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On the Absence of Phase Transition in One-Dimensional Random Fields

II. Superstable Spin Systems

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Summary. It is shown that certain one-dimensional unbounded spin systems, which are superstable in the sense of [12], admit unique "regular" DLR measures, regardless of temperature.

§ 1. Introduction

The present paper is a sequel to [8] (to be referred to as Part I below) and its main purpose is to employ the general theorem obtained there to show that, in the one-dimensional case, certain spin systems that have been discussed in the literature are uniquely determined by their specifications, regardless of "temperature". The spin systems in question are one-dimensional random fields $\dots, X_{-1}, X_0, X_1, \dots$ with state space $\mathbb{R}$, whose distributions (DLR measures) admit superstable specifications generated by a long range pair potential of the form $-J(|i-j|)X_iX_j$ and a self potential $F(X_i)$. In Theorem 2.8 we show that if $\sum_{\rho=1}^{\infty} \rho|J(\rho)| < \infty$, then a specification of this nature admits only one "regular" DLR measure. This result was communicated to the 1983 Swansea Workshop [9].

Before discussing the problem, it is expedient to give the precise definition of the specification. We use the same notation as in Part I. The letter Λ will denote an arbitrary segment of integers, i.e. a finite set of integers of the form $\{i, i+1, \dots, j\}$ $(i \leqq j)$. Let β be a positive number, $J(\cdot)$ a real function on the positive integers such that $|J(\rho)| \geqq |J(\rho+1)|$, $\rho = 1, 2, \dots$ and

$$\sum_{\rho=1}^{\infty} \rho|J(\rho)| < \infty \tag{1}$$

and $F: \mathbb{R} \to \mathbb{R}$ a continuous function satisfying the following condition: there are constants $a > 0$ and c such that, for arbitrary $\Lambda = \{i, i+1, \dots, j\}$ and $\zeta_i \zeta_{i+1} \dots \zeta_j \in \mathbb{R}^{j-i+1}$, $i \leqq j$

$$\sum_{\rho\in\Lambda} F(\zeta_\rho)-\tfrac{1}{2}\sum_{\substack{\rho,l\in\Lambda\\ \rho\neq l}} J(|\rho-l|)\zeta_\rho\zeta_l \geqq \sum_{\rho\in\Lambda}(a\zeta_\rho^2-c). \tag{2}$$

If $w=\zeta_i\zeta_{i+1}\ldots\zeta_j$, $z=\zeta_{j+1}\zeta_{j+2}\ldots$ and $x=\ldots\zeta_{i-2}\zeta_{i-1}$, let dw denote $d\zeta_i d\zeta_{i+1}\ldots d\zeta_j$ and define

$$f_\Lambda(x,w,z)=\exp\{\beta(-\sum_{\rho\in\Lambda}F(\zeta_\rho)+\tfrac{1}{2}\sum_{\substack{\rho,l\in\Lambda\\ \rho\neq l}}J(|\rho-l|)\zeta_\rho\zeta_l+\sum_{\rho\in\Lambda,\, l\notin\Lambda}J(|\rho-l|)\zeta_\rho\zeta_l)\},$$

$$\sigma_\Lambda(x,z)=\int_{\mathbb{R}^{j-i+1}} f_\Lambda(x,w,z)\,dw$$

and

$$q^{[i,j]}_{i-1,j+1}(x,w,z)\equiv q_\Lambda(x,w,z)=\sigma_\Lambda(x,z)^{-1}f_\Lambda(x,w,z).$$

For $B\in\mathscr{B}(\mathbb{R}^{j-i+1})$ let

$$Q^{[i,j]}_{i-1,j+1}(x,B,z)=\int q_\Lambda(x,w,z)\,dw.$$

The function $q_\Lambda(x,\cdot,z)$ and the kernel $Q^{[i,j]}_{i-1,j+1}(x,\cdot,z)$ are defined for all $z\in\vec{T}_0$, $x\in\overleftarrow{T}_0$, where

$$\vec{T}_0=\left\{\zeta_1\zeta_2\ldots:\ \sum_{\rho=1}^{\infty}|J(\rho)|\cdot|\zeta_\rho|<\infty\right\},$$

$$\overleftarrow{T}_0=\left\{\ldots\zeta_{-2}\zeta_{-1}:\ \sum_{\rho=1}^{\infty}|J(\rho)|\cdot|\zeta_{-\rho}|<\infty\right\}.$$

The system of kernels $Q^{[i,j]}_{i-1,j+1}$ determines a specification $\{Q\}=\{Q^{[i,j]}_{l,m}\}$. Specifications of this type can also be introduced for multi-dimensional spin systems, indexed by the elements of the d-dimensional lattice $\mathbb{Z}^d$, and our discussion below contains references to such systems, although we will not define them formally here. A DLR measure (equilibrium state) admitted by the specification $\{Q\}$ is, by definition, the distribution on $\bigtimes_{i=-\infty}^{\infty}\mathbb{R}_i$ ($\mathbb{R}_i=\mathbb{R}$ for all i) of any random field $\ldots,X_{-1},X_0,X_1,\ldots$ satisfying

$$P\left(\sum_{\rho=1}^{\infty}|J(\rho)|\cdot|X_{j+\rho}|<\infty,\ \sum_{\rho=1}^{\infty}|J(\rho)|\cdot|X_{i-\rho}|<\infty \text{ for all } i,j\right)=1$$

and admitting the given specification.

Condition (1) limits the interaction energy (see [10] for the case of a lattice gas), while condition (2) is the superstability condition introduced by Ruelle, who defined a general class of multi-dimensional superstable interactions in [12]. If $F(\zeta)=\alpha\zeta^2$ ($\alpha>0$) then a sufficient condition for the superstability of the above specification is

$$\alpha>\sum_{\rho=1}^{\infty}|J(\rho)| \tag{3}$$

while if, say $F(\zeta)=\alpha|\zeta|^{2+\delta}$ ($\alpha>0$, $\delta>0$), then superstability holds automatically (cf. [2]). In [5] equilibrium states for Ruelle's multidimensional spin systems were obtained as limits of finite volume Gibbs measures. Specifications were employed in the investigation of interactions of the type considered here in [2, 3] and, for a special case, in [1].

In general there may be many DLR measures admitted by the specification described above (phase transition). For the one-dimensional harmonic (Gaussian) case with nearest neighbour interaction ($F(\zeta)=\alpha\zeta^2$, $J(\rho)=0$ for $\rho\geqq 2$, $\alpha>J(1)>0$) the DLR measures were determined in [1], where it was also shown that only one of these DLR measures is translation invariant. (As mentioned in Part I, this is an instance of a universal property shared by all "irreducible" one-dimensional Markovian specifications, as shown in [7]).

In view of the possible occurrence of phase transition, attention was focussed on a class of DLR measures which Ruelle called tempered ([11, 5]) and on the problem of possible uniqueness within this class. These tempered measures are distributions on $\mathop{\times}\limits_{i=-\infty}^{\infty} \mathbb{R}_i$ under which, with probability one, the sequence $\frac{1}{2n+1}\sum_{i=-n}^{n} X_i^2$, $n=1,2,\dots$ is bounded. It will be shown below, in Theorem 2.5, that this requirement is equivalent to the condition $\sup_i E|X_i|<\infty$ used in [2] (and originating from [4]) and also to the condition of tameness employed in Part I. Some further equivalent conditions will be given in Theorem 2.5. DLR measures satisfying these conditions will be called *regular* here.

Uniqueness of tempered DLR measures was proved in [5] for ferromagnetic systems satisfying a special sufficient condition. A broader sufficient condition for uniqueness of regular DLR measures admitted by multi-dimensional spin systems was derived in [2] from Dobrushin's general uniqueness criterion ([4]). In our present context the result of [2] is this: if

$$\sup_{x,z} V(x,z) < \left(2\beta \sum_{\rho=1}^{\infty} |J(\rho)|\right)^{-1} \tag{4}$$

where $V(x,z)$ is the variance of the distribution $Q^{[0]}_{-1,1}(x,\cdot,z)$ on $\mathbb{R}$, then exactly one random field $\dots, X_{-1}, X_0, X_1, \dots$ (up to equivalence in distribution) admitting the specification $\{Q\}$ satisfies $\sup_i E|X_i|<\infty$. (It is assumed in [2] that, for $\zeta>0$, F is of the form $F(\zeta)=\int_0^\zeta G(t)\,dt$, where G is a C^1, convex, positive, increasing function, and analogously for $\zeta<0$.) As will be shown below, condition (4) may be unnecessarily restrictive in the one-dimensional case. If for instance $F(\zeta)=\alpha|\zeta|^{2+\delta}$ ($\alpha>0$, $\delta>0$) then (4) is satisfied when the temperature $1/\beta$ is sufficiently high but will fail for low temperatures, although it is interesting that in the Gaussian case (4) reduces to (3) ([2]).

Our main result (Theorem 2.8) is that in the one-dimensional case superstability and (1) are sufficient to imply that the specification admits exactly one regular DLR measure, regardless of the value of the temperature $1/\beta$.

§ 2. The Results

In $\vec{T}_0$ and $\overleftarrow{T}_0$ we introduce the norms $\|z\|_J = \sum_{\rho=1}^{\infty} |J(\rho)| \cdot |\zeta_\rho|$ and $\|x\|_J = \sum_{\rho=1}^{\infty} |J(\rho)| \cdot |\zeta_{-\rho}|$ respectively, where $z = \zeta_1 \zeta_2 \dots$ and $x = \dots \zeta_{-2} \zeta_{-1}$. (We assume for simplicity that $J(\rho) \neq 0$ for all ρ. If $J(\rho) = 0$ for large ρ, $\|\cdot\|_J$ will not be a norm, but it is obvious how the definition of $\|\cdot\|_J$ should be modified in this case). Ruelle's important estimates ([12]), which play a crucial role here, imply the following lemma.

2.1. Lemma. *For every $b>0$ there are constants $\gamma>0$ and δ such that if $l<i \leqq j<m$, $\|x\|_J \leqq b$, $\|z\|_J \leqq b$ and $w = \zeta_i \zeta_{i+1} \dots \zeta_j \in \mathbb{R}^{j-i+1}$, then*

$$q_{l,m}^{[i,j]}(x, w, z) \leqq \exp\left\{ \sum_{\rho=i}^{j} (-\gamma \zeta_\rho^2 + \delta) \right\},$$

where $q_{l,m}^{[i,j]}(x, \cdot, z)$ denotes the density of $Q_{l,m}^{[i,j]}(x, \cdot, z)$ obtained as a marginal of $q_{l,m}^{[l+1,m-1]}(x, \cdot, z)$.

In fact, as pointed out in [3], if, given $l<m$, the self potential inside $\{l+1, l+2, \dots, m-1\}$ is modified so as to include the interactions with all external spins ζ_ρ, $\rho \leqq l$ or $\rho \geqq m$, (see p. 48 of [3]), then Ruelle's estimates will hold uniformly in $\|x\|_J \leqq b$, $\|z\|_J \leqq b$.

For $v \geqq 1$, $k \geqq 1$ we now set

$$\vec{M}_v(k) = \left\{ \zeta_1 \zeta_2 \dots \zeta_k : \frac{1}{n} \sum_{\rho=1}^{n} \zeta_\rho^2 \leqq v \text{ for } n = 1, 2, \dots, k \right\},$$

$$\overleftarrow{M}_v(k) = \left\{ \zeta_{-k} \zeta_{-k+1} \dots \zeta_{-1} : \frac{1}{n} \sum_{\rho=1}^{n} \zeta_{-\rho}^2 \leqq v \text{ for } n = 1, 2, \dots, k \right\}$$

$$\vec{M}_v = \left\{ \zeta_1 \zeta_2 \dots : \frac{1}{n} \sum_{\rho=1}^{n} \zeta_\rho^2 \leqq v \text{ for all } n \geqq 1 \right\},$$

$$\overleftarrow{M}_v = \left\{ \dots \zeta_{-2} \zeta_{-1} : \frac{1}{n} \sum_{\rho=1}^{n} \zeta_{-\rho}^2 \leqq v \text{ for all } n \geqq 1 \right\},$$

$$M_v(k) = \vec{M}_v(k) \cap \overleftarrow{M}_v(k).$$

Then (cf. [5, Th. 1.1] or [3, p. 49]):

2.2. Corollary. *Given $b>0$ and $\varepsilon>0$, there exists $v \geqq 1$ such that*

$$Q_{l,m}^{[i,j]}(x, M_v(j-i+1), z) > 1 - \varepsilon$$

whenever $l<i \leqq j<m$, $\|x\|_J \leqq b$, $\|z\|_J \leqq b$.

Let now $\Phi(i) = \sum_{\rho=i}^{\infty} |J(\rho)|$, $i = 1, 2, \dots$ and set $K = \sum_{\rho=1}^{\infty} \rho |J(\rho)| = \sum_{i=1}^{\infty} \Phi(i)$. Clearly $\Phi(i) \geqq \Phi(i+1)$, $i = 1, 2, \dots$.

2.3. Lemma. *If* $z=\zeta_1\zeta_2\ldots\in\vec{M}_\nu$, *then*

$$\sum_{\rho=1}^{n}|J(\rho)|\cdot|\zeta_\rho|^r\leqq\nu\sum_{\rho=1}^{n}|J(\rho)|\qquad(r=1,2),$$

$$\sum_{\rho=1}^{n}\Phi(\rho)|\zeta_\rho|^r\leqq\nu\sum_{\rho=1}^{n}\Phi(\rho)\qquad(r=1,2)$$

for all $n\geqq1$.

In fact if we set $\alpha_0=0$, $\alpha_\rho=\sum_{i=1}^{\rho}\zeta_i^2$ $(\rho\geqq1)$, so that $\alpha_\rho\leqq\nu\rho$, then an elementary calculation shows that

$$\begin{aligned}\sum_{\rho=1}^{n}|J(\rho)|\zeta_\rho^2&=\sum_{\rho=1}^{n-1}(|J(\rho)|-|J(\rho+1)|)\alpha_\rho+|J(n)|\alpha_n\\&\leqq\sum_{\rho=1}^{n-1}(|J(\rho)|-|J(\rho+1)|)\nu\rho+|J(n)|\nu n=\nu\sum_{\rho=1}^{n}|J(\rho)|\end{aligned}$$

and similarly for the other inequalities. Note that

$$\frac{1}{n}\sum_{\rho=1}^{n}|\zeta_\rho|\leqq\left(\frac{1}{n}\sum_{\rho=1}^{n}\zeta_\rho^2\right)^{\frac{1}{2}}\leqq\sqrt{\nu}\qquad\text{for }\zeta_1\zeta_2\ldots\in\vec{M}_\nu.$$

A DLR measure Π admitted by the specification $\{Q\}$ is said to be weakly tame with respect to the norm $\|\cdot\|_J$ if for every $\varepsilon>0$ there is a $d>0$ such that $\overleftarrow{\Pi}_i\{x\in\overleftarrow{T}_0:\|x\|_J\leqq d\}\geqq1-\varepsilon$ and $\vec{\Pi}_i\{z\in\vec{T}_0:\|z\|_J\leqq d\}\geqq1-\varepsilon$ for all i. Corollary 2.2 above and Proposition 2.4 of Part I imply that if Π is weakly tame with respect to the norm $\|\cdot\|_J$, then for every $\varepsilon>0$ there is $\nu\geqq1$ such that

$$\overleftarrow{\Pi}_i(\overleftarrow{M}_\nu)\geqq1-\varepsilon\quad\text{and}\quad\vec{\Pi}_i(\vec{M}_\nu)\geqq1-\varepsilon\tag{5}$$

for all i. This, combined with Lemma 2.3 implies that it is sufficient to consider the kernels $Q_{l,m}^{[i,j]}(x,\cdot,z)$ defined for $x\in\overleftarrow{T}$, $z\in\vec{T}$, where

$$\vec{T}=\left\{\zeta_1\zeta_2\ldots:\sum_{\rho=1}^{\infty}\Phi(\rho)|\zeta_\rho|<\infty\right\},$$

$$\overleftarrow{T}=\left\{\ldots\zeta_{-2}\zeta_{-1}:\sum_{\rho=1}^{\infty}\Phi(\rho)|\zeta_{-\rho}|<\infty\right\}.$$

The metrics we introduce in $\vec{T}$ and $\overleftarrow{T}$ are the ones induced by the norms

$$\|z\|_\Phi=\sum_{\rho=1}^{\infty}\Phi(\rho)|\zeta_\rho|\quad\text{and}\quad\|x\|_\Phi=\sum_{\rho=1}^{\infty}\Phi(\rho)|\zeta_{-\rho}|$$

respectively. Notice that if we regard $\Phi(\cdot)$ as a mass function on $\{1,2,\ldots\}$, then it defines a finite measure on this space. The corresponding L_1-space is $\vec{T}$ and $\|\cdot\|_\Phi$ is its L_1-norm. Norm convergence in $\vec{T}$ is equivalent to $\sigma(L_1,L_\infty)$-convergence and therefore ([6, Prop. IV-2-3]) a subset $C\subset\vec{T}$ is metrically relatively compact if and only if its elements are uniformly integrable (=uniformly Φ-summable), i.e. for every $\varepsilon>0$ there is $\delta>0$ such that $\sum_{\rho\in I}\Phi(\rho)|\zeta_\rho|<\varepsilon$ for all $\zeta_1\zeta_2\ldots\in C$ and all subsets I of $\{1,2,\ldots\}$ satisfying $\sum_{\rho\in I}\Phi(\rho)<\delta$.

2.4. Lemma. *The sets* $\vec{M}_\nu$ *and* $\overleftarrow{M}_\nu$ *are compact subsets of* $\vec{T}$ *and* $\overleftarrow{T}$ *respectively, relative to the norms* $\|\cdot\|_\Phi$.

In fact, $\vec{M}_\nu$ is metrically closed and by Lemma 2.3

$$\vec{M}_\nu\subset\left\{\zeta_1\zeta_2\ldots:\ \sum_{\rho=1}^{\infty}\Phi(\rho)\zeta_\rho^2\leqq\nu K\right\},$$

where the set on the right is relatively compact, since its elements are uniformly Φ-summable.

Before proving the main result we link the condition of weak tameness with some other conditions that have been used in the literature ([2, 4, 5, 8, 11, 12]).

2.5. Theorem. *If* Π *is a DLR measure admitted by the specification* $\{Q\}$ *and* $\ldots,X_{-1},X_0,X_1,\ldots$ *is a random field with distribution* Π, *then the following conditions are equivalent.*

(i) *Π is weakly tame with respect to the norm* $\|\cdot\|_J$.

(ii) *For every* $\varepsilon>0$ *there is* $\nu\geqq1$ *such that* (5) *holds for all i.*

(iii) *With probability one, the sequence* $\frac{1}{2n+1}\sum_{i=-n}^{n}X_i^2$, $n=1,2,\ldots$ *is bounded (i.e. Π is tempered).*

(iv) *There are constants* $\gamma>0$ *and* δ *such that for every segment of integers* $\Lambda=\{i,i+1,\ldots,j\}$

$$\Pi^{[i,j]}(d\zeta_i d\zeta_{i+1}\ldots d\zeta_j)\leqq\exp\left\{\sum_{\rho=i}^{j}(-\gamma\zeta_\rho^2+\delta)\right\}d\zeta_i\ldots d\zeta_j.$$

(v) $\sup_i E|X_i|<\infty$, *where E denotes expectation.*

(vi) *Π is tame with respect to the norm* $\|\cdot\|_\Phi$, *in the sense of Definition* 2.3 *of Part I.*

Proof. That (i) implies (ii) was shown above, when (5) was established. Trivially (ii) implies (iii). The deeper implication (iii)⇒(iv) was proved in [5]; a statement of this can also be found in [2]. It is obvious that (iv) implies (v).

We next prove that (v) implies (i). Suppose (v) holds and let $p=\sup_i E|X_i|$. Given $\varepsilon>0$ let $d=\Phi(1)p\varepsilon^{-1}$ and note that, if $(\Omega,\mathscr{F},P)$ is the probability space on which the random field $\ldots,X_{-1},X_0,X_1,\ldots$ is defined then, for any i,

$$P\{\|X_{i+1}X_{i+2}\ldots\|_J>d\}=P\left\{\sum_{\rho=1}^{\infty}|J(\rho)|\cdot|X_{i+\rho}|>d\right\}\leqq d^{-1}E\left(\sum_{\rho=1}^{\infty}|J(\rho)|\cdot|X_{i+\rho}|\right)$$
$$\leqq d^{-1}p\sum_{\rho=1}^{\infty}|J(\rho)|=d^{-1}p\Phi(1)=\varepsilon,$$

establishing (i). To complete the proof of the theorem note that (ii)⇒(vi)⇒(i).

2.6. Corollary. *If Π is translation invariant then it satisfies conditions* (i)–(vi).

In fact if Π is translation invariant, then it satisfies (vi). Thus Theorem 2.5 provides an alternative proof of DeMasi's result in [3] that a translation invariant Π satisfies (iii) and hence also (iv). In connection with the equivalence of (i) and (vi) see also Corollary 2.5 of Part I.

2.7. *Definition.* A DLR measure Π admitted by the specification $\{Q\}$ will be called *regular* if it satisfies any (and hence all) of the conditions (i)–(vi) of Theorem 2.5.

We will now verify that hypotheses I–V of Part I hold with respect to the metric topology induced by $\|\cdot\|_\Phi$. The specification is trivially translation invariant and hence hypothesis I is true. If $\Lambda=\{i,i+1,\dots,j\}$, then by (2)

$$\begin{aligned} f_\Lambda(x,w,z) &\leqq \prod_{\rho\in\Lambda} \exp\{\beta(-a\zeta_\rho^2+(\sum_{l\notin\Lambda} J(|\rho-l|)\zeta_l)\zeta_\rho+c)\} \\ &= \prod_{\rho\in\Lambda} \exp\{\beta(-a\zeta_\rho^2+b(x,z)\zeta_\rho+c)\} \\ &\equiv g_\Lambda(x,w,z) \quad \text{say.} \end{aligned}$$

Now if $x_n\to x$, $z_n\to z$ in the norms $\|\cdot\|_\Phi$ (or even the norms $\|\cdot\|_J$), then $f_\Lambda(x_n,w,z_n)\to f_\Lambda(x,w,z)$, $g_\Lambda(x_n,w,z_n)\to g(x,w,z)$ and $\int g_\Lambda(x_n,w,z_n)\,dw \to\int g_\Lambda(x,w,z)\,dw$ by an explicit calculation. This and the inequality

$$\begin{aligned} f_\Lambda(x_n,w,z_n) \leqq & \min\{f_\Lambda(x_n,w,z_n),\, g_\Lambda(x,w,z)\} \\ &+g_\Lambda(x_n,w,z_n)-\min\{g_\Lambda(x_n,w,z_n),\, g_\Lambda(x,w,z)\} \end{aligned}$$

imply $\int f_\Lambda(x_n,w,z_n)\,dw\to\int f_\Lambda(x,w,z)\,dw$, i.e. $\sigma_\Lambda(x_n,z_n)\to\sigma_\Lambda(x,z)$, and hence $q_\Lambda(x_n,w,z_n)\to q_\Lambda(x,w,z)$ and $\int|q_\Lambda(x_n,w,z_n)-q_\Lambda(x,w,z)|\,dw\to 0$, since the q_Λ's are probability densities. This establishes hypothesis II. The absolute continuity of $Q^{[i,j]}_{i-1,j+1}(x,\cdot,z)$ with respect to $Q^{[i,j]}_{i-1,j+1}(\bar{x},\cdot,\bar{z})$ (hypothesis III) is trivially true. Hypothesis IV is an easy consequence of Corollary 2.2 and Lemma 2.3.

There only remains hypothesis V to verify. First note that if $f_1(w)$, $f_2(w)$ are positive measurable functions on a measure space M, such that $\sigma_1=\int_M f_1(w)\,dw$, $\sigma_2=\int_M f_2(w)\,dw$ are finite, then

$$\begin{aligned} \int_M |\sigma_1^{-1}f_1(w)-\sigma_2^{-1}f_2(w)|\,dw &\leqq \sigma_1^{-1}\int_M |f_1(w)-f_2(w)|\,dw+|\sigma_1^{-1}-\sigma_2^{-1}|\int_M f_2(w)\,dw \\ &\leqq 2\sigma_1^{-1}\int_M |f_1(w)-f_2(w)|\,dw \leqq 2\sup_{w\in M}\left|1-\frac{f_2(w)}{f_1(w)}\right|. \end{aligned}$$

If $\Lambda=\{1,2,\dots,i\}$ and we set $M=M_\mu(i)$, $f_1(w)=f_\Lambda(x,w,z)$, $f_2(w)=f_\Lambda(\bar{x},w,\bar{z})$, then we obtain

$$\begin{aligned} &\left\|\frac{Q^{[1,i]}_{0,i+1}(x,(\cdot)\cap M_\mu(i),z)}{Q^{[1,i]}_{0,i+1}(x,M_\mu(i),z)}-\frac{Q^{[1,i]}_{0,i+1}(\bar{x},(\cdot)\cap M_\mu(i),\bar{z})}{Q^{[1,i]}_{0,i+1}(\bar{x},M_\mu(i),\bar{z})}\right\| \\ &\leqq 2\sup_{w\in M_\mu(i)}|1-\exp\{\beta\sum_{\rho\in\Lambda,\,l\notin\Lambda} J(|\rho-l|)(\bar{\zeta}_l-\zeta_l)\zeta_\rho\}| \end{aligned} \tag{6}$$

where $w=\zeta_1\zeta_2\dots\zeta_i$, $x=\dots\zeta_{-2}\zeta_{-1}$, $\bar{x}=\dots\bar{\zeta}_{-2}\bar{\zeta}_{-1}$, $z=\zeta_{i+1}\zeta_{i+2}\dots$, $\bar{z}=\bar{\zeta}_{i+1}\bar{\zeta}_{i+2}\dots$.

Now suppose A is a metrically compact subset of $\overleftarrow{T}$. Given $\varepsilon>0$ choose r so that $\sum_{j\geqq r}\Phi(j)$ is sufficiently small to imply $\sum_{j\geqq r}\Phi(j)|\eta_{-j}|<\frac{\varepsilon}{8\beta\nu}$ for arbitrary $\ldots\eta_{-2}\eta_{-1}\in A$. Since there is $c_0<\infty$ such that $|\eta_{-r+1}|\leqq c_0$, $|\eta_{-r+2}|\leqq c_0,\ldots,|\eta_{-1}|\leqq c_0$ for arbitrary $\ldots\eta_{-2}\eta_{-1}\in A$, it is clear that if k is sufficiently large then

$$\sum_{j\geqq 1}\Phi(k+j)|\eta_{-j}|<\frac{\varepsilon}{4\beta\nu} \tag{7}$$

and in particular if w, x and $\bar{x}$ above are such that $w\in M_\mu(i)$, $\ldots\zeta_{-k-1}\zeta_{-k}\in A$, $\ldots\bar{\zeta}_{-k-1}\bar{\zeta}_{-k}\in A$ and $\zeta_l=\bar{\zeta}_l$ for $l=-k+1,-k+2,\ldots,0$, then by Lemma 2.3

$$\begin{aligned}|\sum_{l\leqq 0}\sum_{\rho\in A}J(|\rho-l|)(\bar{\zeta}_l-\zeta_l)\zeta_\rho|&\leqq\sum_{l\leqq -k}\{|\bar{\zeta}_l-\zeta_l|\cdot\nu\sum_{\rho>|l|}|J(\rho)|\}\\&=\nu\sum_{l\leqq -k}\Phi(|l|+1)(|\bar{\zeta}_l|+|\zeta_l|)<\frac{\varepsilon}{2\beta}\end{aligned}$$

by (7). This, combined with a similar summation over $l\geqq i+1$, shows that the right-hand side of (6) is less than or equal to $2(e^\varepsilon-1)$. This implies the validity of hypothesis V. Note that there was no need here to impose a restriction on the coordinates ζ_l, $\bar{\zeta}_l$ for $l=-k+1,\ -k+2,\ldots,0$, other than $\zeta_l=\bar{\zeta}_l$.

2.8. Theorem. *If* (1) *holds, then the specification* $\{Q\}$ *admits exactly one regular DLR measure. This DLR measure is translation invariant.*

The existence of such a DLR measure was established in [5]; see also [2]. Uniqueness follows from Theorem 5.1 of Part I and Theorem 2.5 above (cf. Definition 2.7). Translation invariance follows as in Theorem 5.1 of Part I: if Π is a regular DLR measure admitted by $\{Q\}$, then so is any translate of Π, which must therefore coincide with Π.

2.9. Corollary. *If* (1) *holds, then the specification* $\{Q\}$ *admits exactly one translation invariant DLR measure.*

This follows from the fact that every translation invariant DLR measure is regular (Corollary 2.6; see also [3]).

It is worth mentioning that in the present case an alternative approach is to verify hypotheses IV_1, IV_2 and V_1 of Part I, instead of IV and V.

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