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Lindeberg Functions and the Law of the Iterated Logarithm*

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Summary. For a sequence of independent random variables $\{X_n\}$ with zero means and finite variances, define $S_n=\sum_{j=1}^{n} X_j$, $s_n^2=E(S_n^2)$ and $t_n^2=2\log\log s_n^2$; assume $s_n\to\infty$. Kolmogorov's law of the iterated logarithm asserts that $\limsup_{n\to\infty} S_n/(s_n t_n)=1$ a.s. if $t_n|X_n|\leqq\varepsilon_n s_n$ for some real sequence $\varepsilon_n\to 0$. This paper will show that, under the weaker condition $t_n X_n/s_n\to 0$ a.s., the a.s. limiting value of $\limsup_{n\to\infty} S_n/(s_n t_n)$ depends on the limiting behaviour of the modified Lindeberg functions

$$s_n^{-2}\sum_{j=1}^{n} E(X_j^2 I(|X_j|\leqq\varepsilon s_j t_j^{-1})),\qquad \text{where } \varepsilon>0.$$

1. Introduction

Consider a sequence $X_1, X_2, \ldots$ of independent random variables (r.v.) with $E(X_n)=0$ and $E(X_n^2)<\infty$ for $n\geqq 1$. Define $S_n=\sum_{j=1}^{n} X_j$, $s_n^2=E(S_n^2)$ and $t_n=(2\log\log s_n^2)^{1/2}$. Assume $s_n\to\infty$.

According to Kolmogorov's law of the iterated logarithm (LIL), if a positive sequence $\varepsilon_n\to 0$ exists such that

$$t_n|X_n|\leqq\varepsilon_n s_n \text{ almost surely (a.s.) for all } n\geqq 1, \tag{1.1}$$

then

$$\Lambda\equiv\limsup_{n\to\infty} S_n/(2s_n^2\log\log s_n^2)^{1/2}=1\quad\text{a.s.}. \tag{1.2}$$

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This paper will investigate the value of Λ when (1.1) is replaced by the weaker condition:

$$t_n X_n / s_n \to 0 \quad \text{a.s.} \tag{1.3}$$

or, equivalently (by the Borel Zero-One Law),

$$\sum_{n=1}^{\infty} P[|X_n| > \varepsilon s_n t_n^{-1}] < \infty \qquad \text{for every } \varepsilon > 0. \tag{1.4}$$

Under the less restrictive assumption (1.3), Λ need not be one. For example, suppose $P[X_n = \pm n] = (2n^2)^{-1}$ and $P[X_n = 0] = 1 - n^{-2}$ for $n \geqq 1$. Then $s_n^2 = n$ and, for $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P[|X_n| > \varepsilon s_n t_n^{-1}] \leqq \sum_{n=1}^{\infty} P[X_n \neq 0] = \sum_{n=1}^{\infty} n^{-2} < \infty,$$

establishing (1.4). But, since $\sum_{n=1}^{\infty} P[X_n \neq 0] < \infty$, $P[X_n \neq 0$ infinitely often (i.o.)$] = 0$ by the Borel-Cantelli lemma. Consequently, $\sum_{n=1}^{\infty} X_n$ converges a.s. so that, trivially, $S_n/(s_n t_n) \to 0$ a.s.; i.e. $\Lambda = 0$.

Theorem 1 of Teicher [6] implicitly suggests a relationship between the value of Λ and the Lindeberg functions

$$L_n(x) = s_n^{-2} \sum_{j=1}^{n} E(X_j^2 I(|X_j| > x s_j)),$$

where $I(A)$ denotes the indicator function of the event A. Teicher's LIL asserts that $\Lambda = 1$ if, for some $\delta > 0$, $\sum_{n=1}^{\infty} P[|X_n| > \delta s_n t_n] < \infty$ and, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} (s_n t_n)^{-2} E(X_n^2 I(\varepsilon s_n t_n^{-1} < |X_n| \leqq \delta s_n t_n) < \infty$$

and

$$\lim_{n \to \infty} L_n(\varepsilon t_n^{-1}) = 0. \tag{1.5}$$

Since (1.4) implies Teicher's first two hypotheses, (1.2) must hold when (1.3) and (1.5) hold.

Rather than deal with $L_n(x)$ or $L_n(x t_n^{-1})$, it will be more convenient in this paper to work with the functions

$$H_n(x) \equiv s_n^{-2} \sum_{j=1}^{n} E(X_j^2 I(|X_j| \leqq x s_j t_j^{-1})). \tag{1.6}$$

Here is the main result of the paper.

Theorem 1.1. *Let $X_1, X_2, \ldots$ be independent r.v. with $E(X_n) = 0$ and $E(X_n^2) < \infty$ for all $n \geqq 1$. For $n \geqq 1$, define $S_n = \sum_{j=1}^{n} X_j$ and $s_n^2 = E(S_n^2)$. Suppose $s_n \to \infty$ and*

$(\log\log s_n^2)^{\frac{1}{2}}|X_n|/s_n \to 0$ *a.s. Then*

$$H_- \leqq \limsup_{n\to\infty} S_n/(2s_n^2 \log\log s_n^2)^{\frac{1}{2}} \leqq H_+ \quad a.s., \tag{1.7}$$

where H_- *and* H_+ *are numbers satisfying* $0 \leqq H_- \leqq H_+ \leqq 1$ *and* (*cf.* (1.6))

$$H_-^2 \equiv \liminf_{n\to\infty} H_n(x), \qquad H_+^2 = \limsup_{n\to\infty} H_n(x); \tag{1.8}$$

the values H_- *and* H_+ *are independent of* x.

Theorem 1.1 will be proved in Sect. 3, following the establishment in Sect. 2 of several lemmas, some of which may be of interest in themselves. Section 4 will contain a number of examples which pertain to the main result.

2. Some Preliminary Results

The following lemma is known (cf. Egorov [2], p. 512).

Lemma 2.1. *Let* $\{a_n(\varepsilon),\ n \geqq 1\}$ *be a sequence of non-negative functions defined for all* $\varepsilon > 0$.

(i) *If* $\lim_{n\to\infty} a_n(\varepsilon) = 0$ *for every* $\varepsilon > 0$, *then a sequence* $\{\varepsilon_n\}$ *exists such that* $\varepsilon_n \downarrow 0$ *and* $a_n(\varepsilon_n) \to 0$ *as* $n \to \infty$.

(ii) *If* $\sum_{n=1}^{\infty} a_n(\varepsilon) < \infty$ *for every* $\varepsilon > 0$, *then there exists a sequence* $\{\varepsilon_n\}$ *such that* $\varepsilon_n \downarrow 0$ *and* $\sum_{n=1}^{\infty} a_n(\varepsilon_n) < \infty$.

Lemma 2.2. *Let* $\{a_n(\varepsilon)\}$ *be a sequence of non-negative functions, defined for all* $\varepsilon > 0$. *Define* $a^* = \liminf_{\varepsilon\downarrow 0} \liminf_{n\to\infty} a_n(\varepsilon)$. *Then there exists a sequence* $\{\varepsilon_n\}$ *such that* $\varepsilon_n \downarrow 0$ *and* $\liminf_{n\to\infty} a_n(\varepsilon_n) \geqq a^*$. *Moreover, if* $a_n(\varepsilon)$ *is a non-decreasing function of* ε *for each* $n \geqq 1$, *then* $\liminf_{n\to\infty} a_n(\delta_n) \leqq a^*$ *for every real sequence* $\{\delta_n\}$ *satisfying* $\delta_n \downarrow 0$.

Proof. Define $f_n(\varepsilon) = \inf_{m \geqq n} a_m(\varepsilon)$ for $n \geqq 1$ and let $a(\varepsilon) = \liminf_{n\to\infty} a_n(\varepsilon)$ where $\varepsilon > 0$. Then $a(\varepsilon) - f_n(\varepsilon) \downarrow 0$ as $n \to \infty$ for every $\varepsilon > 0$. By Lemma 2.1 (i), a sequence $\{\varepsilon_n\}$ exists such that $\varepsilon_n \downarrow 0$ and $\lim_{n\to\infty} [a(\varepsilon_n) - f_n(\varepsilon_n)] = 0$. However, $\liminf_{n\to\infty} a(\varepsilon_n) \geqq a^*$, so $\lim_{n\to\infty} f_n(\varepsilon_n) \geqq a^*$. However, $a_n(\varepsilon_n) \geqq f_n(\varepsilon_n)$ so $\liminf_{n\to\infty} a_n(\varepsilon_n) \geqq a^*$.

If $a_n(\varepsilon)$ is non-decreasing for each $n \geqq 1$, then, for every $\varepsilon > 0$, $a_n(\delta_n) \leqq a_n(\varepsilon)$ for all large n if $\delta_n \downarrow 0$. Hence $\liminf_{n\to\infty} a_n(\delta_n) \leqq \liminf_{n\to\infty} a_n(\varepsilon)$ and the proof is completed by letting $\varepsilon \downarrow 0$. □

Using Hartman's modificiation [3] of Kolmogorov's techniques, the final lemma will establish a refinement of Corollary 1 of Teicher [8] and of Lemma 1 of Tomkins [9].

Lemma 2.3. *Let $(M_n, \mathscr{F}_n, n \geqq 1)$ be a submartingale and let $\{\alpha_n\}$, $\{B_n\}$ and $\{c_n\}$ be positive real sequences. Suppose $B_n \uparrow \infty$ and define $\alpha = \limsup_{n\to\infty} \alpha_n$, $\gamma = \limsup_{n\to\infty} (\log\log B_n^2)^{\frac{1}{2}} c_n$, and $g(x) = x^{-2}(e^{-x} - 1 - x)$. Assume $\alpha < \infty$ and $\gamma < \infty$.*

If positive numbers C, N and T exist such that

$$E \exp\{t M_n/(\alpha_n B_n)\} \leqq C \exp\{t^2 g(t c_n)\} \tag{2.1}$$

whenever $n \geqq N$ and $0 \leqq t c_n < T$, then

$$\limsup_{n\to\infty} \frac{M_n}{(B_n^2 \log\log B_n^2)^{\frac{1}{2}}} \leqq \alpha K_\gamma \quad a.s. \tag{2.2}$$

where

$$K_0 = 2^{\frac{1}{2}} \quad and \quad K_\gamma = \min_{0 < b \leqq T\gamma^{-1}} (b^{-1} + b g(\gamma b)) \quad for \quad \gamma > 0. \tag{2.3}$$

Proof. Fix b such that $0 < b \leqq T/\gamma$ ($b > 0$ if $\gamma = 0$). Pick any $\alpha' > \alpha$ and let $\delta = \alpha'(b^{-1} + b g(\gamma b))$. Then choose $c > 1$ so close to 1 that $\delta' \equiv \delta c^{-2} > \alpha'(b^{-1} + b g(\gamma b))$; consequently $b\delta'/\alpha' - b^2 g(\gamma b) > 1$. Define $n_0 = 1$ and, for $k \geqq 1$, $n_k = \min\{n \mid B_n \geqq c B_{n_{k-1}}\}$, $M_k^* = \max_{n_{k-1} \leqq n < n_k} M_n$ and $m_k = n_k - 1$. Let $b_n = (\log\log B_n^2)^{\frac{1}{2}}$ for $n \geqq 1$.

Choose any $\gamma' > \gamma$ so close to γ that $\lambda \equiv b\delta'/\alpha' - b^2 g(\gamma' b) > 1$. Then, for all k so large that $m_k > N$, $\alpha_{m_k} < \alpha'$ and $b_{m_k} c_{m_k} < \gamma'$, it follows from Markov's inequality, Doob's inequality ([1], p. 314) and (2.1) that

$$\begin{aligned}
&P[M_k^* \geqq \delta' B_{m_k} b_{m_k}] \\
&\quad \leqq P[\exp\{b b_{m_k} M_k^*/(\alpha_{m_k} B_{m_k})\} \geqq \exp\{b\delta' b_{m_k}^2/\alpha'\}] \\
&\quad \leqq C \exp\{-b\delta' b_{m_k}^2/\alpha' + b^2 b_{m_k}^2 g(b b_{m_k} c_{m_k})\} \\
&\quad \leqq C \exp\{-b\delta' b_{m_k}^2/\alpha' + b^2 b_{m_k}^2 g(\gamma' b)\} \\
&\quad = C \exp\{-\lambda b_{m_k}^2\} = C(\log B_{m_k}^2)^{-\lambda} \leqq C(\log B_{n_{k-1}}^2)^{-\lambda} = O((k-1)^{-\lambda}).
\end{aligned}$$

Since $\lambda > 1$, $P[M_k^* \geqq \delta' B_{m_k} b_{m_k}$ i.o.$] = 0$. But $B_{n_{k-1}} \leqq B_{m_k} < c B_{n_{k-1}}$, so $b_{m_k} \sim b_{n_{k-1}}$ (i.e. $b_{m_k}/b_{n_{k-1}} \to 1$) as $k \to \infty$ and

$$\begin{aligned}
P[M_n \geqq \delta B_n b_n \text{ i.o.}] &\leqq P[M_k^* \geqq \delta B_{n_{k-1}} b_{n_{k-1}} \text{ i.o.}] \\
&\leqq P[M_k^* \geqq \delta c^{-1} B_{m_k} b_{n_{k-1}} \text{ i.o.}] \\
&\leqq P[M_k^* \geqq \delta' B_{m_k} b_{m_k} \text{ i.o.}] = 0.
\end{aligned}$$

Therefore, $\limsup_{n\to\infty} M_n/(B_n b_n) \leqq \delta = \alpha'(b^{-1} + b g(\gamma b))$ for all $\alpha' > \alpha$ and $0 < b \leqq T/\gamma$ (where $T/\gamma \equiv \infty$ when $\gamma = 0$), proving (2.2). Note that, when $\gamma = 0$, $K_0 = \min_{b>0} (b^{-1} + b g(0)) = 2^{\frac{1}{2}}$ as shown by Teicher [8]. □

Remark. Lemma 2.3 remains valid under the less stringent hypothesis (cf. [1], p. 295) that $(e^{tM_n}, \mathscr{F}_n, n \geqq 1)$ is a submartingale for every $t > 0$.

Corollary 2.4 (cf. Corollary 1 of [8]). *Let $\{X_n\}$ be a sequence of independent r.v. with zero means and finite variances. Define $S_n = \sum_{j=1}^{n} X_j$ and suppose*

$s_n^2 \equiv E(S_n^2) \to \infty$. *If* $X_n \leqq c_n s_n$ *for some positive sequence* $\{c_n\}$ *and all* $n \geqq 1$ *and* $a \equiv \limsup_{n\to\infty} c_n(\log\log s_n^2)^{\frac{1}{2}} < \infty$, *then* $\limsup_{n\to\infty} S_n/(s_n^2 \log\log s_n^2)^{\frac{1}{2}} \leqq K_a$ *a.s. where* K_a *is defined by* (2.3).

Proof. By Lemma 1 (i) of Teicher [8], (2.1) holds with $M_n = S_n$, $\alpha_n = 1$, $B_n = s_n$, $N = C = 1$ and any $T > 0$. Since $\alpha = 1$ and $\gamma = a$ in this case, the desired result is clear from Lemma 2.3. □

Remark. Corollary 2.4 shows that the hypothesis in Corollary 1 (i) of [8] that $E(X_n^2) = o(s_n^2)$ is unnecessary.

3. Proof of Theorem 1.1.

First, it will be shown that, for any pair $\varepsilon_2 > \varepsilon_1 > 0$,

$$\lim_{n\to\infty} (H_n(\varepsilon_2) - H_n(\varepsilon_1)) = 0; \tag{3.1}$$

this fact establishes the independence of H_- and H_+ from the value of x in (1.8). For $n \geqq 1$, define the event $A_n = [\varepsilon_1 s_n < t_n |X_n| \leqq \varepsilon_2 s_n]$ and the r.v. $Y_n = X_n I(A_n)$. Using (1.4),

$$\sum_{n=1}^{\infty} t_n^2 E(Y_n^2)/s_n^2 \leqq \varepsilon_2^2 \sum_{n=1}^{\infty} P(A_n) < \infty,$$

so $(t_n/s_n)^2 \sum_{j=1}^{n} E(Y_j^2) \to 0$ by Kronecker's lemma. (3.1) is now clear because

$$H_n(\varepsilon_2) - H_n(\varepsilon_1) = s_n^{-2} \sum_{j=1}^{n} E(Y_j^2).$$

Turn now to the proof of (1.7). Let $\varepsilon > 0$. For $j \geqq 1$, define $P_j = P[|X_j| > \varepsilon s_j t_j^{-1}]$. Then, using the Cauchy-Schwarz inequality twice,

$$\begin{aligned} \sum_{j=1}^{n} E(|X_j|\, I(|X_j| > \varepsilon s_j t_j^{-1})) &\leqq \sum_{j=1}^{n} (E(X_j^2))^{\frac{1}{2}} P_j^{\frac{1}{2}} \\ &\leqq \left(\left(\sum_{j=1}^{n} E(X_j^2)\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} P_j\right)\right)^{\frac{1}{2}} \leqq s_n \left(\sum_{j=1}^{\infty} P_j\right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, in view of (1.4),

$$(s_n t_n)^{-1} \sum_{j=1}^{n} E(|X_j|\, I(|X_j| > \varepsilon s_j t_j^{-1})) \to 0 \tag{3.2}$$

for every $\varepsilon > 0$.

Since (1.4) and (3.2) hold, Lemmas 2.1 and 2.2 ensure the existence of a sequence $\varepsilon_n \downarrow 0$ satisfying

$$\sum_{n=1}^{\infty} P[|X_n| > \varepsilon_n s_n t_n^{-1}] < \infty, \tag{3.3}$$

$$(s_n t_n)^{-1} \sum_{j=1}^{n} E(|X_j| I(|X_j| > \varepsilon_j s_j t_j^{-1})) \to 0, \tag{3.4}$$

and

$$\liminf_{n\to\infty} H_n(\varepsilon_n) = H_-^2. \tag{3.5}$$

(Actually, these lemmas yield possibly distinct sequences ε_n', ε_n'' and ε_n''' in (3.3), (3.4) and (3.5) respectively, but all three statements remain true using $\varepsilon_n = \max(\varepsilon_n', \varepsilon_n'', \varepsilon_n''')$.)

Now, for $n \geqq 1$, define $X_n' = X_n I(|X_n| \leqq \varepsilon_n s_n t_n^{-1})$,

$$S_n' = \sum_{j=1}^{n} X_j', \quad (s_n')^2 = \operatorname{Var}(S_n') \quad \text{and} \quad t_n' = (2 \log\log s_n')^{\frac{1}{2}}.$$

Using (3.3) and the Borel-Cantelli lemma, $P[X_n \neq X_n' \text{ i.o.}] = 0$, so $\lim_{n\to\infty}(S_n - S_n')$ is finite a.s. and, hence, $(S_n - S_n')/(s_n t_n) \to 0$ a.s. Furthermore, $E(S_n - S_n')/(s_n t_n) \to 0$ by virtue of (3.4). Thus it will suffice to prove that

$$H_- \leqq \Lambda' \equiv \limsup_{n\to\infty} (S_n' - E(S_n'))/(s_n t_n) \leqq H_+ \text{ a.s.} \tag{3.6}$$

Because $E(X_j) = 0$,

$$\sum_{j=1}^{n} (E(X_j'))^2 \leqq \sum_{j=1}^{n} (\varepsilon_j s_j t_j^{-1}) |E(X_j - X_j')| \leqq \varepsilon_1 s_n t_n^{-1} \sum_{j=1}^{n} E|X_j - X_j'|$$

so $s_n^{-2} \sum_{j=1}^{n} (E(X_j'))^2 \to 0$ by (3.4). This means that

$$(s_n'/s_n)^2 = s_n^{-2} \sum_{j=1}^{n} E(X_j^2 I(|X_j| \leqq \varepsilon_j s_j t_j^{-1}) + o(1). \tag{3.7}$$

For $\varepsilon > 0$, choose N such that $\varepsilon_n < \varepsilon$ when $n \geqq N$. If $n \geqq N$,

$$\begin{aligned} H_n(\varepsilon_n) &\leqq s_n^{-2} \sum_{j=1}^{n} E(X_j^2 I(|X_j| \leqq \varepsilon_j s_j t_j^{-1})) \\ &\leqq s_n^{-2} \sum_{j=1}^{N} E(X_j^2 I(\varepsilon < t_j |X_j|/s_j \leqq \varepsilon_j) + H_n(\varepsilon), \end{aligned}$$

so (3.5) and (3.7) yield

$$\liminf_{n\to\infty} s_n'/s_n \geqq H_- \quad \text{and} \quad \limsup_{n\to\infty} (s_n'/s_n)^2 \leqq \limsup_{n\to\infty} H_n(\varepsilon) \quad \text{for all } \varepsilon.$$

Letting $\varepsilon \downarrow 0$,

$$H_- \leqq \liminf_{n\to\infty} s_n'/s_n \leqq \limsup_{n\to\infty} s_n'/s_n \leqq H_+. \tag{3.8}$$

If s_n' converges then $S_n' - E(S_n')$ converges a.s. by the Kolmogorov Convergence Theorem (Loève [4], p. 248) so $\Lambda' = 0 = H_-$ a.s. trivially, since $s_n \to \infty$. So it may be assumed hereinafter that $s_n' \to \infty$.

Let $m>1$ and $\lambda>H_+$. Define $Y_n=X_n'$ for $n\geqq m$ and $Y_n=X_n I(|X_n|\leqq\varepsilon_m s_n t_n^{-1})$ for $n<m$, and then let $R_n=\sum_{j=1}^{n}(Y_j-E(Y_j))$ and $r_n^2=E(R_n^2)$. Since $S_n'-\sum_{j=1}^{n} Y_j$ depends only on $X_1, X_2, \ldots, X_m$, clearly

$$(S_n'-E(S_n')-R_n)/(s_n t_n)\to 0 \text{ a.s.,} \tag{3.9}$$

and $r_n/s_n'\to 1$. In view of (3.8), then, and integer N exists such that $r_n\leqq\lambda s_n$ for $n\geqq N$.

Now, $|Y_n-E(Y_n)|\leqq 2\varepsilon_m s_n t_n^{-1}$, $n\geqq 1$. Since $\varepsilon_m s_n t_n^{-1}$ is non-decreasing as $n\to\infty$, Lemma 1(i) of Teicher [8] implies

$$E\exp\{t R_n/r_n\}\leqq\exp\{t^2 g(2t\varepsilon_m s_n/(r_n t_n))\}$$

for every $t>0$ and $n\geqq 1$, where $g(x)=x^{-2}(e^{-x}-1-x)$. Replacing t by $t r_n/(\lambda s_n)$ yields (2.1) for $n\geqq N$, with $M_n=R_n$, $C=1$, $\alpha_n=\lambda$, $B_n=s_n$, $c_n=2\varepsilon_m/(\lambda t_n)$ and any $T>0$. Since $\gamma=\gamma_m=\limsup_{n\to\infty} t_n c_n=2\varepsilon_m/\lambda<\infty$, Lemma 2.3 implies

$$\limsup_{n\to\infty} R_n/(s_n t_n)\leqq 2^{-\frac{1}{2}}\lambda K_{\gamma_m} \text{ a.s.}$$

for every $m>1$ and $\lambda>H_+$. In view of (3.9), then,

$$\limsup_{n\to\infty}(S_n'-E(S_n'))/(s_n t_n)\leqq 2^{-\frac{1}{2}} H_+ K_{\gamma_m} \text{ a.s.}$$

for every $m>1$. The right-hand side of (3.6) now obtains by letting $m\to\infty$, which implies $\gamma_m\to 0$ and $K_{\gamma_m}\to 2^{\frac{1}{2}}$.

By Čebyšev's inequality and (3.8), $P[|S_n'-E(S_n')|\geqq\varepsilon s_n t_n]\leqq(s_n'/(\varepsilon s_n t_n))^2\to 0$ as $n\to\infty$ for every $\varepsilon>0$. Hence $(S_n'-E(S_n'))/(s_n t_n)\to 0$ in probability, so $\Lambda'\geqq 0$. Therefore, (3.6) is true when $H_-=0$, so suppose $H_->0$. Since

$$|X_n'-E(X_n')|/s_n'\leqq 2\varepsilon_n s_n/(s_n' t_n)\equiv a_n \quad\text{and}\quad a_n t_n'=O(\varepsilon_n t_n' t_n^{-1} H_-^{-1})=O(\varepsilon_n),$$

Kolmogorov's LIL implies $\limsup_{n\to\infty}(S_n'-E(S_n'))/(s_n' t_n')=1$ a.s. Noting that $s_n H_-/2<s_n'<2s_n H_+$ for all large n when $H_->0$ (cf. (3.8)), clearly $t_n'\sim t_n$. Since $\liminf_{n\to\infty} s_n'/s_n\geqq H_-$, the left side of (3.6) follows. □

4. Some Illustrative Examples

The three examples to follow emphasize the importance of (1.3) in Theorem 1.1.

Example 4.1. Consider any numbers a, b such that $0\leqq a\leqq b\leqq 1$. An example will be constructed in which Theorem 1.1 applies with $H_-=a$ and $H_+=b$. To this end, let $\{\delta_n\}$ be a sequence with $\delta_n=0$ or 1, $a^2=\liminf_{n\to\infty}\sum_{j=1}^{n}\delta_j/n$ and $b^2=\limsup_{n\to\infty}\sum_{j=1}^{n}\delta_j/n$. Let $\{Y_n\}$, $\{Z_n\}$ be independent sequences, independent of each other, such that $P[Y_n = \pm 1] = 1/2$, $P[Z_n = \pm n] = 1/(2n^2)$ and $P[Z_n = 0] = 1 - n^{-2}$. Then let $X_n = \delta_n Y_n + (1-\delta_n) Z_n$. Clearly $E(X_n) = 0$, $E(X_n^2) = 1$ so $s_n^2 = n$. Moreover, for $\varepsilon > 0$, $X_n I(|X_n| \leqq \varepsilon s_n t_n^{-1}) = \delta_n Y_n$ for all n so large that $t_n < \varepsilon s_n < n t_n$, so $E(X_n^2 I(|X_n| \leqq \varepsilon s_n t_n^{-1})) = \delta_n^2 = \delta_n$ for all such n. Hence

$$\lim_{n\to\infty} \left| H_n(\varepsilon) - \sum_{j=1}^{n} \delta_j / n \right| = 0, \qquad \text{so } H_- = a,\ H_+ = b.$$

Note that (1.3) holds in this example, but (1.1) does not if $a < 1$. Indeed, if $a < 1$, then there is a subsequence $\{\delta_{n_k}\}$ such that $\delta_{n_k} = 0$ for all $k \geqq 1$. But then

$$|X_{n_k}| / s_{n_k} = |Z_{n_k}| / n_k^{\frac{1}{2}} = n_k^{\frac{1}{2}}$$

on the event $[Z_{n_k} \neq 0]$, so (1.1) is false. But, for $\varepsilon > 0$,

$$\sum_{n=N}^{\infty} P[|X_n| \geqq \varepsilon s_n t_n^{-1}] \leqq \sum_{n=N}^{\infty} P[Z_n \neq 0] < \infty,$$

where $N = \min\{n \mid \varepsilon s_n > t_n\}$, so (1.3) holds.

Example 4.2. It will be shown that (3.1) may fail when (1.3) doesn't hold by means of a well-known example (see Marcinkiewicz and Zygmund [5] or Theorem 5 of Teicher [6]). Let $Y_1, Y_2, \ldots$ be independent r.v. with $P[Y_n = \pm 1] = 1/2$ for all $n \geqq 1$. For $\lambda > 1$, let $\sigma_n^2 \equiv (\log n)^{-1} \exp(2\lambda n / \log n)$ and $X_n = \sigma_n Y_n$. Since $s_n^2 \sim (\log n)\sigma_n^2/(2\lambda)$ and, hence, $t_n^2 \sim 2 \log n$ (cf. [5] p. 219) in this case, the (constant) sequence $t_n^2 X_n^2 / s_n^2 \to 4\lambda$. Hence (1.3) is false, $H_n(\varepsilon) \to 1$ if $\varepsilon^2 > 4\lambda$, and $H_n(\varepsilon) \to 0$ if $\varepsilon^2 < 4\lambda$.

Example 4.3. Suppose $X_1, X_2, \ldots$ are independent r.v. such that X_n is normal with mean zero and variance n^n. Then Hartman's LIL [3] implies (1.2). But $s_n^2 \sim n^n$ so, for every $\varepsilon > 0$,

$$P[|X_n| > \varepsilon s_n t_n^{-1}] = P[Z > \varepsilon s_n / (t_n n^{n/2})] \to P[Z > 0] = 1/2,$$

where Z is a standard normal r.v. Hence (1.4) fails in this case. This shows that (1.4) is not necessary for (1.2).

Teicher [7] showed that (1.2) always implies $\limsup_{n\to\infty} X_n/(s_n t_n) \leqq 1$ a.s. In fact, the example above shows that Teicher's necessary condition is the best possible, in the sense that $\limsup_{n\to\infty} X_n/(s_n t_n) = 1$ a.s. above. It follows from the well-known fact that $P[Z > x] \sim e^{-x^2/2}/(2\pi x^2)^{\frac{1}{2}}$ as $x \to \infty$ that, for $\varepsilon > 0$,

$$P[|X_n| > \varepsilon s_n t_n] \sim (\pi \varepsilon^2 t_n^2 / 2)^{\frac{1}{2}} \exp\{-\varepsilon^2 s_n^2 \log\log s_n^2 / n^n\};$$

it follows readily that $\sum_{n=1}^{\infty} P[|X_n| > \varepsilon s_n t_n]$ converges if $\varepsilon > 1$ and diverges if $\varepsilon < 1$.

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