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On the Law of Iterated Logarithm for Occupation Measures of Empirical Processes

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Summary. Let $\{K(s,t)\colon 0 \leqq s \leqq 1,\ t \geqq 0\}$ be a Kiefer process. Let

$$L_t(E) = \int_0^1 I_E((t/\log\log t)^{-1/2} K(s,t))\, ds$$

denote the occupation distribution. Using the ideas of Mogul'skii, Donsker and Varadhan, the limit behavior of L_t is studied. These and strong approximation results are then used to derive LIL in Chung's form for various functions of empirical processes.

Introduction

Let $\{X_n\}$ be i.i.d. random variables with common distribution $F(x)=x$ for $0<x<1$. Let F_n denote the empirical distribution of $X_1, \ldots, X_n$. Recently Mogul'skii [4] proved that

$$\liminf_{n\to\infty} \sup_{0\leqq s\leqq 1} |F_n(s)-s| \sqrt{n \log\log n} = \frac{\pi}{\sqrt{8}} \quad \text{a.e.} \tag{1}$$

This result follows from strong approximations and a similar result for Kiefer process K, namely,

$$\liminf_{t\to\infty} \sup_{0\leqq s\leqq 1} |K(s,t)\, a(t)| = \frac{\pi}{\sqrt{8}} \quad \text{a.e.,} \tag{2}$$

where $a(t)=(t/\log\log t)^{-1/2}$. A Kiefer process $\{K(s,t)\colon 0\leqq s\leqq 1, t\geqq 0\}$ is a mean zero Gaussian process with continuous paths and satisfying

$$E(K(s,t)\,K(s',t')) = \{\min(s,s') - ss'\} \min(t,t').$$

The strong approximation result mentioned here is stated below for easy reference.

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Theorem A. (See Theorem 4.4.3 of [5]) *$F_n, n \geqq 1$ and K can be defined on the same probability space such that a.e.*

$$\sup_{0 \leqq s \leqq 1} |[n(F_n(s)-s)-K(s,n)]| \ll (\log n)^2. \tag{3}$$

In the case of Wiener process $\{W(s),\ s \geqq 0\}$, we have the following result due to Jain and Pruitt [3],

$$\liminf_{t \to \infty} \sup_{0 \leqq s \leqq t} |a(t)\, W(s)| = \frac{\pi}{\sqrt{8}} \quad \text{a.e.} \tag{4}$$

Using their powerful theory on Markov processes, Donsker and Varadhan [2] obtained results about the occupation distribution of $\{a(t)\, W(s)\colon 0 \leqq s \leqq t\}$. One of the consequences of these results is (4). In the same way we shall generalize (2) by studying the occupation measures of $\{K(s,t)\, a(t)\colon 0 \leqq s \leqq 1\}$.

As $\{K(s,1),\ 0 \leqq s \leqq 1\}$ is not a Markov process one cannot use Donskar-Varadhan theory directly. We use combination of several methods developed by Mogul'skii, Donsker and Vardhan, to study L_t defined by

$$L_t(E) = \int_0^1 I_E(a(t)\, K(s,t))\, ds.$$

L_t can be viewed as an element in the space M of sub-probability measures λ on $R(\lambda(R) \leqq 1)$, with the topology of vague convergence. Let $\mathscr{U}$ denote the class of infinitely differentiable functions u on R, which are constant outside a compact set and $0 < a \leqq u \leqq b < \infty$, where a, b depend on u. For $\lambda \in M$, let

$$I(\lambda) = -\inf_{u \in \mathscr{U}} \int \frac{u''}{2u}(x)\, \lambda(dx).$$

We shall show that the set of limit points of L_t is the set $C = \{\lambda \in M\colon I(\lambda) \leqq 1\}$. The I appearing here is the same I-function for the Brownian motion introduced by Donskar and Varadhan. It then follows for suitable functions Φ on M.

$$\limsup_{t \to \infty} \Phi(L_t) = \sup_{\lambda \in C} \Phi(\lambda). \quad \text{a.e.}$$

By taking $\Phi(\lambda) = \inf\{a\colon \lambda(x\colon |x| \leqq a) = 1\}$, we deduce (1) and (2). Several other consequences are given in the last section.

The Results

Theorem 1. *For almost all ω, the set of limit points, as $t \to \infty$, of L_t is $C = \{\lambda \in M\colon I(\beta) \leqq 1\}$.*

Proof of this theorem will be given later.

This has the following Corollary.

Corollary 2. *If Φ is a functional on M which is lower semi-continuous on M, then $\limsup_{t \to \infty} \Phi(L_t) \geqq \sup_{\lambda \in C} \Phi(\lambda)$ a.s. The inequality gets reversed, if instead, Φ is upper semi-continuous. In particular, equality holds if Φ is continuous on M.*

From Theorems A and 1 and Corollary 2, we have the following

Corollary 3. *The limit results of Theorem* 1 *and Corollary* 2 *hold, as* $n\to\infty$, *if* L_t *is replaced by* $\bar{L}_n$, *where*

$$\bar{L}_n(E)=\int_0^1 I_E(\sqrt{n\log\log n}(F_n(s)-s))\,ds.$$

Theorem 4 (Upper bound). *Let* D *be a closed set in* M. *Then*

$$\limsup_{t\to\infty}\frac{1}{t}\log P(A_t\in D)\leqq -\inf_{\lambda\in D} I(\lambda),$$

where $A_t(E)=\int_0^1 I_E(K(s,t))\,ds$.

To prove Theorem 4, we require the following two lemmas. First some notation. Put $\lambda(f)=\int f(x)\,\lambda(dx)$, for $\lambda\in M$.

Lemma 5. *Let* $J(\lambda,x)=\inf_{u\in\mathscr{U}}\lambda\left(\frac{u''}{2u}-x\frac{u'}{u}\right)$. *Then*

$$J(\lambda,x)\leqq -I(\lambda)-\tfrac{1}{2}x^2\lambda(R), \tag{5}$$

for all $x\in R$ *and* $\lambda\in M$.

Proof. If $\lambda(R)=0$, then the result is obvious. Suppose $\lambda(R)>0$ and λ has a density f, which is continuously differentiable and $f(y)>0$ for all $y\in R$. Now following the lines of proof of Lemma 2.2 of [1] (see Eqs. 2.33-2.35) we obtain

$$\int g(y)\,(f'(y)-2xf(y))\,dy\leqq\sqrt{-8J(\lambda,x)}\sqrt{g^2(y)\,f(y)\,dy}, \tag{6}$$

for all continuous functions g with compact support. Suppose ϕ has all derivatives, has compact support and $0\leqq\phi(y)\leqq 1$ for all $y\in R$. By putting $g=(f'-2xf)\,\phi/f$, we conclude

$$J(\lambda,x)\leqq -\tfrac{1}{8}\int\left(\frac{f'-2xf}{f}\right)^2(y)\,\phi(y)\,f(y)\,dy.$$

This holds for all such ϕ, so

$$J(\lambda,x)\leqq -\tfrac{1}{8}\int\frac{(f')^2}{f}(y)\,dy-\tfrac{1}{2}x^2\lambda(R)=-I(\lambda)-\tfrac{1}{2}x^2\lambda(R).$$

The last equality follows from Lemma 2.2 of [1]. Now suppose $\lambda\in M$ and $\lambda(R)>0$. For any $\varepsilon>0$, let Ψ_ε be the Gaussian distribution with mean 0 and variance ε. For our given λ, define $\lambda_\varepsilon=\lambda*\Psi_\varepsilon$. By concavity and the argument given above we have

$$\begin{aligned} J(\lambda,x)&\leqq J(\lambda_\varepsilon,x)\leqq -I(\lambda_\varepsilon)-\tfrac{1}{2}x^2\lambda_\varepsilon(R)\\ &=-I(\lambda_\varepsilon)-\tfrac{1}{2}x^2\lambda(R). \end{aligned} \tag{7}$$

Since I is lower semicontinuous on M and since λ_ε converges vaguely, we have $\liminf_{\varepsilon\to 0} I(\lambda_\varepsilon)\geqq I(\lambda)$. The lemma now follows from (7).

Before stating the next lemma, we note that for any $t>0, \{K(s,t)\colon 0 \leqq s \leqq 1\}$ and $\{W(s,t)-sW(t)\colon 0 \leqq s \leqq 1\}$ have the same distribution. Define

$$B_t(E)=\int_0^1 I_E(W(s\cdot t)-sW(t))\,ds,$$

and

$$\mathscr{C}=\{f\colon f \text{ is continuous on } R \text{ and } f(x)\to 0 \text{ as } |x|\to\infty\}.$$

Lemma 6. *Let $(a,b)\subset R$, $\lambda\in M$ and V a neighborhood of λ in M. Then there exists a open set N containing λ such that $N\subset V$ and*

$$\limsup_{t\to\infty}(1/t)\log P(B_t\in N,\ at<W(t)<bt) \leqq \inf_{u\in\mathscr{U}}\Big[\sup_{a\leqq y\leqq b}\big(\sup_{\beta\in V} G(\beta,y,u)\big)\Big],$$

where $G(\beta,y,u)=\beta\left(\frac{u''}{2u}-y\frac{u'}{u}\right)$.

Proof. Let $D(t,y)(E)=\frac{1}{t}\int_0^t I_E(W(s)-sy)\,ds$. First note that there exist $\varepsilon>0$, $f_1,\ldots,f_k\in\mathscr{C}$ such that

$$\{\beta\in M\colon |\beta(f_i)-\lambda(f_i)|<2\varepsilon,\ 1\leqq i\leqq k\}\subset V.$$

Put $N=\{\beta\in M\colon |\beta(f_i)-\lambda(f_i)|<\varepsilon,\ 1\leqq i\leqq k\}$. Since f_i are uniformly continuous, there exists a $\delta\in(0,1)$ such that

$$\max_{1\leqq i\leqq k}|f_i(x)-f_i(y)|<\varepsilon,$$

whenever $|x-y|<\delta$. So

$$(B_t\in N,\ at<W(t)<bt) \subset \bigcup_{at\leqq r\delta\leqq bt}(D(t,r\delta/t)\in V)\cap(|W(t)-r\delta|<\delta).$$

Hence

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log P(B_t\in N,\ at<W(t)<bt) &\leqq \limsup_{t\to\infty}\left(\sup_{a\leqq y\leqq b}\left(\frac{1}{t}\log[(2+|a|+|b|)(t/\delta)P(D(t,y)\in V)]\right)\right)\\ &=\limsup_{t\to\infty}\left[\sup_{a\leqq y\leqq b}\left(\frac{1}{t}\log P(D(t,y)\in V)\right)\right] \qquad (8)\end{aligned}$$

Because $\{W(s)-sy\colon s\geq 0\}$ is a Markov process with generator $\frac{1}{2}\frac{\partial^2}{\partial x^2}-y\frac{\partial}{\partial x}$, we have using Feynman-Kac formula that for any $u\in\mathscr{U}$,

$$E\left\{u(W(t)-ty)\exp\left[-\int_0^t\left(\frac{u''}{2u}-y\frac{u'}{u}\right)(W(s)-sy)\,ds\right]\right\}=u(o).$$

So

$$P(D(t,y)\in V)\leqq[\sup_x u(x)/\inf_x u(x)]\exp\left(t\sup_{\beta\in V}\beta\left(\frac{u''}{2u}-y\frac{u'}{u}\right)\right).$$

Thus

$$\limsup_{t\to\infty}\left[\sup_{a\leqq y\leqq b}\left(\frac{1}{t}\log P(D(t,y)\in V)\right)\right]$$
$$\leqq\inf_{u\in\mathcal{U}}(\sup_{a\leqq y\leqq b}(\sup_{\beta\in V}G(\beta,y,u))). \tag{9}$$

The lemma follws now from (8) and (9)

Proof of Theorem 4. Let $d>0$, $\varepsilon>0$ and

$$\sup_{|y|\leqq d}\sup_{\lambda\in D}\inf_{u\in\mathcal{U}}G(\lambda,y,u)=h<\infty$$

(if $h=\infty$ there is nothing to prove.) Note that for each $u\in\mathcal{U}$, $G(\lambda,y,u)$ is continuous in (λ,y). So for each $y\in[-d,d]$ and $\lambda\in D$, there exist an open interval I_y containing y, a neighborhood V_λ of λ and $u_{\lambda,y}\in\mathcal{U}$ such that $G(\beta,x,u_{\lambda,y})<h+\varepsilon$ for all $x\in I_y$ and $\beta\in V_\lambda$. Now by Lemma 6, there exists a open set $N_\lambda\subset V_\lambda$ and $\lambda\in N_\lambda$ such that

$$\limsup_{t\to\infty}\frac{1}{t}\log P(B_t\in N_\lambda,(W(t)/t)\in I_y)\leqq h+\varepsilon. \tag{10}$$

Since D is closed and M is compact, D is compact. As $D\times[-d,d]$ is compact for any $d>0$, we can choose $\lambda_1,\ldots,\lambda_k\in D$ and $y_1,\ldots,y_k\in[-d,d]$ such that

$$D\times[-d,d]\subset\bigcup_{i=1}^{k}(N_{\lambda_i}\times I_{y_i}). \tag{11}$$

From (10) and (11) we obtain, for any $d>2$, that

$$\limsup_{t\to\infty}\frac{1}{t}\log P(B_t\in D,|W(t)|\leqq td)<h+\varepsilon$$
$$=\sup_{|y|\leqq d}[\sup_{\beta\in D}J(\beta,y)]+\varepsilon$$
$$\leqq-\inf_{\beta\in D}I(\beta)+\varepsilon$$

the last inequality follows from Lemma 5. Since $\varepsilon>0$ is arbitrary and for $d>2$,

$$\limsup_{t\to\infty}\frac{1}{t}\log P(|W(t)|\geqq td)\leqq-\tfrac{1}{2}d^2<-d,$$

we have

$$\limsup_{t\to\infty}\frac{1}{t}\log P(B_t\in D)\leqq\max(-d,-\inf_{\beta\in D}I(\beta)).$$

The theorem now follows as $d>2$ is arbitrary and as A_t and B_t have the same distribution.

Theorem 7. (Lower bound) *Let $\mathcal{M}$ denote the space of probability measures on R with the topology of weak convergence. Let $\lambda \in \mathcal{M}$ with $\lambda(x\colon |x| \leqq a) = 1$ and N be a weak neighborhood of λ in $\mathcal{M}$, then for any $d > a > 0$*

$$\liminf_{t\to\infty} \frac{1}{t} \log P(A_t \in N, |K(s,t)| \leqq d;\ 0 \leqq s \leqq 1) \geqq -I(\lambda). \tag{12}$$

Proof. Without loss of generality we may take $I(\lambda) < \infty$ and

$$N = \{\beta \in \mathcal{M}\colon\ |\lambda(f_i) - \beta(f_i)| < \varepsilon,\ 1 \leqq i \leqq k\},$$

where $\varepsilon > 0$ and f_i are uniformly continuous and bounded functions on R. Let $|f_i(x)| \leqq H$ for all $x \in R, 1 \leqq i \leqq k$. By uniform continuity of f_i there exists $\delta > 0$ such that $|f_i(x) - f_i(y)| < (\varepsilon/4)$, whenever $|x - y| < \delta$, $1 \leqq i \leqq k$. Since $\{K(s,t)\colon 0 \leqq s \leqq 1\}$ has the same distribution as $\{W(s\cdot t) - s W(t)\colon 0 \leqq s \leqq 1\}$, the probability in (12) dominates

$$P\left(\left|\frac{1}{t}\int_0^t f_i(W(s))\,ds - \lambda(f_i)\right| < (3\varepsilon/4),\right.$$
$$\left. 1 \leqq i \leqq k,\ |W(s)| \leqq d - \delta,\ 0 \leqq s < t,\ |W(t)| < \delta\right). \tag{13}$$

Let t_0 be such that $8H < \varepsilon(t_0 - 1)$, $a < b < d$ and $0 < \theta < \min(\delta, d - b)/2$. Since W has independent increments and since for $t \geqq t_0$,

$$\left|\frac{1}{t}\int_0^t f_i(W(s))\,ds - \frac{1}{t-1}\int_0^{t-1} f_i(W(s))\,ds\right| \leqq \frac{2H}{t-1} < \frac{\varepsilon}{4},$$

the probability in (13) dominates

$$\begin{aligned}
&P\left(\left|\frac{1}{t-1}\int_0^{t-1} f_i(W(s))\,ds - \lambda(f_i)\right| < \frac{\varepsilon}{2};\ 1 \leqq i \leqq k, |W(s)| < b + \theta,\right.\\
&\quad \left. 0 \leqq s < t,\ |W(t)| \leqq \theta\right)\\
&\quad \geqq P\left(\left|\frac{1}{t-1}\int_0^{t-1} f_i(W(s))\,ds - \lambda(f_i)\right| < \frac{\varepsilon}{2};\ 1 \leqq i \leqq k,\ |W(s)| < b,\ 0 \leqq s \leqq t-1\right)\\
&\qquad \times \inf_{|z| \leqq b} P\left(|W(s) + z| < b + \theta, 0 \leqq s < 1, |W(1) + z| \leqq \theta\right)\\
&\quad = M_t \cdot J \quad \text{(say)}.
\end{aligned} \tag{14}$$

Clearly

$$J \geqq \inf_{|z| \leqq b} P(|W(s) + sz| < \theta,\ 0 \leqq s \leqq 1) > 0$$

and by Lemma 2.12 of [2],

$$\liminf_{t\to\infty} \frac{1}{t} \log M_t \geqq -I(\lambda).$$

The theorem now follows from (14).

Theorem 8. *Let $\lambda \in \mathscr{M}$ be such that $\lambda\{x: |x| \leqq a\} = 1$ and $I(\lambda) < 1$. Let N be a weak neighborhood of λ in $\mathscr{M}$. For $d > a$, let $E_t = (L_t \in N,\ |a(t)\,K(s,t)| \leqq d;\ 0 \leqq s \leqq 1)$. Then, for almost all ω, $\omega \in E_t$ for a sequence of times t increasing to infinity.*

Proof. Let $p > 1$ be such that $pI(\lambda) < 1$ and let t_n denote the integral part of $\exp(n^p)$. We shall show that $P(E_{t_n}$ i.o.$) = 1$. Since $K(.,t)$ has the same distribution as that of $\sqrt{t}\,K(.,1)$, we have for any $\eta > 0$

$$\begin{aligned} P(\sup_{0 \leqq s \leqq 1} |K(s,t_n)| > 2\eta\sqrt{t_n \log t_n}) &= P(\sup_{0 \leqq s \leqq 1} |K(s,1)| > 2\eta\sqrt{\log t_n}) \\ &\leqq 2P(\sup_{0 \leqq s \leqq 1} |W(s)| > \eta\sqrt{\log t_n}) \ll n^{-2}. \end{aligned}$$

So by Borel-Cantelli lemma,

$$\sup_{0 \leqq s \leqq 1} |K(s,t_n)\,(t_n \log t_n)^{-1/2}| \to 0 \quad \text{a.e.} \tag{15}$$

Since $(\log t_n)^2\, t_{n-1}/t_n \to 0$ as $n \to \infty$, (15) implies that

$$\sup_{0 \leqq s \leqq 1} |K(s,t_{n-1})| s_n \to 0 \quad \text{a.e.,} \tag{16}$$

where $s_n = ((\log\log t_n)/t_n)^{1/2}$.

Let N be as in the proof of Theorem 7 and $a < b < d$. Define

$$H_n = \left\{ \left| \int_0^1 f_i[s_n(K(s,t_n) - K(s,t_{n-1}))]\,ds - \lambda(f_i) \right| < \frac{\varepsilon}{2},\ 1 \leqq i \leqq k, \right.$$
$$\left. s_n |K(s,t_n) - K(s,t_{n-1})| < b \ \text{ for } \ 0 \leqq s \leqq 1 \right\}.$$

By (16) for almost all ω, n sufficiently large, if H_n occurs, E_{t_n} occurs. As $\{K(.,t_n) - K(.,t_{n-1}):\ n \geqq 1\}$ are independent ($K(.,m)$ has the same distribution as the sum of m independent brownian bridges), the events H_n are independent. So by Borel-Cantelli it is enough to show that $\Sigma P(H_n) = \infty$. Now by Theorem 7

$$P(H_n) = P\left(\left| \int_0^1 f_i[K(s, s_n^2(t_n - t_{n-1}))]\,ds - \lambda(f_i) \right| < \frac{\varepsilon}{2};\ 1 \leqq i \leqq k, \right.$$
$$\left. |K(s, s_n^2(t_n - t_{n-1}))| < b \ \text{ for } \ 0 \leqq s \leqq 1 \right)$$
$$\geqq \exp[-I(\lambda)\, s_n^2(t_n - t_{n-1}) + o(s_n^2(t_n - t_{n-1}))].$$

Since $s_n^2(t_n - t_{n-1}) \sim \log\log t_n \sim p \log n$, we have

$$P(H_n) \geqq n^{-pI(\lambda) + o(1)}.$$

As $pI(\lambda) < 1$, it follows that $\Sigma P(H_n) = \infty$. This completes the proof.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let N be an open set containing C. As N^c is compact $\theta = \inf_{\beta \in N^c} I(\beta) > 1$. Let $0 < p < 1$ be such that $\theta p > 1$. Put $t_n = [e^{n^p}]$. Clearly for any $k \geqq 1$, $(t_n - t_{n-1})$ $(\log\log t_n)^k$ $t_{n-1}^{-1} \to 0$ as $n \to \infty$. By a representation of Kiefer process (see Sect. 1.15 of [5]) and by the Corollaries 1.12.4 and 1.15.1 of [5], we have

$$\begin{aligned} &\sup_{t_{n-1} \leqq t \leqq t_n} \left(\sup_{0 \leqq s \leqq 1} |a(t)\, K(s,t) - a(t_n)\, K(s,t_n)| \right) \\ &\leqq \sup_{t_{n-1} \leqq t \leqq t_n} |a(t) - a(t_n)| \sup_{0 \leqq s \leqq 1} |K(s,t_n)| \\ &\quad + a(t_{n-1}) \sup_{t_{n-1} \leqq t \leqq t_n} \left(\sup_{0 \leqq s \leqq 1} |K(s,t) - K(s,t_n)| \right) \\ &\to 0 \quad \text{a.e.} \end{aligned} \tag{17}$$

Now as in the proof of Theorem 2.8 of [2], it follows from (17) and Theorem 4, that

$$\bigcap_T \bigcup_{t \leqq T} L_t \subset C \quad \text{a.e.}$$

The converse follows from Lemma 2.16 of [2] and Theorem 8.

Applications

The following statements are immediate consequences of results in Sect. 4 of [2], Theorem 1 and Corollaries 2 and 3.

A.1. Let g be a continuous function on R such that $g(x) \to 0$ as $|x| \to \infty$. Then

$$\limsup_{t \to \infty} \int_0^1 g(a(t)\, K(s,t))\, ds = \sup_{\lambda \in C} \lambda(g) \quad \text{a.e.}$$

and

$$\limsup_{n \to \infty} \int_0^1 g(\sqrt{n \log\log n}\, (F_n(s) - s))\, ds = \sup_{\lambda \in C} \lambda(g) \quad \text{a.e.}$$

A.2. Let g be a continuous function on R with $g(x) \to \infty$ as $|x| \to \infty$. Define Φ on M as

$$\begin{aligned} \Phi(\lambda) &= \lambda(g) \quad \text{if } \lambda \in \mathcal{M} \quad \text{and if } \int |g(x)|\, \lambda(dx) < \infty \\ &= \infty \quad \text{otherwise.} \end{aligned}$$

Then

$$\liminf_{t \to \infty} \int_0^1 g(a(t)\, K(s,t))\, ds = \inf_{\lambda \in C \cap \mathcal{M}} \lambda(g) \quad \text{a.e.}$$

and

$$\liminf_{n \to \infty} \int_0^1 g(\sqrt{n \log\log n}\, (F_n(s) - s))\, ds = \inf_{\lambda \in C \cap \mathcal{M}} \lambda(g) \quad \text{a.e.}$$

Inparticular these results hold if $g(x) = |x|^\theta$, $\theta > 0$. As in [2] if we take $g(x) = x^2$ we get

$$\liminf_{t \to \infty} \int_0^1 \left(\frac{\log\log t}{t} \right) K^2(s,t)\, ds = \tfrac{1}{8} \quad \text{a.e.}$$

and

$$\liminf_{n\to\infty}(n\log\log n)\int_0^1 (F_n(s)-s)^2\,ds=\tfrac{1}{8}\quad\text{a.e.}$$

A.3. For each $a>0$, there exists $k(a)$ such that

$$\limsup_{t\to\infty}\text{Leb meas}\{s\colon\ 0\leqq s\leqq 1, |K(s,t)|\,a(t)\leqq a\}=k(a)\quad\text{a.e.}$$

and

$$\limsup_{n\to\infty}\text{Leb meas}\{s\colon\ 0\leqq s\leqq 1, \sqrt{n\log\log n}\,(F_n(s)-s)\leqq a\}=k(a)\quad\text{a.e.}$$

A.4. Using Theorem 8, we obtain

$$\liminf_{t\to\infty} a(t)\sup_{0\leqq s\leqq 1}|K(s,t)|=\frac{\pi}{\sqrt{8}}\quad\text{a.e.}$$

and

$$\liminf_{n\to\infty}(\sqrt{n\log\log n})\sup_{0\leqq s\leqq 1}|F_n(s)-s|=\frac{\pi}{\sqrt{8}}\quad\text{a.e.}$$

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