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A Class of Games that Evolve

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1. In many real life situations which might be described in terms of extensive games from a theoretical point of view, the payoff is split in each step of the game but not in the terminal part. This is what happens in many parts of the economy. In these situations the standard description for extensive games seems inadequate since the players have to use foresight at each step.

Here we intend to introduce a brief description of such games, together with some remarks about them.

2. Let us consider a finite set of natural numbers $T=\{1,2,\ldots,n\}$ which describes the set of *discrete times* during which the process will be carried on. For the sake of simplicity, we consider only the case of two players $N=\{1,2\}$ and we recursively define the following kind of two-person game, which will be called briefly a two-person *evolutionary game*, by

$$\Gamma_t(\sigma(1,\ldots,t-1))=\{\Sigma_{1,t}(\sigma(1,\ldots,t-1)),\Sigma_{2,t}(\sigma(1,\ldots,t-1)); \\ A_{1,t}(\sigma(1,\ldots,t-1))(\cdot,\cdot),A_{2,t}(\sigma(1,\ldots,t-1))(\cdot,\cdot)\}$$

for $t\in T$. The set of strategies available to player $i\in N$ at time $t\in T$: $\Sigma_{i,t}(\sigma(1,\ldots,t-1))$ is a non-empty subset of a euclidean space $R^{n_{i,t}}$, which depends on the previous choices $\sigma(1,\ldots,t-1)=(\sigma(1),\ldots,\sigma(t-1))$. For $s\leqq n$

$$\sigma(s)=(\sigma_1(s),\sigma_2(s))\in\Sigma_s(\sigma(1,\ldots,s-1))=\Sigma_{1,s}(\sigma(1,\ldots,s-1))\times\Sigma_{2,s}(\sigma(1,\ldots,s-1)).$$

The payoffs

$$A_{i,t}(\sigma(1,\ldots,t-1)):\ \Sigma_t(\sigma(1,\ldots,t-1))\to R,$$

which also depend upon the strategies just chosen, are real functions.

In the following, we assume, for the sake of simplicity, that the multivalued functions $\Sigma_{i,t}$ have convex and compact images and are both lower and upper-semicontinuous. This lower and upper-semicontinuity is with respect to the variable $\sigma(1,\ldots,t-1)$ in the graph $\mathscr{I}_{\Sigma_{t-1}}$ of those strategies for which $\sigma(s)\in\Sigma_s(\sigma(1,\ldots,s-1))$ for all $s\leqq t$. Furthermore, the payoffs are continuous on the corresponding graph. Also for notational ease, we drop the arguments.

We are concerned with the simplest case of *zero-sum* games, when all the payoffs of the second player are $A_{2,t}=-A_{1,t}$.

At each time the players are in a conflict of interest situation in which the intensity of interaction is measured by the payoffs. Now, at a given time the players

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have not only to take into account the possible amount of winning or losing at the present juncture but also the strategical possibilities in the *future*. Otherwise, what appears to be a good action in the present may result in a disadvantageous position at some subsequent time. Indeed, this observation about foresight arises from a common form of behavior in real life. But, of course, such foresight is nothing more than the estimation of the importance of the unit of the gain in future times seen at the present time. We measure it at time t for the player $i \in N$ by

$$\lambda_{i,t} = (\lambda_{i,t}^0, \ldots, \lambda_{i,t}^{n-t}) \geqq 0 \quad \text{with} \quad \sum_{k=0}^{n-t} \lambda_{i,t}^k = 1.$$

In other words, $\lambda_{i,t}^k$ is the value of time t of the unit of payoff at time $t+k$ for the player i. The λ's are personal criteria and outside the formal rules of the games. We call them the *weights* at time t.

Under the weight $\lambda_{i,t}$ at time t, it is reasonable to believe that the safe level for the first player is given by

$$\begin{aligned}
v_{1,t} = & \max_{\sigma_1(t)\in\Sigma_{1,t}} \min_{\sigma_2(t)\in\Sigma_{2,t}} \{\lambda_{1,t}^0 A_{1,t}(\sigma_1(t), \sigma_2(t)) \\
+ & \max_{\sigma_1(t+1)\in\Sigma_{1,t+1}(\sigma(t))} \min_{\sigma_2(t+1)\in\Sigma_{2,t+1}(\sigma(t))} \{\lambda_{1,t}^1 A_{1,t+1}(\sigma(t))(\sigma_1(t+1), \sigma_2(t+1)) \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
+ & \max_{\sigma_1(n-1)\in\Sigma_{1,n-1}(\sigma(t,\ldots,n-2))} \min_{\sigma_2(t-1)\in\Sigma_{2,n-1}(\sigma(t,\ldots,n-2))} \{\lambda_{1,t}^{n-t-1} A_{1,n-1}(\sigma(t, \ldots, n-2)) \\
& \cdot(\sigma_1(n-1), \sigma_2(n-1)) \\
+ & \max_{\sigma_1(n)\in\Sigma_{1,n}(\sigma(1,\ldots,n-1))} \min_{\sigma_2(n)\in\Sigma_{2,n}(\sigma(1,\ldots,n-1))} \lambda_{1,t}^{n-t} A_{1,n}(\sigma(1, \ldots, n-1))(\sigma_1(n), \sigma_2(n))\} \cdots\}
\end{aligned}$$

or more concisely

$$\begin{aligned}
v_{1,t} &= \max_{\sigma_1(t)\in\Sigma_{1,t}} \min_{\sigma_2(t)\in\Sigma_{2,t}} \{\lambda_{1,t}^0 A_{1,t}(\sigma_1(t), \sigma_2(t)) + v_{1,t}^1(\sigma_1(t), \sigma_2(t))\} \\
&= \max_{\sigma_1(t)\in\Sigma_{1,t}} \min_{\sigma_2(t)\in\Sigma_{2,t}} \tilde{v}_{1,1}^1(\sigma_1(t), \sigma_2(t)),
\end{aligned}$$

where the meaning of the expression $v_{1,t}^1$ and $\tilde{v}_{1,1}^1$ is clear.

We notice that in virtue of the assumptions made on the strategy sets and the payoff functions, the expressions just considered are well defined.

We want to show that indeed this value is the safe value in the t-th step of the game. In order to verify this we have to introduce the game in which our player finds himself at time $t \in T$, under his own weights $\lambda_{1,t}$. Such a zero-sum game is given by

$$\Lambda_t^1 = \{C_{1,t}, C_{2,t}; \mathscr{C}_{1,t}(\cdot,\cdot) = \sum_{s=0}^{n-t} \lambda_{1,t}^k B_{i,t+s}(\cdot,\cdot)\}$$

where the strategy spaces are the cartesian products

$$\begin{aligned}
C_{i,t} &= C_{i,t}(\sigma(1, \ldots, t-1)) \\
&= \Sigma_{1,t}(\sigma(1, \ldots, t-1)) \times \Sigma_{i,t,t+1}^*(\sigma(1, \ldots, t-1)) \times \cdots \times \Sigma_{i,t,n}^*(\sigma(1, \ldots, t-1)).
\end{aligned}$$

The sets Σ^* are defined recursively as follows: the first one $\Sigma_{i,t,t+1}^*$ is the set of all the functions $f_{i,t,t+1}$ defined on Σ_t such that $f_{i,t,t+1}(\sigma(t)) \in \Sigma_{i,t+1}(\sigma(1, \ldots, t))$. If

$f_{t,t+k}$ denotes the pair of functions, $(f_{1,t,t+k}, f_{2,t,t+k})$, then the k-th set $\Sigma^*_{i,t,t+k}$ is the set of all the functions $f_{i,t,t+k}$ which assign to the vector $(\sigma(t), f_{t,t+1}, \ldots, f_{t,t+k-1})$ an element

$$\begin{aligned} &f_{i,t,t+k}(\sigma(t), f_{t,t+1}, \ldots, f_{t,t+k-1}) \\ &\quad \in \Sigma_{i,t+k}\big(\sigma(1, \ldots, t-1), \sigma(t), f_{t,t+1}(\sigma(t)), f_{t,t+2}(\sigma(t), f_{t,t+1}(\sigma(t))), \ldots, \\ &\quad f_{t,t+k-1}(\sigma(t), f_{t,t+1}(\sigma(t)), \ldots, f_{t,t+k-2}(\sigma(t), f_{t,t+1}(\sigma(t), \ldots)))\big). \end{aligned}$$

Similarly, the modified payoffs $B_{i,t+s}$ are defined on $\Sigma_t \times \underset{r \leqq s}{\times} \Sigma^*_{t,t+s}$ and are given by

$$\begin{aligned} B_{i,t+s}(\sigma(t), f_{t,t+1}, \ldots, f_{t,t+s}) &= A_{i,t+s}\big(\sigma(t), f_{t,t+1}(\sigma(t)), f_{t,t+2}(\sigma(t), f_{t,t+1}(\sigma(t))), \\ &\ldots, f_{t,t+s}(\sigma(t), f_{t,t+1}(\sigma(t)), \ldots, f_{t,t+s-1}(\sigma(t), f_{t,t+1}(\sigma(t), \ldots)))\big). \end{aligned}$$

From this, we see that the only functions in the sum payoff of the new game that actually determine a variation with respect to $f_{t,t+s}, \ldots, f_{t,n}$ are the last $n-(s+t)+1$ of them; that is, $B_{i,t+s}, \ldots, B_{i,n}$. Then, it follows at once that with $i \neq j$:

$$\begin{aligned} &\min_{(\sigma_j(t), f_{j,t,t+1}, \ldots, f_{j,t,n}) \in C_{j,t}} \mathscr{C}_{i,t}(\sigma_1(t), f_{i,t,t+1}, \ldots, f_{i,t,n}, \sigma_j(t), f_{j,t,t+1}, \ldots, f_{j,t,n}) \\ &\quad = \min_{\sigma_j(t) \in \Sigma_{j,t}} \{\lambda^0_{1,t} B_{i,t}(\sigma_i(t), \sigma_j(t)) \\ &\quad + \min_{f_{j,t,t+1} \in \Sigma^*_{j,t,t+1}} \{\lambda^1_{i,t} B_{i,t}(\sigma_i(t), f_{i,t,t+1}, f_{j,t,t+1}) \\ &\quad + \cdots + \\ &\quad + \min_{f_{j,t,n} \in \Sigma^*_{j,t,n}} \{\lambda^{n-t}_{i,t} B_{i,n}(\sigma(t), f_{t,t+1}, \ldots, f_{i,t,n}, f_{j,t,n})\} \cdots\}. \end{aligned}$$

On the other hand, from the relation between the B's and A's the last number equals the minimum of the primitive payoff function

$$\begin{aligned} &A_{i,n}(\sigma(t), f_{t,t+1}(\sigma(t)), \ldots, f_{t,n-1}(\sigma(t), f_{t,t+1}(\sigma(t)), \\ &\quad \ldots, f_{t,n-2}(\sigma(t), \ldots,)), f_{i,t,n}(\sigma(t), f_{t,t+1}(\sigma(t)), \ldots, f_{t,n-1}(\sigma(t), f_{t,t+1}(\sigma(t)), \\ &\quad \ldots, f_{t,n-2}(\sigma(t), \ldots)), \sigma_j(n)\big), \end{aligned}$$

on the variable $\sigma_j(n)$. In a similar fashion, by applying analogous arguments to the previous minima and after taking the maximum on the set $C_{1,t}$ one sees that the safe value of game A_t is exactly the same as the value $v_{1,t}$. Therefore we state the following:

Lemma 1. *The safe value at time t in the evolutionary game Γ_t with weight $\lambda_{1,t}$ is given by $v_{1,t}$.*

Similarly, one can obtain the expression for the corresponding minimax value $w_{1,t}$ at the t-th step by an appropriate change of indices and a reversal of the order of the min and the max. Such a value is to be considered with respect to the weights of the same player and verifies $w_{1,t} \geqq v_{1,t}$.

3. Now, it is reasonable to ask whether the behavior of both players is optimal. In general the problem is complicated because of the shape of the corresponding payoff faced at time t. First, we will consider one of the simplest cases; namely, that in which the players' foresight at any given step is limited to the next two steps, that is, when the weights $\lambda_{1,t}^k$ at time $t \leqq n-2$ are all zero for $k \geqq 2$. In such a case the modified payoff of our player is given by

$$\lambda_{1,t}^0 A_{1,t}(\sigma_1(t), \sigma_2(t)) + \max_{\sigma_1(t+1)\in\Sigma_{1,t+1}} \min_{\sigma_2(t+1)\in\Sigma_{2,t+1}} \lambda_{1,t}^1 A_{1,t+1}(\sigma(t), \sigma_1(t+1), \sigma_2(t+1)).$$

We need the following definition. We say that the payoff $A_{1,t+1}$ is a *M-pseudo concave function* in $\sigma_2(t+1)$ with respect to $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_1(t+1)$ if for all $\sigma_2(t)\in\Sigma_{2,t}$; $\sigma_1(t)$ and $\bar\sigma_1(t)\in\Sigma_{1,t}$; $0\leqq\rho\leqq 1$ and all

$$\sigma_1(t+1)\in\Sigma_{1,t+1}(\sigma_1(t), \sigma_2(t)), \qquad \bar\sigma_1(t+1)\in\Sigma_{1,t+1}(\bar\sigma_1(t), \sigma_2(t))$$

there exist elements

$$\bar\sigma_2(t+1)\in\Sigma_{2,t+1}(\bar\sigma_1(t), \sigma_2(t)), \qquad \sigma_2(t+1)\in\Sigma_{2,t+1}(\sigma_1(t), \sigma_2(t))$$

and

$$\sigma_1^\rho(t+1)\in\Sigma_{1,t+1}(\sigma_1^\rho(t), \sigma_2(t))$$

where $\sigma_1^\rho(t)=\rho\,\sigma_1(t)+(1-\rho)\,\bar\sigma_1(t)$, such that for all

$$\sigma_2^\rho(t+1)\in\Sigma_{2,t+1}(\sigma_1^\rho(t), \sigma_2(t))$$

the following is satisfied

$$\begin{aligned} &A_{1,t+1}(\sigma_1^\rho(t), \sigma_2(t), \sigma_1^\rho(t+1), \sigma_2^\rho(t+1)) \\ &\geqq \rho A_{1,t+1}(\sigma_1(t), \sigma_2(t), \sigma_1(t+1), \sigma_2(t+1)) + (1-\rho) A_{1,t+1}(\bar\sigma_1(t), \sigma_2(t), \bar\sigma_1(t+1), \bar\sigma_2(t+1)). \end{aligned}$$

We note that the order of the variables in the above definition is important.

The following result concerns the concavity of the function $M_{1,t}^1$ defined by

$$M_{1,t}^1(\sigma_1(t), \sigma_2(t)) = \max_{\sigma_1(t+1)\in\Sigma_{1,t+1}(\sigma(t))} \min_{\sigma_2(t+1)\in\Sigma_{2,t+1}(\sigma(t))} A_{1,t+1}(\sigma(t), \sigma_1(t+1), \sigma_2(t+1)).$$

Lemma 2. *A necessary and sufficient condition for the concavity in $\sigma_1(t)$ for fixed $\sigma_2(t)$ of the function $M_{1,t}^1$ is that the payoff function $A_{1,t+1}$ be M-pseudo concave in $\sigma_2(t+1)$ with respect to $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_1(t+1)$.*

Proof. From the definition of the function $M_{1,t}^1$, we have that for an arbitrary pair of strategies $\sigma_1(t)$ and $\bar\sigma_1(t)$, for all

$$\sigma_1(t+1)\in\Sigma_{1,t+1}(\sigma_1(t), \sigma_2(t)); \qquad \bar\sigma_1(t+1)\in\Sigma_{1,t+1}(\bar\sigma_1(t), \sigma_2(t))$$

the following two inequalities hold:

$$M_{1,t}^1(\sigma_1(t), \sigma_2(t)) \geqq \min_{\sigma_2(t+1)\in\Sigma_{2,t+1}(\sigma_1(t),\sigma_2(t))} A_{1,t+1}(\sigma_1(t), \sigma_2(t), \sigma_1(t+1), \sigma_2(t+1))$$

and

$$M_{1,t}^1(\bar\sigma_1(t), \sigma_2(t)) \geqq \min_{\sigma_2(t+1)\in\Sigma_{2,t+1}(\bar\sigma_1(t),\sigma_2(t))} A_{1,t+1}(\bar\sigma_1(t), \sigma_2(t), \bar\sigma_1(t+1), \sigma_2(t+1)).$$

On the other hand, if $\sigma_1^\rho \in \Sigma_{1,t+1}(\sigma_1^\rho(t), \sigma_2(t))$ where $\sigma_1^\rho = \rho\, \sigma_1(t) + (1-\rho)\, \bar{\sigma}_1(t)$, $0 \leqq \rho \leqq 1$, is a point on which the maximum $M_{1,t}^1(\sigma_1^\rho(t), \sigma_2(t))$ is reached, then the concavity of $M_{1,t}^1$ implies that

$$\min_{\sigma_2(t+1)\in\Sigma_{2,t+1}(\sigma_1^\rho(t),\sigma_2(t))} A_{1,t+1}(\sigma_1^\rho(t), \sigma_2(t), \sigma_1^\rho(t+1), \sigma_2(t+1))$$

$$\geqq \rho \min_{\sigma_2(t+1)\in\Sigma_{2,t+1}(\sigma_1(t),\sigma_2(t))} A_{1,t+1}(\sigma_1(t), \sigma_2(t), \sigma_1(t+1), \sigma_2(t+1))$$

$$+(1-\rho) \min_{\sigma_2(t+1)\in\Sigma_{2,t+1}(\bar{\sigma}_1(t),\sigma_2(t))} A_{1,t+1}(\bar{\sigma}_1(t), \sigma_2(t), \bar{\sigma}_1(t+1), \bar{\sigma}_2(t+1)).$$

Now, if

$$\sigma_2(t+1)\in\Sigma_{2,t+1}(\sigma_1(t), \sigma_2(t)) \quad \text{and} \quad \bar{\sigma}_2(t+1)\in\Sigma_{2,t+1}(\bar{\sigma}_1(t), \sigma_2(t))$$

are two strategies such that the respective minima on the right hand inequality are attained, then the assumption of the lemma holds. Similarly, the condition guarantees the above property regarding the minima. It is easy to see that this is equivalent to the required concavity. (q.e.d.)

We note that for $\sigma_1(t) = \bar{\sigma}_1(t)$ the condition is always fulfilled. Indeed, take $\sigma_1^\rho(t+1)$ to be a maximum strategy in $\Sigma_{1,t+1}(\sigma_1(t), \sigma_2(t))$ and $\sigma_2(t+1)$, $\bar{\sigma}_2(t+1)$ in $\Sigma_{2,t+1}(\sigma_1(t), \sigma_2(t))$ to be some strategies where the corresponding minimum is reached on $\sigma_1(t)$ and $\sigma_2(t)$ respectively. Therefore, the condition links relations between different points $\sigma_1(t)$ and $\bar{\sigma}_1(t)$.

Similarly, the function $A_{1,t+1}$ will be called a *M-pseudo convex function* in $\sigma_2(t+1)$ with respect to $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_1(t+1)$ if for all $\sigma_1(t)\in\Sigma_{1,t}$, $\sigma_2(t)$ and $\bar{\sigma}_2(t)\in\Sigma_{2,t}$; $0 \leqq \rho \leqq 1$ there exist strategies

$$\sigma_1(t+1)\in\Sigma_{2,t+1}(\sigma_1(t), \sigma_2(t)), \qquad \bar{\sigma}_1(t+1)\in\Sigma_{1,t+1}(\sigma_1(t), \bar{\sigma}_2(t))$$

such that for all

$$\sigma_2(t+1)\in\Sigma_{2,t+1}(\sigma_1(t), \sigma_2(t)), \qquad \bar{\sigma}_2(t+1)\in\Sigma_2(\sigma_1(t), \bar{\sigma}_2(t))$$

and

$$\sigma_1^\rho(t+1)\in\Sigma_{1,t+1}(\sigma_1(t), \sigma_2^\rho(t))$$

where $\sigma_2^\rho = \rho\, \sigma_2(t) + (1-\rho)\, \bar{\sigma}_2(t)$, there exists an element

$$\sigma_2^\rho(t+1)\in\Sigma_{2,t+1}(\sigma_1(t), \sigma_2^\rho(t))$$

with the property

$$A_{1,t+1}(\sigma_1(t), \sigma_2^\rho(t), \sigma_1^\rho(t+1), \sigma_2^\rho(t+1)) \leqq \rho\, A_{1,t+1}(\sigma_1(t), \sigma_2(t), \sigma_1(t+1), \sigma_2(t+1))$$

$$+(1-\rho)\, A_{1,t+1}(\sigma_1(t), \bar{\sigma}_2(t), \bar{\sigma}_1(t+1), \bar{\sigma}_2(t+1)).$$

Now, we present a simple result regarding convexity of the function $M_{1,t}^1$ which will be used later. Its proof is similar to that of the result just considered.

Lemma 3. *If the payoff function $A_{1,t+1}$ is M-pseudo convex in $\sigma_2(t+1)$ with respect to $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_1(t+1)$, then the function $M_{1,t+1}^1(\sigma_1(t), \cdot)$ is convex for any $\sigma_1(t)$.*

Using the properties described in the statements of the lemmas we can derive immediately the following result which guarantees the optimal behavior.

Theorem 4. *Under the conditions of Lemmas 2 and 3, if the payoff functions $A_{1,t}$ and $A_{1,t+1}$ are concave in $\sigma_1(t)$ and $\sigma_1(t+1)$ and convex in $\sigma_2(t)$ and $\sigma_2(t+1)$ respectively, then $v_{1,t}=w_{1,t}$.*

Proof. By virtue of Lemmas 2 and 3 the function $v_{1,t}^1$ is concave in $\sigma_1(t)$ and convex in $\sigma_2(t)$. Therefore the minimax theorem guarantees the existence of a saddle point for the function inside the expression for $v_{1,t}$. Since at such a point $\sigma^*(t)$ we have

$$\max_{\sigma_1(t+1)\in\Sigma_{1,t}(\sigma^*(t))} \min_{\sigma_2(t+1)\in\Sigma_{2,t+1}(\sigma^*(t))} A_{1,t+1}(\sigma^*(t), \sigma_1(t+1), \sigma_2(t+1))$$
$$= \min_{\sigma_2(t+1)\in\Sigma_{2,t+1}(\sigma^*(t))} \max_{\sigma_1(t+1)\in\Sigma_{1,t+1}(\sigma^*(t))} A_{1,t+1}(\sigma^*(t), \sigma_1(t+1), \sigma_2(t+1)),$$

the equality $v_{1,t}=w_{1,t}$ is obtained. (q.e.d.)

It should be pointed out that the conditions of concavity and convexity cannot simply be replaced by more general ones of quasi-concavity and quasi-convexity since convex combination of functions with these properties do not have the same property in general.

Consequently, we get the following simple condition for the set of strategies satisfying Lemme 2.

Given the strategies $\sigma_1(t)$, $\bar{\sigma}_1(t)$, $\sigma_1(t+1)$, $\bar{\sigma}_1(t+1)$, let $\mathscr{H}(\sigma_1(t), \bar{\sigma}_1(t), \sigma_1(t+1), \bar{\sigma}_1(t+1))$ be the set of elements

$$(\sigma_1^\rho(t+1), \sigma_2(t+1), \bar{\sigma}_2(t+1))\in\Sigma_{1,t+1}(\sigma_1^\rho(t), \sigma_2(t))$$
$$\times\Sigma_{1,t+1}(\sigma_1(t), \sigma_2(t))\times\Sigma_{1,t+1}(\bar{\sigma}_1(t), \sigma_2(t))$$

such that for all $\sigma_2^\varrho(t+1)\in\Sigma_{2,t+1}(\sigma_1^\rho(t+1), \sigma_2(t))$:

$$A_{1,t+1}(\sigma_1^\rho(t), \sigma_2(t), \sigma_1^\rho(t+1), \sigma_2^\varrho(t+1)$$
$$\geqq\rho A_{1,t+1}(\sigma_1(t), \sigma_2(t), \sigma_1(t+1), \sigma_2(t+1))+(1-\rho)A_{1,t+1}(\bar{\sigma}_1(t), \sigma_2(t), \sigma_1(t+1), \sigma_2(t+1))$$

is convex.

Corollary 5. *Under the conditions of Theorem 4, the set $\mathscr{H}(\sigma_1(t), \bar{\sigma}_1(t), \sigma_1(t+1), \bar{\sigma}_1(t+1))$ is convex.*

Proof. Indeed, for such a pair of points $(\tilde{\sigma}_1^\rho(t+1), \tilde{\sigma}_2(t+1), \tilde{\bar{\sigma}}(t+1))$ and $(\hat{\sigma}_1^\rho(t+1), \tilde{\sigma}_2(t+1), \hat{\bar{\sigma}}(t+1))$ and $0\leqq\mu\leqq1$, we have

$$\begin{aligned}
A_{1,t+1}&(\sigma_1^\rho(t), \sigma_2(t), \mu\,\tilde{\sigma}_1^\rho(t+1)+(1-\mu)\,\hat{\sigma}_1^\rho(t+1), \sigma_2^\varrho(t+1))\\
&\geqq\mu A_{1,t+1}(\sigma_1^\rho(t), \sigma_2(t), \tilde{\sigma}_1^\rho(t+1), \sigma_2^\varrho(t+1))\\
&\quad+(1-\mu)\,A_{1,t+1}(\sigma_1^\rho(t), \sigma_2(t), \hat{\sigma}_1^\rho(t+1), \sigma_2^\varrho(t+1))\\
&\geqq\rho\{\mu A_{1,t+1}(\sigma_1(t), \sigma_2(t), \sigma_1(t+1), \tilde{\sigma}_2(t+1))\\
&\quad+(1-\mu)\,A_{1,t+1}(\sigma_1(t), \sigma_2(t), \sigma_1(t+1), \hat{\sigma}_2(t+1))\}\\
&\quad+(1-\rho)\,\{\mu A_{1,t+1}(\bar{\sigma}_1(t), \sigma_2(t), \bar{\sigma}_1(t+1), \tilde{\bar{\sigma}}_2(t+1))\\
&\quad+(1-\mu)\,A_{1,t+1}(\bar{\sigma}_1(t), \sigma_2(t), \bar{\sigma}_1(t+1), \hat{\bar{\sigma}}_2(t+1))\}\\
&\geqq\rho A_{1,t+1}(\sigma_1(t), \sigma_2(t), \sigma_1(t+1), \mu\,\tilde{\sigma}_2(t+1)+(1-\mu)\,\hat{\sigma}_2(t+1))\\
&\quad+(1-\rho)\,A_{1,t+1}(\bar{\sigma}_1(t), \sigma_2(t), \bar{\sigma}_1(t+1), \mu\,\tilde{\bar{\sigma}}_2(t+1)+(1-\mu)\,\hat{\bar{\sigma}}_2(t+1)).
\end{aligned}$$

The first inequality follows from the concavity of the payoff function. The second is a consequence of the assumption and the last are as follows from the convexity in the remaining variable. Therefore, the set under consideration is convex. (q.e.d.)

It seems natural that an adequate rational concept of optimal behavior for the whole evolutionary game is that for which at every step, with the corresponding weight, the players choose optimal strategies for the game Λ_t^1. If the weight at time t is $\lambda_{1,t}$ and $\bar{\sigma}(t)$ is the played saddle point at this time then our player will get $A_{1,t}(\bar{\sigma}(1, \ldots, t-1), \bar{\sigma}(t))$. Now, if instead of considering the second player as attacking the first one, one considers him with his own weights then we are faced with a time t with the two person game

$$\Lambda_t = \left\{ C_{1,t}, C_{2,t}; \sum_{s=0}^{n-t} \lambda_{1,t}^s B_{1,t+s}, \sum_{s=0}^{n-t} \lambda_{2,t}^s B_{2,t+s} \right\}$$

which is *non-zero sum*. This is very interesting since the primitive game was zero-sum. The actual situation is transformed into a non-zero sum. Of course, this change has been produced by the different foresights of each player. When their foresights are equal, that is, $\lambda_{1,t}^k = \lambda_{2,t}^k$ then the actual game Λ_t at time t is zero-sum. If both players play a maximin strategy in their respective situations, then both will win more than the corresponding values at each time.

We have considered that both players are in a conflict of interest situation since from the beginning, the game was zero-sum. But, since the players have different foresights, the entire situation will be *non-zero sum*. Therefore, it is natural to consider some further concepts of what a solution is to be. We notice that for which the joint strategy $\bar{\sigma}(t)$ is a saddle point in the game

$$\{\Sigma_{1,t}(\bar{\sigma}(1, \ldots, t-1)), \Sigma_{2,t}(\bar{\sigma}(1, \ldots, t-1)); A_{1,t}(\bar{\sigma}(1, \ldots, t-1))(\cdot, \cdot)\}$$

does not fit our requirements, since it is independent of the foresights. One concept of solution which would fit the intuitive idea is that one for which the joint strategy $\bar{\sigma}(t)$ is an equilibrium point in the game

$$\Pi_t = \left\{ \Sigma_{1,t}, \Sigma_{2,t}; \sum_{s=0}^{n-t} \lambda_{1,t}^s A_{1,t+s}, \sum_{s=0}^{n-t} \lambda_{2,t}^s A_{2,t+s} \right\}$$

evaluated on the point $\bar{\sigma}(t+1, \ldots, n)$ having the same property. Such a point $\bar{\sigma}(1, \ldots, n)$ will be called a *complex equilibrium point*. An existence theorem for such points will be presented in the following paragraph where we will consider that the multivalued functions of the strategy sets are given by a natural but more restricted kind of function. Indeed, in order to prove it in the generality we have used up to now, one might have to introduce many restrictive assumptions on the payoffs.

A more adequate concept which is neither as stable as the first one nor as instable as that of the complex equilibrium point is just a point $\bar{\sigma}(1, \ldots, n)$ which is an equilibrium point in the game

$$\Theta_t = \{C_{1,t}, C_{2,t}; \tilde{v}_{1,t}^1, \tilde{v}_{2,t}^1\}$$

considered as the permissible joint strategy $\bar{\sigma}(1, \ldots, t-1)$. When $\lambda_{1,t}=\lambda_{2,t}$ this coincides with the first concept. Perhaps, such an *intermediate point* better fits the intuitive idea of solution as it commonly is understood in real life.

In the case with which we are dealing, the foresights have only two values different than zero. Therefore the functions $v_{i,t}^1$ are just the maximum functions $M_{i,t}^1$. An existence result which is essentially Theorem 4 is as follows:

Theorem 6. *Under the condition of Lemmas* 2 *and* 3, *if the payoffs* $A_{1,t}$ *and* $A_{1,t+1}$ *are concave in* $\sigma_1(t)$ *and* $\sigma_1(t+1)$ *and* $A_{2,t}, A_{2,t+1}$ *are convex in* $\sigma_2(t)$ *and* $\sigma_2(t+1)$ *respectively, then there exists an intermediate point.*

We note that this is still true under the non-zero condition.

It is interesting to compare situations having different foresights. If the weight at time t is $\lambda_{1,t}$ and at time $t+1$, $\bar{\lambda}_{1,t+1}$ which is given by $\bar{\lambda}_{1,t+1}^k=\lambda_{1,t}^{k+1}/1-\lambda_{1,t}^0$, then we get that $v_{1,t+1}(\bar{\sigma}(t))=v_{1,t}^1(\bar{\sigma}(t))/1-\lambda_{1,t}^0$. Therefore given a foresight at time t the best thing to do with respect to this weight is to proceed with the new modified weight obtained by dividing by $\lambda-\lambda_{1,t}^0$, which means following the same foresight since the division is only a matter of homogeneity, and so on. Indeed, if one changes the foresight then with respect to the old foresight he will be sure to receive a value which is less than the safe value. From this, we deduce that after changing foresights one will lose winnings *with respect to the old point of view*. Again, we recall that the weights are not intrinsic components of the game. This is just the same as what happens in real life.

From a technical point of view we only have treated situations having at most states two stages in the weights. For them we derived some results. One can go further and get similar results by examining the properties of the payoffs just as easily as we have done above. We will not go through this. Even though it is easy, it is technically quite involved. We prefer to show in the next paragraph that this complication can be eliminated by introducing some adequate functions for the strategy sets.

If all the payoff functions $A_{1,t}$ for $t \leqq n-1$ are identically zero, then one can derive the existence of optimal behavior for games with perfect information, by only considering the weights $\lambda_{1,t}^k=0$ for $k \neq n-t$ and $\lambda_{1,t}^{n-t}=1$. In this case we have

$$v_{1,t}=\max_{\sigma_1(t)\in\Sigma_{1,t}} \min_{\sigma_2(t)=\Sigma_{2,t}} v_{1,t+1}(\sigma_1(t), \sigma_2(t)).$$

One can obtain the Zermelo-von Neumann-Kuhn theorem (see [1, 2], and [4]) for such games with perfect information which differ by having no discrete sets of strategy.

4. We now proceed with our treatment by considering that the strategy sets are obtained from the following functions which will be defined recursively by $\hat{F}_{t+1}$ whose domain is

$$\hat{F}_t(\hat{F}_{t-1}(\ldots(\hat{F}_2(S_1, S_2), S_3), \ldots, S_t)) \times S_{t+1}=\mathscr{S}_t \times S_{t+1}$$

with values in $R^{n_{t+1}}=R^{n_{1,t+1}} \times R^{n_{2,t+1}}$, where the set $S_t=S_{1,t} \times S_{2,t}$ is non-empty, compact and convex. The image

$$\begin{aligned}
&\hat{F}_{t+1}(\hat{F}_t(\ldots(\hat{F}_2(s(1), s(2)), s(3)), \ldots), s(t+1)) \\
&\quad=\hat{F}_{1,t+1}(\hat{F}_t(\ldots(\hat{F}_2(s(1), s(2)), s(3)), \ldots, s(t)), s(t+1)) \\
&\qquad\times \hat{F}_{2,t+1}(\hat{F}_t(\ldots(\hat{F}_2(s(1), s(2)), s(3)), \ldots, s(t)), s(t+1))
\end{aligned}$$

is such that it gives a convex set $\hat{F}_{t+1}(\hat{F}_t(\ldots(\hat{F}_2(s(1), s(2)), s(3)), \ldots, s(t)), S_{t+1})$, which is the joint strategy set $\Sigma_{t+1}(s(1, \ldots, t))$. Moreover, $\hat{F}_{t+1}$ is an homeomorphism. The first joint strategy set is just $\Sigma_1 = S_1$. From the fact that the functions $\hat{F}_t$ are homeomorphisms, we have that the sets $\mathscr{I}_t$ are contractible and the corresponding multi-valued functions are upper and lower semi-continuous.

Now, by introducing the functions F_{t+1} which equal $\hat{F}_{t+1}$ but only are defined on $S(t+1) = \prod_{k=1}^{t+1} S_k$, by

$$F_{t+1}(s(1, \ldots, t+1)) = \hat{F}_{t+1}(\hat{F}_t \ldots(\hat{F}_2(s(1), s(2)), s(3)), \ldots), s(t+1))$$

we can see that the joint strategy sets are homeomorphic to $S(t+1)$. Thus, the whole evolutionary game is given by

$$\Gamma_t^m = \{S_{1,t}, S_{2,t}; A_{1,t}^m\}$$

where the new payoff function is given by

$$A_{1,t}^m(s(1, \ldots, t)) = A_{1,t}(s(1), F_2(s(1), s(2)), \ldots, F_t(s(1), \ldots, s(t)))$$

for $s(1, \ldots, t) \in S(t)$.

From here on, we only have to consider the maximin, minimax, etc. values on the fixed sets S_t. Then, the existence of stable points can be established in this general case by examining the *lower* and *upper* sets (introduced in [3]) of the corresponding partial games at each step A_t or its equivalent. We note that we have to observe them in their respective *order*.

For the complex equilibrium points we derive the following:

Theorem 7. *If for all t and all $\sigma(1, \ldots, n)$ the image by the homeomorphism of the set of the points $\bar{\sigma}_1(t)$ and the set of strategy $\bar{\sigma}_2(t)$ where*

$$A_{1,t}(\sigma(1, \ldots, t-1), \cdot, \sigma_2(t), \sigma(t+1, \ldots, n))$$

and

$$A_{1,t}(\sigma(1, \ldots, t-1), \sigma_1(t), \cdot, \sigma(t+1, \ldots, n))$$

reach their maximum and minimum respectively, are convex, then the evolution game has a complex equilibrium point.

Proof. For a joint strategy $s(1, \ldots, n)$ in $S(n)$, let us consider for the step t, the set

$$\psi_t(s(1, \ldots, n)) = \psi_{1,t}(s(1, \ldots, n)) \times \psi_{2,t}(s(1, \ldots, n)) \subset S_t$$

to be the set of all the points $\bar{s}(t) \in S_t$ such that on them the respective payoffs of game Π_t considered at $s(1, \ldots, t-1, t+1, \ldots, n)$ and with the new functions $A_{1,t}^m$ and $A_{2,t}^m$, reach their maximum. Such a set is convex by the condition stated in the hypothesis. From the continuity of the payoff the graph of ψ_t is closed. If ψ indicates the cartesian product $\prod_{t=1}^{n} \psi_t$, by the Kakutani's fixed point theorem we have the existence of a point $\bar{s}(1, \ldots, n)$ such that for all t: $\bar{s}(t) \in \psi_t(\bar{s}(1, \ldots, n))$. Such a point is a complex equilibrium point. (q.e.d.)

Finally, we remark that one can treat intermediate points in a related way by examining some lower and upper sets.

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