distribution function. Let $Y_1 = X_1^2, \ldots, Y_n = X_n^2$, write $S_n = \Sigma_1^n Y_i$ and define $\eta_r = \Sigma_1^r Y_i/S_n$, $r = 1, 2, \ldots, n-1$. Then the η_r act like $(n-1)$ order statistics of $(n-1)$ independent $U(0,1)$ random variables if and only if the X_i have the Weibull density*

$$(2X)\, B^{-1} \exp(-x^2/B), \qquad x>0,\ B>0,$$

written as $W(2,B)$ from now on.

Proof. From straightforward change of variable technique we get that the X_i are $W(2,B)$ if and only if the Y_i are $\operatorname{Exp}(0,B)$, and Theorem 1 gives the Y_i $\operatorname{Exp}(0,B)$ if and only if the η_r act like $(n-1)$ ordered independent $U(0,1)$ random variables.

Corollary 4. *Let $X_1, X_2, \ldots, X_n$, $n = 2k+3$, $k \geqq 2$, be independent identically distributed positive random variables with finite mean and variance and an absolutely continuous distribution function. Let $Y_1 = \log X_1, \ldots, Y_n = \log X_n$.*

Let

$$Z_1 = (Y_1 - Y_2)/\sqrt{2},\ Z_2 = (Y_1 + Y_2 - 2Y_3)/\sqrt{6},\ \ldots,$$
$$Z_{n-1} = (Y_1 + \cdots + Y_{n-1} - (n-1)\, Y_n)/\sqrt{n(n-1)},\ Z_n = (Y_1 + \cdots + Y_n)/\sqrt{n},$$

and define

$$V_1 = Z_1^2 + Z_2^2,\ V_2 = Z_3^2 + Z_4^2,\ \ldots,\ V_{k+1} = Z_{n-2}^2 + Z_{n-1}^2, \qquad k+1 = \frac{n-1}{2}.$$

Further let $S_{k+1} = \Sigma_1^{k+1} V_i$ and define $\eta_r^* = \Sigma_1^r V_i/S_{k+1}$, $r = 1, 2, \ldots, k$. Then the η_r^* act like k order statistics of k independent $U(0,1)$ random variables if and only if the X_i are distributed according to the logarithmic normal law, that is for $x>0$ their density function is

$$x^{-1}(2\pi\sigma^2)^{-\frac{1}{2}} \exp(-1/2\,\sigma^2(\log x - \mu))^2, \qquad -\infty < \mu + \infty,\ \sigma > 0. \tag{5.1}$$

Proof. The Y_i are $N(\mu, \sigma^2)$ if and only if the X_i have density as in (5.1). The rest of the proof is identical to that of Theorem 7.

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