Z. Wahrscheinlichkeitstheorie verw. Gebiete
65, 61–72 (1983)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Two-Sided Bounds for Nonuniform Rates of Convergence in the Central Limit Theorem

Peter Hall

Australian National University, Dept. of Statistics, p.o. Box 4, Canberra ACT 2600, Australia

Summary. Two-sided bounds are derived for nonuniform rates of convergence in the central limit theorem, and are applied to solve several problems on rates of convergence in nonuniform metrics. In particular, an expression for the fastest rate of convergence is obtained, and rates of convergence using different norming constants are compared. A useful estimate of the rate of convergence in the $(1+|x|^{2+\varepsilon})$-metric is derived, which does not require the assumption of $(2+\varepsilon)$'th order moments.

1. Introduction and Summary

One of our purposes in this paper is to solve three problems concerning nonuniform rates of convergence in the central limit theorem:

(i) What is the optimal rate of convergence in the central limit theorem in a nonuniform metric, and with what norming constants is this rate achieved?

(ii) In what way does the rate of convergence depend on the norming constants?

(iii) Is it possible to provide a useful estimate of the rate of convergence in the metric, $\|f-g\| = \sup_{-\infty<x<\infty} (1+|x|^{2+\varepsilon})|f(x)-g(x)|$, without assuming the existence of $(2+\varepsilon)$'th order moments?

We shall answer these questions by deriving two-sided bounds for nonuniform rates of convergence. Before we consider the solutions, let us briefly describe the motivation behind problems (i)–(iii). We shall assume that $X, X_1, X_2, \ldots$ are independent and identically distributed random variables with zero mean and unit variance, and set $S_n = \sum_1^n X_j$. We let Φ denote the standard normal distribution function.

Many results on nonuniform rates of convergence consist of showing that descriptions of uniform rates may be applied directly to nonuniform rates. For example, Berry [2] and Esséen [5] demonstrated that for an absolute constant C,

$$\sup_{-\infty<x<\infty} |P(S_n \leq n^{\frac{1}{2}}x) - \Phi(x)| \leq Cn^{-\frac{1}{2}} E|X_1|^3,$$

and Nagaev [17] derived the analogous nonuniform estimate,

$$\sup_{-\infty<x<\infty} (1+|x|^3)|P(S_n \leq n^{\frac{1}{2}}x) - \Phi(x)| \leq Cn^{-\frac{1}{2}} E|X_1|^3.$$

(See van Beek [1] and Michel [16] for values of the constants C.) As a second example, Heyde [11] proved that the two conditions,

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_{-\infty<x<\infty} |P(S_n \leq n^{\frac{1}{2}}x) - \Phi(x)| < \infty$$

and $E|X|^{2+\delta} < \infty$, are equivalent whenever $0<\delta<1$, while Maejima [13] showed that they are also equivalent to the nonuniform constraint,

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_{-\infty<x<\infty} (1+x^2)|P(S_n \leq n^{\frac{1}{2}}x) - \Phi(x)| < \infty.$$

By obtaining upper and lower bounds for nonuniform rates of convergence, we are able to predict conditions under which nonuniform rates *differ* from uniform rates, as well as conditions under which they are the same. In particular, it is possible to compare optimal rates of convergence in uniform and nonuniform metrics. Much of the literature on nonuniform rates of convergence concentrates on upper bounds; see for example [3–6, 12–15, 18, 19].

Define the minimum distance in a nonuniform metric by

$$\Delta_{n1}(\alpha) = \inf_{c>0,d} \sup_{-\infty<x<\infty} (1+|x|^\alpha)|P(S_n \leq cx+d) - \Phi(x)|,$$

where $\alpha \geq 0$. It is known that when $\alpha=0$ (the uniform case), this distance function is of precise order

$$\delta_{n1} = nP(|X|>n^{\frac{1}{2}}) + n^{-1}E\{X^4 I(|X| \leq n^{\frac{1}{2}})\} + n^{-\frac{1}{2}}|E\{X^3 I(|X| \leq n^{\frac{1}{2}})\}|,$$

up to terms of order $n^{-\frac{1}{2}}$. That is, the ratio $(\Delta_{n1}+n^{-\frac{1}{2}})/(\delta_{n1}+n^{-\frac{1}{2}})$ is bounded away from zero and infinity as $n\to\infty$; see [8, 20, 21]. A corollary of Theorem 1 in the next section is that when $\alpha>0$, $\Delta_{n1}(\alpha)$ is of precise order

$$\delta_{n1}(\alpha) = n\{\sup_{x\geq 1} x^\alpha P(|X|>n^{\frac{1}{2}}x)\} + n^{-1}E\{X^4 I(|X| \leq n^{\frac{1}{2}})\}$$
$$+ n^{-\frac{1}{2}}|E\{X^3 I(|X| \leq n^{\frac{1}{2}})\}|,$$

up to terms of order $n^{-\frac{1}{2}}(\log n)^\beta$, where $\beta=\max(0, \alpha/2-1)$. This rate of convergence is achieved with the norming constants $c=n^{\frac{1}{2}}\sigma_n$ and $d=nv_n$, where

$$\sigma_n^2 = E\{X^2 I(|X| \leq n^{\frac{1}{2}})\} \quad \text{and} \quad v_n = E\{XI(|X| \leq n^{\frac{1}{2}})\}.$$

These results provide an answer to the first question posed in the introductory paragraph. They substantially improve upon earlier results derived in [9], in which attention was confined entirely to the less interesting case $\alpha<2$, and where optimal rates of convergence in nonuniform metrics were not considered.

We shall also investigate rates of convergence using the classical norming sequences, in which the scale and location constants are taken as $c=n^{\frac{1}{2}}$ and $d=0$, respectively. This leads to an interesting and unexpected conclusion: If $\alpha>2$ then the rate of convergence of the nonuniform estimate,

$$\Delta_n(\alpha; c_n, d_n) = \sup_{-\infty<x<\infty} (1+|x|^\alpha)|P(S_n \leqq c_n x + d_n) - \Phi(x)|, \tag{1.1}$$

is *not* improved by replacing the classical norming constants $(c_n, d_n)=(n^{\frac{1}{2}}, 0)$ by the "optimal" pair, $(c_n, d_n)=(n^{\frac{1}{2}}\sigma_n, v_n)$, up to terms of order $n^{-\frac{1}{2}}(\log n)^\beta$. On the other hand, in the case $\alpha\leqq 2$, an improvement *can* be achieved. In this way we may derive an answer to question (ii). The results of our main theorems can be applied to obtain characterisations of rates of convergence in nonuniform metrics, for instance of the type derived by Heyde [11]. An example is given in Sect. 2.

Note that $\delta_{n1}(\alpha)$ is finite and converges to zero for many distributions with infinite α'th moments. In particular, if $P(|X|>x)\sim\text{const.}\,x^{-\alpha}$ for large values of x, where $2<\alpha<3$, then we may deduce from Theorems 1 and 2 below that

$$C_1 n^{1-\alpha/2} \leqq \Delta_{n1}(\alpha) \leqq \Delta_n(\alpha; n^{\frac{1}{2}}, 0) \leqq C_2 n^{1-\alpha/2}$$

for positive constants C_1 and C_2. Since the random variable $|X|$ does not have finite α'th moment, many classical estimates of the size of $\Delta_n(\alpha; n^{\frac{1}{2}}, 0)$ (see for example Bikyalis [4]) provide only an infinite upper bound. Our bounds to rates of convergence allow us to give a positive answer to the third question posed in the introduction.

The techniques used during our proofs involve two stages: we derive an estimate for one-sided large deviation probabilities (see Theorem 3), and then we apply a leading term approach to rates of convergence, developed in [9, 10]. We have tried to keep our notation close to that of [9]. It is possible to derive very similar results for rates of convergence in local limit theorems, and for rates of convergence in Chebyshev-Edgeworth-Cramér expansions.

2. Results

The notation introduced in Sect. 1 will be assumed throughout. Note in particular the definition of $\Delta_n(\alpha; c_n, d_n)$ at (1.1). Our first result describes rates of convergence using "optimal" norming constants.

Theorem 1. *Let $\beta=\max(0, \alpha/2-1)$. Then for all $\alpha\geqq 0$,*

$$\Delta_n(\alpha; n^{\frac{1}{2}}\sigma_n, n v_n) = O\{\delta_{n1}(\alpha) + n^{-\frac{1}{2}}(\log n)^\beta\} \tag{2.1}$$

as $n\to\infty$, and

$$\liminf_{n\to\infty} \{\Delta_{n1}(\alpha) + n^{-\frac{1}{2}}(\log n)^\beta\}/\delta_{n1}(\alpha) > 0, \tag{2.2}$$

provided only that the ratio ∞/∞ is interpreted as unity.

The next result describes rates of convergence using the classical norming constants, $c_n = n^{\frac{1}{2}}$ and $d_n = 0$. We define

$$\delta_n(\alpha) = E\{X^2 I(|X| > n^{\frac{1}{2}})\} + n^{-1} E\{X^4 I(|X| \leqq n^{\frac{1}{2}})\} + n^{-\frac{1}{2}} |E\{X^3 I(|X| \leqq n^{\frac{1}{2}})\}|$$

for $0 \leqq \alpha \leqq 2$, and $\delta_n(\alpha) = \delta_{n1}(\alpha)$ for $\alpha > 2$.

Theorem 2. *For all $\alpha \geqq 0$, the ratio*

$$\{\Delta_n(\alpha; n^{\frac{1}{2}}, 0) + n^{-\frac{1}{2}}(\log n)^\beta\} / \{\delta_n(\alpha) + n^{-\frac{1}{2}}(\log n)^\beta\}$$

is bounded away from zero and infinity as $n \to \infty$, provided only that the ratio ∞/∞ is interpreted as unity.

Remarks. (i) It is easily proved that for any $\alpha > 2$,

$$\delta_{n1}(\alpha) = \delta_n(\alpha) \geqq \{(\alpha - 2)/\alpha\}\, \delta_n(0).$$

(ii) If X has a lattice distribution, or if the distribution of X satisfies Cramér's condition, $\limsup_{|t| \to \infty} |E(e^{itX})| < 1$, then the constant β appearing in Theorems 1 and 2 can be replaced by zero. In this case, the ratios

$$\{\Delta_{n1}(\alpha) + n^{-\frac{1}{2}}\} / \{\delta_{n1}(\alpha) + n^{-\frac{1}{2}}\} \quad \text{and} \quad \{\Delta_n(\alpha; n^{\frac{1}{2}}, 0) + n^{-\frac{1}{2}}\} / \{\delta_n(\alpha) + n^{-\frac{1}{2}}\}$$

are both bounded away from zero and infinity as $n \to \infty$. This sharpening can be accomplished after relatively minor modifications to the proofs given in Sect. 3, and using Theorems 4.2 and 4.3, pages 162–164 of [9]. Distributions which are neither lattice nor satisfy Cramér's condition are encountered very infrequently in practice.

(iii) As an application of Theorem 2, let us derive necessary and sufficient conditions for convergence of the series,

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \Delta_n(\alpha; n^{\frac{1}{2}}, 0), \tag{2.3}$$

in which we take $\delta \in (0, 1)$. It follows from results of Heyde [11] that a necessary condition for convergence is $E|X|^{2+\delta} < \infty$, and that this constraint is sufficient when $\alpha = 0$. Maejima [13] showed that the condition is also sufficient in the case $0 < \alpha < 2$. Now suppose $\alpha > 2$, and $E|X|^{2+\delta} < \infty$. We may deduce from Theorem 2 that the series in (2.3) will converge if and only if

$$\sum_{n=1}^{\infty} n^{\delta/2} \{\sup_{x \geqq 1} x^\alpha P(|X| > n^{\frac{1}{2}} x)\} < \infty,$$

and by making an integral approximation to the series we see that this condition is equivalent to

$$\int_1^{\infty} u^{\delta+1-\alpha} \{\sup_{x \geqq u} x^\alpha P(|X| > x)\}\, du < \infty. \tag{2.4}$$

If the function $x^\alpha P(|X|>x)$ is finite and nonincreasing for $x \geqq x_0$, or if $\alpha<2+\delta$, then condition (2.4) is equivalent to $E|X|^{2+\delta}<\infty$. However, if $\alpha=2+\delta$ then there exist distributions for which $E|X|^{2+\delta}<\infty$ but condition (2.4) fails. To see this, let Y be a discrete random variable with atoms at the points 2^{n^2}, $n \geqq 1$, and distribution given by

$$P(Y \geqq 2^{n^2}) = 2^\alpha / 2^{\alpha n^2} n^2, \qquad n \geqq 1.$$

Set $c^2 = E(Y^2)$, let $|X|$ have the distribution of Y/c, and let X be symmetric. Then $E|X|^\alpha < \infty$, and if $2^{n^2} < cx < 2^{(n+1)^2}$,

$$\begin{aligned} c^\alpha \sup_{u \geqq x} u^\alpha P(|X|>u) &= \sup_{u \geqq cx} u^\alpha P(Y>u) \\ &\geqq 2^{\alpha(n+1)^2} P(Y \geqq 2^{(n+1)^2}) = 2^\alpha/(n+1)^2. \end{aligned}$$

Hence if $0<u_0<2/c$,

$$\begin{aligned} & c^\alpha \int_{u_0}^{\infty} u^{\delta+1-\alpha} \{\sup_{x \geqq u} x^\alpha P(|X|>x)\}\, du \\ & \geqq 2^\alpha \sum_{n=1}^{\infty} (n+1)^{-2} \int_{2^{n^2}/c}^{2^{(n+1)^2}/c} u^{-1}\, du = \infty. \end{aligned}$$

In summary, the conditions

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \Delta_n(\alpha; n^{\frac{1}{2}}, 0) \quad \text{and} \quad E|X|^{2+\delta}<\infty$$

are equivalent for $0 \leqq \alpha < 2+\delta$, but not for $\alpha \geqq 2+\delta$. Similar results may be obtained in the case $\delta=0$, and for alternative characterisations of rates of convergence.

(iv) It was shown in [9, Examples 3.4.4 and 3.4.5, pages 134–142] that, even when $\alpha<2$, $\delta_{n1}(0)$ can be negligible in comparison with $\delta_{n1}(\alpha)$, and $\delta_{n1}(\alpha)$ can be negligible in comparison with $\delta_n(\alpha)$.

We conclude with a new result on one-sided large deviation probabilities, which is used as a lemma in our proofs, but which seems to be of independent interest. This result is related to Theorem 2 of Michel [15].

Theorem 3. *Suppose* $E(X^+)^{2+\delta}<\infty$ *for some* $\delta>0$. *Given* $k>1$ *we may choose constants* C, ε *and* $\lambda>0$ *such that*

$$P(S_n > n^{\frac{1}{2}} x) \leqq n P(X > n^{\frac{1}{2}} \varepsilon x) + C(nx)^{-k}$$

whenever $x > (\lambda \log n)^{\frac{1}{2}}$.

3. Proofs

Throughout the proofs the symbol C denotes a generic positive constant.

Proof of Theorem 3. Observe that for any $0<\xi<\eta<1$ and $z>\eta^{-1}$,

$$P(S_n > 2n^{\frac{1}{2}}z) \leqq nP(X > n^{\frac{1}{2}}\eta z) + P\left\{\sum_1^n X_i I(X_i \leqq n^{\frac{1}{2}}\eta z) > 2n^{\frac{1}{2}}z\right\}$$
$$\leqq nP(X > n^{\frac{1}{2}}\eta z) + P\left\{\sum_1^n X_i I(X_i \leqq \xi n^{\frac{1}{2}}) > n^{\frac{1}{2}}z\right\}$$
$$+ P\left\{\sum_1^n X_i I(\xi n^{\frac{1}{2}} < X_i \leqq n^{\frac{1}{2}}\eta z) > n^{\frac{1}{2}}z\right\}. \qquad (3.1)$$

Let Y, $Y_1, \ldots, Y_n$ be independent and identically distributed random variables with $\operatorname{ess\,sup} Y = y < \infty$, and finite variance. Then

$$P\left(\sum_1^n Y_i > n^{\frac{1}{2}}z\right) \leqq \psi^n \exp(-tn^{\frac{1}{2}}z) \qquad (3.2)$$

for any $t > 0$, where $\psi = E\exp(tY)$. Now,

$$\exp(tY) = 1 + tY + \tfrac{1}{2}(tY)^2 + r(t, Y),$$

where $r(t, Y) \leqq 0$ if $Y \leqq 0$, and $0 < r(t, Y) \leqq \frac{1}{6}(tY)^3 \exp(ty)$ almost surely if $Y > 0$. Therefore

$$\psi \leqq 1 + t|EY| + \tfrac{1}{2}t^2 E(Y^2) + \tfrac{1}{6}t^3 E\{Y^3 I(Y>0)\}\exp(ty)$$
$$\leqq \exp[t|EY| + \tfrac{1}{2}t^2 E(Y^2) + \tfrac{1}{6}t^3 E\{Y^3 I(Y>0)\}\exp(ty)]. \qquad (3.3)$$

We first take $Y = XI(X \leqq \xi n^{\frac{1}{2}})$ and $t = n^{-\frac{1}{2}}\theta$, where $\theta > 0$, and observe that

$$|EY| = E\{XI(X > \xi n^{\frac{1}{2}})\} \leqq \xi^{-1} n^{-\frac{1}{2}} E\{X^2 I(X > \xi n^{\frac{1}{2}})\}.$$

Consequently

$$\psi^n \leqq \exp[\theta\xi^{-1} E\{X^2 I(X > \xi n^{\frac{1}{2}})\} + \tfrac{1}{2}\theta^2 + \theta^3 e^{\xi\theta}\Delta], \qquad (3.4)$$

where $\Delta = n^{-\frac{1}{2}} E\{X^3 I(0 < X \leqq \xi n^{\frac{1}{2}})\}$. Let ρ be a large positive constant. If $1 \leqq z \leqq 2\rho\log n$ then we take $\theta = z$. Since

$$E\{X^2 I(X > \xi n^{\frac{1}{2}})\} + n^{-\frac{1}{2}} E\{X^3 I(0 < X \leqq \xi n^{\frac{1}{2}})\} = O(n^{-\delta/2}) \qquad (3.5)$$

under the condition $E(X^+)^{2+\delta} < \infty$, then provided ξ is so small that $2\xi\rho < \delta/2$,

$$\theta\xi^{-1} E\{X^2 I(X > \xi n^{\frac{1}{2}})\} + \tfrac{1}{2}\theta^2 + \theta^3 e^{\xi\theta}\Delta$$
$$\leqq \tfrac{1}{2}z^2 + O\{n^{-\delta/2 + 2\xi\rho}(\log n)^3\} \leqq \tfrac{1}{2}z^2 + C.$$

We may now deduce from (3.2) and (3.4) that

$$P\left\{\sum_1^n X_i I(X_i \leqq \xi n^{\frac{1}{2}}) > n^{\frac{1}{2}}z\right\} \leqq Ce^{-z^2/2}.$$

If $z > (\lambda\log n)^{\frac{1}{2}}$ and λ is sufficiently large, this implies that

$$P\left\{\sum_1^n X_i I(X_i \leqq \xi n^{\frac{1}{2}}) > n^{\frac{1}{2}}z\right\} \leqq C(nz)^{-k}. \qquad (3.6)$$

When $z > 2\rho\log n$ we take $\theta = \rho(\log z - \log\Delta)$. Since $-\log\Delta \leqq \frac{1}{2}\log n + C$ for large n then

$$\begin{aligned}&\xi^{-1} E\{X^2 I(X>\xi n^{\frac{1}{2}})\}+\tfrac{1}{2}\theta+\theta^2 e^{\xi\theta}\Delta\\ &\leqq \tfrac{1}{2}\rho\log z+\tfrac{1}{4}\rho\log n+2\rho^2\{(\log z)^2+(\log \Delta)^2\}\,\Delta^{1-\xi\rho} z^{\xi\rho}+C\\ &\leqq \tfrac{1}{2}z,\end{aligned}$$

provided n is large and ξ is so small that $\xi\rho<1$. Therefore by (3.2) and (3.4),

$$P\left\{\sum_1^n X_i I(X_i\leqq \xi n^{\frac{1}{2}})>n^{\frac{1}{2}}z\right\}\leqq e^{-\theta z/2}=(\Delta/z)^{-\rho z/2}.$$

Therefore (3.6) also holds in this case.

Next we take $Y=XI(\xi n^{\frac{1}{2}}<X\leqq \eta n^{\frac{1}{2}}z)$ and $t=(n^{\frac{1}{2}}z)^{-1}\theta$ in (3.3), from which we deduce that

$$\psi^n\leqq \exp\{(\xi^{-2}z^{-1}\theta+\tfrac{1}{2}\xi^{-1}z^{-2}\theta^2)\Delta+\tfrac{1}{6}z^{-3}\theta^3 e^{\eta\theta}\Delta\},\tag{3.7}$$

where on this occasion, $\Delta=n^{-\frac{1}{2}}E\{X^3 I(\xi n^{\frac{1}{2}}<X\leqq \eta n^{\frac{1}{2}}z)\}$. Let $\theta=\rho(\log z-\log\Delta)$ and observe that for any $r>0$,

$$z^{-r}\theta^r\Delta\leqq Cz^{-r}\{(\log z)^r+|\log\Delta|^r\}\,\Delta$$

and

$$z^{-3}\theta^3 e^{\eta\theta}\Delta\leqq Cz^{-3}\{(\log z)^3+|\log\Delta|^3\}\,\Delta^{1-\eta\rho}z^{\eta\rho}.$$

Provided η is chosen so small that $\eta\rho<1$, both these quantities are bounded uniformly in $z>(\log n)^{\frac{1}{2}}$ and large n. In this case $\psi^n\leqq C$, and so by (3.2) and (3.7),

$$P\{\sum X_i I(\xi n^{\frac{1}{2}}<X_i\leqq \eta n^{\frac{1}{2}}z)>n^{\frac{1}{2}}z\}\leqq Ce^{-\theta}=C(\Delta/z)^{-\rho}.$$

Therefore if $z>(\log n)^{\frac{1}{2}}$ and ρ is sufficiently large, we may deduce via (3.5) that

$$P\left\{\sum_1^n X_i I(\xi n^{\frac{1}{2}}<X_i\leqq \eta n^{\frac{1}{2}})>n^{\frac{1}{2}}z\right\}\leqq C(nz)^{-k}.$$

Theorem 3 follows on combining this result with (3.1) and (3.6).

Proof of Theorem 1. In the case $0\leqq\alpha\leqq 2$, the upper bound (2.1) follows from Theorems 3.2 and 3.5, pages 89–90 and 121–122 of [9]. Therefore we may confine attention to the case $\alpha>2$. In this situation we may assume that $E|X|^{2+\delta}<\infty$ for some $\delta>0$, for otherwise both $\Delta_{n1}(\alpha)$ and $\delta_{n1}(\alpha)$ are infinite, and the result is trivial. When $E|X|^{2+\delta}<\infty$, we have

$$E\{X^2 I(|X|>n^{\frac{1}{2}})\}+n^{-\frac{1}{2}}E\{|X|^3 I(|X|\leqq n^{\frac{1}{2}})\}=O(n^{-\delta/2})\tag{3.8}$$

as $n\to\infty$. We may deduce from Theorem 3 that for sufficiently large λ,

$$\begin{aligned}&\sup_{x>(2\lambda\log n)^{1/2}}(1+x^\alpha)\,|P(S_n\leqq n^{\frac{1}{2}}\sigma_n x+n\nu_n)-\Phi(x)|\\ &\quad\leqq \sup_{x>(2\lambda\log n)^{1/2}}(1+x^\alpha)\,P(S_n>n^{\frac{1}{2}}\sigma_n x+n\nu_n)\\ &\qquad+C_1(\log n)^{(\alpha-1)/2}\exp(-\lambda\log n)\\ &\quad\leqq C_2 n\{\sup_{x>(\varepsilon\log n)^{1/2}}(1+x^\alpha)\,P(X>n^{\frac{1}{2}}x)\}+O(n^{-\frac{1}{2}})\end{aligned}$$

for some $\varepsilon>0$, noting that $\sigma_n\to 1$ and $n^{\frac{1}{2}}v_n\to 0$. A similar result holds for negative x, and so

$$\sup_{|x|>(2\lambda\log n)^{1/2}}(1+|x|^{\alpha})\,|P(S_n\leqq n^{\frac{1}{2}}\sigma_n x+nv_n)-\Phi(x)|=O\{\delta_{n1}(\alpha)+n^{-\frac{1}{2}}\} \tag{3.9}$$

as $n\to\infty$.

From Theorems 3.2 and 3.5 of [9] we see that

$$\sup_{-\infty<x<\infty}(1+x^2)\,|P(S_n\leqq n^{\frac{1}{2}}\sigma_n x+nv_n)-\Phi(x)-L_{n11}(x)|=O\{\delta_{n1}(\alpha)\,\delta_n+n^{-\frac{1}{2}}\}, \tag{3.10}$$

where

$$L_{n11}(x)=nE\{\Phi(x-X/n^{\frac{1}{2}}\sigma_n)-\Phi(x)\}+n^{\frac{1}{2}}v_n\sigma_n^{-1}-\tfrac{1}{2}\phi'(x)$$

and $\delta_n=\delta_n(0)$. It follows from Theorem 3.4, page 120 of [9] that

$$\sup_{-\infty<x<\infty}|L_{n11}(x)|=O\{\delta_{n1}(0)\}, \tag{3.11}$$

and from (3.8) and (3.10) that

$$\sup_{|x|\leqq(2\lambda\log n)^{1/2}}(1+|x|^{\alpha})\,|P(S_n\leqq n^{\frac{1}{2}}\sigma_n x+nv_n)-\Phi(x)-L_{n11}(x)|$$
$$=O\{\delta_{n1}(\alpha)+n^{-\frac{1}{2}}(\log n)^{-1+\alpha/2}\}. \tag{3.12}$$

We shall prove next that

$$\sup_{|x|>2}(1+|x|^{\alpha})\,|L_{n11}(x)|=O\{\delta_{n1}(\alpha)\}. \tag{3.13}$$

The results (3.9), (3.12) and (3.13) combine to imply the first part of Theorem 1.

For $x>2$ we have

$$\begin{aligned}
|L_{n11}(x)|\leqq\, & nE[\{\Phi(x)-\Phi(x-X/n^{\frac{1}{2}}\sigma_n)\}\,I(X>n^{\frac{1}{2}})]\\
&+nE[\{\Phi(x-X/n^{\frac{1}{2}}\sigma_n)-\Phi(x)\}\,I(X<-n^{\frac{1}{2}})]\\
&+nE\left\{\left|\Phi(x-X/n^{\frac{1}{2}}\sigma_n)-\Phi(x)-\sum_{j=1}^{3}(-X/n^{\frac{1}{2}}\sigma_n)^j\,\phi^{(j-1)}(x)/j!\right|\right.\\
&\left.\cdot I(|X|\leqq n^{\frac{1}{2}})\right\}\\
&+n\tfrac{1}{6}\,|\phi''(x)|\,(n^{\frac{1}{2}}\sigma_n)^{-3}\,|E\{X^3 I(|X|\leqq n^{\frac{1}{2}})\}|\\
\leqq\, & nE[\{\Phi(x)-\Phi(x-X/n^{\frac{1}{2}}\sigma_n)\}\,I(1<X/n^{\frac{1}{2}}\leqq\tfrac{1}{2}x)]\\
&+nP(X>n^{\frac{1}{2}}x/2)\\
&+\{1-\Phi(x)+\sigma_n^{-4}\sup_{|y|\leqq 1}|\phi^{(3)}(x+y)|+\sigma_n^{-3}\,|\phi''(x)|\}\,\delta_{n1}(0)\\
\leqq\, & nP(X>n^{\frac{1}{2}}x/2)+C(1+x^{\alpha})^{-1}\,\delta_{n1}(0).
\end{aligned} \tag{3.14}$$

Therefore

$$\sup_{x>2}(1+x^{\alpha})\,|L_{n11}(x)|\leqq n\{\sup_{x>2}(1+x^{\alpha})\,P(X>n^{\frac{1}{2}}x/2)\}+C\,\delta_{n1}(0),$$

and a similar result holds for negative x. This proves (3.13).

We now turn to the proof of (2.2). Choose constants $c_n>0$ and d_n such that

$$2\Delta_{n1}(\alpha)\geqq \sup_{-\infty<x<\infty}(1+|x|^\alpha)\,|P(S_n\leqq c_n x+d_n)-\Phi(x)|. \tag{3.15}$$

By the Convergence of Types Theorem (see Theorem 2, page 42 of [7]) we necessarily have $a_n\equiv n^{\frac{1}{2}}\sigma_n/c_n-1\to 0$ and $b_n\equiv(n v_n-d_n)/c_n\to 0$ as $n\to\infty$. Changing variable from x to $y=(c_n x+d_n-n v_n)/n^{\frac{1}{2}}\sigma_n$ in (3.15), we obtain

$$C\Delta_{n1}(\alpha)\geqq \sup_{-\infty<y<\infty}(1+|y|^\alpha)\,|P(S_n\leqq n^{\frac{1}{2}}\sigma_n y+n v_n)-\Phi(y+a_n y+b_n)|. \tag{3.16}$$

Since $(1+x^\alpha)\{1-\Phi(x+a_n x+b_n)\}=O(n^{-\frac{1}{2}})$ uniformly in $x>(2\log n)^{\frac{1}{2}}$,

$$\begin{aligned}&\sup_{x>(2\log n)^{1/2}}(1+x^\alpha)\,|P(S_n\leqq n^{\frac{1}{2}}\sigma_n x+n v_n)-\Phi(x+a_n x+b_n)|\\ &\quad=\sup_{x>(2\log n)^{1/2}}(1+x^\alpha)\,P(S_n>n^{\frac{1}{2}}\sigma_n x+n v_n)+O(n^{-\frac{1}{2}}).\end{aligned} \tag{3.17}$$

For any $z>0$,

$$\begin{aligned}P(S_n>n^{\frac{1}{2}}z)&\geqq P\Big[\bigcup_{j=1}^{n}\{X_i\leqq n^{\frac{1}{2}}z \text{ for } i\leqq j-1;\ X_j>n^{\frac{1}{2}}z;\ \sum_{\substack{1\leqq i\leqq n,\\ i\neq j}}X_i\geqq 0\}\Big]\\
&\geqq\sum_{j=1}^{n}\{P(X_j>n^{\frac{1}{2}}z;\ \sum_{\substack{1\leqq i\leqq n,\\ i\neq j}}X_i\geqq 0)\\
&\qquad -P(X_j>n^{\frac{1}{2}}z;\ X_i>n^{\frac{1}{2}}z \text{ for some } 1\leqq i\leqq j-1)\}\\
&\geqq\sum_{j=1}^{n}\{P(X>n^{\frac{1}{2}}z)\,P(S_{n-1}\geqq 0)-P(X>n^{\frac{1}{2}}z)(n-1)\,P(X>n^{\frac{1}{2}}z)\}\\
&\geqq nP(X>n^{\frac{1}{2}}z)\,\{P(S_{n-1}\geqq 0)-nP(X>n^{\frac{1}{2}}z)\}.\end{aligned} \tag{3.18}$$

Since $nP\{X>n^{\frac{1}{2}}(\log n)^{\frac{1}{2}}\}\to 0$ as $n\to\infty$, and $P(S_{n-1}\geqq 0)\to\frac{1}{2}$, we may deduce from (3.18) that for all large n,

$$\begin{aligned}&\sup_{x>(2\log n)^{1/2}}(1+x^\alpha)\,P(S_n>n^{\frac{1}{2}}\sigma_n x+n v_n)\\ &\quad\geqq\tfrac{1}{4}n\{\sup_{x>(2\log n)^{1/2}}(1+x^\alpha)\,P(X>2n^{\frac{1}{2}}x)\}.\end{aligned} \tag{3.19}$$

Next observe that for $x>0$,

$$|L_{n11}(x)-nE[\{\Phi(x-X/n^{\frac{1}{2}}\sigma_n)-\Phi(x)\}\,I(X>n^{\frac{1}{2}})]|\leqq C(1+x^\alpha)^{-1}\,\delta_{n1}(0), \tag{3.20}$$

using the argument leading to (3.14). If $0\leqq\alpha\leqq 2$, we may deduce from (3.10) and (3.20) that

$$\begin{aligned}&\sup_{1<x\leqq(2\log n)^{1/2}}x^\alpha\,|P(S_n\leqq n^{\frac{1}{2}}\sigma_n x+n\mu_n)-\Phi(x)-nE[\{\Phi(x-X/n^{\frac{1}{2}}\sigma_n)\\ &\qquad-\Phi(x)\}\,I(X>n^{\frac{1}{2}})]|\\ &\quad=O\{\delta_{n1}(0)+n^{-\frac{1}{2}}\}+o\{\delta_{n1}(\alpha)\},\end{aligned} \tag{3.21}$$

while if $\alpha > 2$, it follows from (3.8), (3.10) and (3.20) that (3.21) continues to hold provided the term $n^{-\frac{1}{2}}$ on the right hand side is replaced by $n^{-\frac{1}{2}}(\log n)^{-1+\alpha/2}$. It was proved in [8, 21] that

$$\delta_{n1}(0) = O\{\Delta_{n1}(0) + n^{-\frac{1}{2}}\}, \tag{3.22}$$

and by combining the results (3.16), (3.21) and (3.22) we obtain,

$$\begin{aligned} \sup_{1 < x < (2\log n)^{1/2}} & x^\alpha |nE[\{\Phi(x - X/n^{\frac{1}{2}}\sigma_n) - \Phi(x)\}\, I(X > n^{\frac{1}{2}})] \\ & + \Phi(x) - \Phi(x + a_n x + b_n)| \\ & = O\{\Delta_{n1}(\alpha) + n^{-\frac{1}{2}}(\log n)^\beta\} + o\{\delta_{n1}(\alpha)\}. \end{aligned} \tag{3.23}$$

Suppose n is so large that $|a_n|, |b_n| \leqq \frac{1}{4}$, and that $x \geqq 1$. We may write $\Phi(x + a_n x + b_n) - \Phi(x) = (a_n x + b_n)\, \phi(x_n)$, where x_n lies between x and $x + a_n x + b_n$ and so satisfies $x_n \geqq x - \frac{1}{4}x - \frac{1}{4} \geqq \frac{1}{2}x$. Therefore

$$|\Phi(x + a_n x + b_n) - \Phi(x)| \leqq (|a_n| + |b_n|)\, x e^{-x^2/8},$$

and

$$\begin{aligned} & x^\alpha (nE[\{\Phi(x) - \Phi(x - X/n^{\frac{1}{2}}\sigma_n)\}\, I(X > n^{\frac{1}{2}})] + \Phi(x + a_n x + b_n) - \Phi(x)) \\ & \quad \geqq n\{\Phi(1) - \Phi(0)\}\, x^\alpha P(X > 2n^{\frac{1}{2}}x) - (|a_n| + |b_n|)\, x^{\alpha+1} e^{-x^2/8}. \end{aligned}$$

We may now deduce from (3.23) that for any $\lambda > 1$,

$$\begin{aligned} & C\{\Delta_{n1}(\alpha) + n^{-\frac{1}{2}}(\log n)^\beta\} + o\{\delta_{n1}(\alpha)\} \\ & \quad \geqq n\{\Phi(1) - \Phi(0)\} \sup_{\lambda < x \leqq (2\log n)^{1/2}} x^\alpha P(X > 2n^{\frac{1}{2}}x) - (|a_n| + |b_n|) \sup_{x > \lambda} x^{\alpha+1} e^{-x^2/8}. \end{aligned}$$

Combining this estimate with (3.16), (3.17) and (3.19) we find that

$$\begin{aligned} & C\{\Delta_{n1}(\alpha) + n^{-\frac{1}{2}}(\log n)^\beta\} + o\{\delta_{n1}(\alpha)\} \\ & \quad \geqq n\{\Phi(1) - \Phi(0)\} \sup_{x > \lambda} x^\alpha P(X > 2n^{\frac{1}{2}}x) - (|a_n| + |b_n|) \sup_{x > \lambda} x^{\alpha+1} e^{-x^2/8} \\ & \quad \geqq n\{\Phi(1) - \Phi(0)\}\, 2^{-\alpha} \sup_{x \geqq 1} x^\alpha P(X > n^{\frac{1}{2}}x) \\ & \qquad - (|a_n| + |b_n|) \sup_{x > \lambda} x^{\alpha+1} e^{-x^2/8} - \lambda^\alpha P(X > n^{\frac{1}{2}}). \end{aligned}$$

Using (3.22) again, we deduce that

$$\begin{aligned} & C(\lambda)\{\Delta_{n1}(\alpha) + n^{-\frac{1}{2}}(\log n)^\beta\} + o\{\delta_{n1}(\alpha)\} \\ & \quad \geqq 2^{-\alpha}\{\Phi(1) - \Phi(0)\}\, n\, \{\sup_{x \geqq 1} x^\alpha P(X > n^{\frac{1}{2}}x)\} - (|a_n| + |b_n|) \sup_{x > \lambda} x^{\alpha+1} e^{-x^2/8}. \end{aligned}$$

The case of negative x may be treated in the same way, and so

$$\begin{aligned} & C(\lambda)\{\Delta_{n1}(\alpha) + n^{-\frac{1}{2}}(\log n)^\beta\} + o\{\delta_{n1}(\alpha)\} \\ & \quad \geqq 2^{-\alpha}\{\Phi(1) - \Phi(0)\}\, n\, \{\sup_{x \geqq 1} x^\alpha P(|X| > n^{\frac{1}{2}}x)\} - 2(|a_n| + |b_n|) \sup_{x > \lambda} x^{\alpha+1} e^{-x^2/8}. \end{aligned} \tag{3.24}$$

If the result (2.2) is false, we may choose a sequence $\{n_k\}$ diverging to infinity such that

$$\{\Delta_{n1}(\alpha)+n^{-\frac{1}{2}}(\log n)^{\beta}\}/\delta_{n1}(\alpha)\to 0 \tag{3.25}$$

as $n\to\infty$ along this subsequence. In view of (3.22), this entails

$$\delta_{n1}(\alpha)/n\{\sup_{x\geqq 1} x^{\alpha}P(|X|>n^{\frac{1}{2}}x)\}\to 1$$

as $n\to\infty$ along $\{n_k\}$. Since (3.24) is true for all large values of λ, we must necessarily have

$$(|a_n|+|b_n|)/\delta_{n1}(\alpha)\to\infty \tag{3.26}$$

as $n\to\infty$ along $\{n_k\}$. However,

$$\begin{aligned}|a_n|+|b_n| &\leqq C\sup_{-\infty<x<\infty}|\Phi(x+a_nx+b_n)-\Phi(x)|\\ &\leqq C\{\sup_{-\infty<x<\infty}|P(S_n\leqq n^{\frac{1}{2}}\sigma_n x+n\nu_n)-\Phi(x+a_nx+b_n)|\\ &\quad+\sup_{-\infty<x<\infty}|P(S_n\leqq n^{\frac{1}{2}}\sigma_n x+n\nu_n)-\Phi(x)|\}\\ &=O\{\Delta_{n1}(\alpha)+\delta_{n1}(\alpha)+n^{-\frac{1}{2}}(\log n)^{\beta}\},\end{aligned} \tag{3.27}$$

using (2.1). Results (3.26) and (3.27) together imply that

$$\{\Delta_{n1}(\alpha)+n^{-\frac{1}{2}}(\log n)^{\beta}\}/\delta_{n1}(\alpha)\to\infty$$

as $n\to\infty$ along $\{n_k\}$, contradicting (3.25) and completing the proof of Theorem 1.

Theorem 2 in the case $0\leqq\alpha\leqq 2$ follows from Theorems 2.2, 2.3 and 2.4, pages 25, 44 and 46 of [9]. The proof in the case $\alpha>2$ is similar in many respects to that of Theorem 1, and so will not be given here.

References

1. van Beek, P.: An application of Fourier methods to the problem of sharpening the Berry-Esséen inequality. Z. Wahrscheinlichkeitstheorie verw. Gebiete **23**, 187–196 (1972)
2. Berry, A.C.: The accuracy of the Gaussian approximation to the sum of independent variates. Trans. Amer. Math. Soc. **49**, 122–136 (1941)
3. Bikyalis, A.: On an estimate of the remainder term in the central limit theorem. Litovsk. Mat. Sb. **7**, 571–582 (1964). (In Russian)
4. Bikyalis, A.: Estimates of the remainder term in the central limit theorem. Litovsk. Mat. Sb. **6**, 323–346 (1966). (In Russian)
5. Esséen, C.-G.: Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law. Acta Math. **77**, 1–125 (1945)
6. Feller, W.: On the Berry-Esséen theorem. Z. Wahrscheinlichkeitstheorie verw. Gebiete **10**, 261–268 (1968)
7. Gnedenko, B.V., Kolmogorov, A.N.: Limit Distributions for Sums of Independent Random Variables. Reading, Mass: Addison-Wesley 1954

8. Hall, P.: Characterizing the rate of convergence in the central limit theorem. Ann. Probability **8**, 1037–1048 (1980)
9. Hall, P.: Rates of Convergence in the Central Limit Theorem. London: Pitman 1982
10. Hall, P.: Fast rates of convergence in the central limit theorem. Z. Wahrscheinlichkeitstheorie verw. Gebiete **62**, 491–507 (1983)
11. Heyde, C.C.: On the influence of moments on the rate of convergence to the normal distribution. Z. Wahrscheinlichkeitstheorie verw. Gebiete **8**, 12–18 (1967)
12. Maejima, M.: Non-uniform estimates in the central limit theorem. Yokohama Math. J. **26**, 137–149 (1978)
13. Maejima, M.: Some L_p versions of the central limit theorem. Ann. Probability **6**, 341–344 (1978)
14. Michel, R.: On the accuracy of nonuniform Gaussian approximations to the distribution functions of sums of independent and identically distributed random variables. Z. Wahrscheinlichkeitstheorie verw. Gebiete **35**, 337–347 (1976)
15. Michel, R.: Nonuniform central limit bounds with applications to probabilities of deviations. Ann. Probability **4**, 102–106 (1976)
16. Michel, R.: On the constant in the nonuniform version of the Berry-Esséen theorem. Z. Wahrscheinlichkeitstheorie verw. Gebiete **55**, 109–117 (1981)
17. Nagaev, S.V.: Some limit theorems for large deviations. Theor. Probability Appl. **10**, 214–235 (1965)
18. Osipov, L.V., Petrov, V.V.: On an estimate of the remainder term in the central limit theorem. Theory Probability Appl. **12**, 281–286 (1967)
19. Rotar', V.I.: Non-classical estimates for the rate of convergence in the multivariate central limit theorem. I. Theory Probability Appl. **22**, 755–772 (1977)
20. Rozovskii, L.V.: A lower bound to the remainder term in the central limit theorem. Mat. Zamet. **24**, 403–410 (1978). (In Russian)
21. Rozovskii, L.V.: On the precision of an estimate of the remainder term in the central limit theorem. Theory Probability Appl. **23**, 712–730 (1978)

Received September 10, 1982