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On a Strong Tauberian Result

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Summary. We consider the determination of the behavior of a distribution function F at its endpoints in terms of the behavior of its Laplace-Stieltjes transform ω at the limits of its interval of convergence. The results extend various known strong and weak results to a larger class of distributions via a relatively straightforward technique based on the weak convergence of suitably normalized associated distributions. An application and examples are considered briefly.

1. Introduction

Consider a distribution function F on the real line $\mathbb{R}$ and denote its two-sided Laplace-Stieltjes transform by ω:

$$\omega(r)=\int_{-\infty}^{\infty} e^{-rx}F(dx); \tag{1}$$

where we assume that the interval of convergence of ω is (R,S),

$$-\infty \leqq R < S \leqq \infty. \tag{2}$$

We are interested in determining the behaviour of F at its left endpoint e_L,

$$e_L=\inf\{x\colon F(x)>0\} \geqq -\infty; \tag{3}$$

based on the behaviour of ω at S, i.e., as $r \to S$. Such a result is called a Tauberian theorem or an inverse Abelian theorem depending on whether some "extra" condition is placed on F or on ω respectively (see Lew (1973)). Karamata's famous Tauberian theorem is an example and deals with the case of regular variation of ω at ∞ (see, for example, Feller (1971)).

As pointed out in Remark 1 below the results are easily translated to the behaviour of $1-F$ at the right end point, e_R say.

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For the case that ω is not regularly varying at ∞ but $\log\omega$ is (or has some other smooth behaviour at ∞), the literature contains a series of so-called *weak* Tauberian theorems which relate the behaviour of $\log F$ at e_L to that of $\log\omega$ at ∞. See the papers by Bingham and Teugels (1975), Balkema et al. (1979), Wagner (1966, 1968), and Kohlbecker (1958).

However, these weak results are often not satisfactory: for example, they do not provide sufficient conditions for the domain of attraction of the minimum $W_n = \min(X_1, \ldots, X_n)$, for $\{X_i\}$ independent and identically distributed with distribution F, to be the Gumbel law Λ,

$$\Lambda(x) = 1 - \exp(-e^x), \qquad -\infty < x < \infty. \tag{4}$$

(See, for example, de Haan (1970) on the topic of weak convergence of sample extremes.)

Here we provide a strong Tauberian theorem and a more useful (strong) inverse Abelian theorem. The latter theorem generalizes one due to Berg (1960). We have simplified the assumptions, thereby extracting the essense of the proof, and have also extended the theorem's applicability. We show, in fact, that the result derives from the weak convergence of distributions *associated* with F (see Feller (1971, p. 549)) when this convergence is also local; that is, when the relevant densities converge. For the asymptotically normal case, this result may also be considered as a converse of the Abelian theorem of Balkema et al. (1979, pp. 408,9), viz. Remark 6 below.

For the class of distribution functions F having a density f corresponding to a *convolution kernel*, that is for which

$$-\log\omega(r) = -cr^2 + br + \sum_k \{\log(1 - r/a_k) + r/a_k\}, \ \sum_k a_k^{-2} < \infty, \tag{5}$$

Hirschman and Widder (1955) have provided much information on the behaviour of f, including asymptotic expansions at the endpoints. One may thus also regard the present work as an extension of their asymptotic results to a wider class of distributions, still using some of their basic methodology.

In Sect. 2, after introducing some notation we present and prove the main results with some brief remarks at the end. In Sect. 3 we discuss applications and examples.

2. Main Results

In addition to the quantities defined in Eqs. (1), (2), and (3) we also consider the derivatives of $\log\omega(r)$ and denote them as follows:

$$\xi(r) = -(d/dr)\log\omega(r), \qquad R < r < S \tag{6}$$

and

$$\eta^2(r) = -(d/dr)\,\xi(r), \qquad R < r < S. \tag{7}$$

Their properties of interest to us are summarized in the following lemma, the proof of which is straightforward and delegated to the Appendix.

Lemma 2.1. *Under the assumption* (2), *the following hold for* $R<r<S$:

(i) $\omega^{(\alpha)}(r) \equiv (d^\alpha/dr^\alpha)(\omega(r)) = \int_{-\infty}^{\infty} (-1)^\alpha x^\alpha e^{-rx} F(dx)$ exists, $\alpha=0, 1, \ldots$;

(ii) $\int_{-\infty}^{\infty} x^\alpha e^{-rx} F(dx) = \delta(\alpha) - \alpha \int_{-\infty}^{\infty} x^{\alpha-1} e^{-rx} \tilde{F}(x)\,dx + r \int_{-\infty}^{\infty} x^\alpha e^{-rx} \tilde{F}(x)\,dx$,
$\alpha=0, 1, \ldots$,

where

$$\tilde{F}(x) = \begin{cases} F(x) & x<0 \\ F(x)-1 & x \geqq 0 \end{cases}$$

and

$$\delta(\alpha) = \begin{cases} 1 & \alpha=0 \\ 0 & \textit{otherwise}; \end{cases}$$

(iii) $\xi(r)$ *and* $\eta^2(r)$ *are finite and* $\eta^2(r) \geqq 0$;

(iv) $\xi(r) \to e_L$ *(monotonically) as* $r \to S$. □

We also record the following result, the essence of which is well known. An outline of the proof appears in the Appendix.

Lemma 2.2. *Suppose there exists a sequence of non-negative functions* $\{u_n\}$ *and constants* $\{B_n\}$ *such that*

(i) $u_n(x)$ *is* log-*concave for* $x<B_n$;

(ii) $B_n \nearrow \infty$;

(iii) $\int_a^b u_n(x)\,dx \to M(b)-M(a)$ *all* $-\infty<a<b<\infty$ *where* M *is an absolutely continuous distribution function with density* m.

Then $u_n(x) \to m(x)$ *at all continuity points* x *of* m *for which* $m(x)>0$. □

We now introduce a slow variation condition which has also been used in Tauberian theorems of the Wiener type (see Bloom (1976)). We say that $\eta(r)$ satisfies *Condition B* if:

$$\exists h>0 \quad \text{such that} \quad \eta(r+\theta/\eta(r))/\eta(r) \to 1 \quad \text{as} \quad r \to S \quad \text{for all } |\theta|<h. \tag{8}$$

For $S=\infty$, this condition is virtually equivalent to requiring that $1/\eta(r)$ be *Beurling slowly varying*. It has been investigated by Bloom (1976) and more recently by Bingham and Goldie (1982). The Condition B is also referred to by Balkema et al. (1979, p. 409) in the context of their Abelian counterpart of our Corollary 1.

The following result, due to Bloom (1976), will be required in what follows.

Lemma 2.3. *For* $\eta(r)$ *as defined in* (7), *if Condition B holds then*

(i) *the convergence in* (8) *is uniform on* $\theta \in (-h, h)$;

(ii) $r\eta(r) \to \infty$ *as* $r \to S$, *whenever* $S>0$. □ (9)

The proof is again indicated in the Appendix.

Theorems 1 and 2 each involve two parts: the first is common to both and involves proving the weak convergence of the normalized associated distributions; the second part involves using either an extra condition on F (the Tauberian theorem) or an extra condition on ω (the inverse Abelian theorem) to ensure that the densities converge as well.

We precede the results and their proofs by defining some extra quantities. Consider the associated density, for $R<r<S$,

$$re^{-rx}F(x)/\omega(r), \qquad x\in\mathbb{R}$$

which has expectation $\xi(r)+r^{-1}$ and variance $\eta^2(r)+r^{-2}$ (see Lemma 2.1 (iii) and (iv)). We normalize this density and so define a sequence of distributions $\{M_r\}$ with densities $\{m_r\}$ and Laplace-Stieltjes transforms $\{\mu_r\}$ given by:

$$m_r(x)=r\cdot\eta\cdot[\exp\{-r(\xi+x\eta)\}]\cdot F(\xi+x\eta)/\omega(r), \qquad R<r<S \tag{10}$$

and

$$\mu_r(s)=[1+s/(r\eta)]^{-1}\cdot\exp\{s\xi/\eta\}\cdot\omega(r+s/\eta)/\omega(r), \qquad R<r(1+s/r\eta), \qquad r<S.$$

The argument, r, of ξ and η has been omitted for typographical clarity. We also note that the characteristic functions of the distributions $\{M_r\}$ are given by $\{\mu_r(-is)\}$.

The key to the following results is the simple observation that if, as $r\to S$, $m_r(0)$ converges to a non-zero quantity then, from (10), we have established an asymptotic relationship between $F(\xi(r))$ and $\omega(r)$ as $r\to S$. The results below are concerned with conditions on $\{\mu_r(\cdot)\}$ and possibly on F itself which ensure this convergence of the sequence of densities $\{m_r(\cdot)\}$.

We first turn to the inverse Abelian results involving only conditions on $\omega(r)$ and its derivatives.

Theorem 1. *Suppose μ is the Laplace-Stieltjes transform of a distribution function M, and $\mu(s)$ exists for $s\in[-h,h]$, $h>0$. If*

(i) *$\omega(r)$ satisfies* (2);

(ii) *$\mu_r(s)\to\mu(s)$ as $r\to S$, for $s\in[-h,h]$;*

(iii) *$\{\mu_r(-is)\}$ is dominated by $g\in L^1(-\infty,\infty)$* (11)

then, as $r\to S$,

$$F(\xi(r))\sim\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mu(-is)\,ds\right)\cdot\omega(r)\cdot\exp\{r\xi(r)\}/\{r\eta(r)\}. \tag{12}$$

Proof. The continuity theorem for moment generating functions (see Billingsley (1979, p. 345)) ensures that (ii) implies

$$M_r\Rightarrow M,$$

which, with (ii), implies

$$\mu_r(-is)\to\mu(-is); \qquad s\in\mathbb{R}.$$

The hypothesis (iii) and Lebesgue's dominated convergence result then give

$$m_r(x) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} e^{-ixs} \mu_r(-is)\, ds \to \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} e^{-ixs} \mu(-is)\, ds$$

so that (12) follows on substituting $x=0$ in (10). □

One important special case of this result is when M corresponds to the standard normal distribution. The following corollary gives a sufficient condition for this case.

Corollary 1. *If* (i) *and* (iii) *of the theorem hold,* $S>0$ *and* (iiB) $\eta(r)$ *satisifes Condition B*;

then

$$F(\xi(r)) \sim \omega(r) \cdot \exp\{r\xi(r)\}/\{r\eta(r)\sqrt{2\pi}\}, \qquad r \to S. \tag{13}$$

Proof. From the *uniform* convergence result of Lemma 2.3, we see that (iiB) implies that for some $R_0(h)$ and all $s<h$, $(r+s/\eta(r))<S$ for $r>R_0$.

A Taylor expansion of $\log \mu_r(s)$ about $s=0$ yields, for all $s \in [-h, h]$,

$$\log \mu_r(s) = -\log(1+s/(r\eta)) + (s^2/2)\eta^2(r+\theta s/\eta)/\eta^2(r), \qquad r>R_0$$

with $\theta=\theta(r,s) \in [0,1]$. Again, Condition B and the consequent uniformity of convergence ensures that

$$\mu_r(s) \to \exp(s^2/2), \qquad r \to S$$

and so (ii) of the Theorem is established with $M=N(0,1)$. The result (13) then follows on substituting $\exp(-s^2/2)$ for $\mu(-is)$ in (12). □

Berg (1960) examined conditions which are sufficient to ensure that Condition B as well as (ii) hold.

We now turn to a Tauberian counterpart to the above results. The hypothesis (iiia) may be considered as the Tauberian condition.

Theorem 2. *Suppose that* μ *is the Laplace-Stieltjes transform of a distribution function* M, *that* M *has a density* m *positive and continuous at* 0, *and that* $\mu(s)$ *exists for* $s \in [-h, h]$, $h>0$. *If*

(i) $\omega(r)$ *satisfies* (2),

(ii) $\mu_r(s) \to \mu(s)$ *as* $r \to S$, *for* $s \in [-h, h]$,

(iiia) F *is* log-*concave in an interval* $[e_L, \Delta]$, $\Delta > e_L$; *then* (12) *holds.*

Proof. As for Theorem 1, (i) and (ii) imply that $M_r \Rightarrow M$.

In order to show that the densities m_r also converge pointwise to the density m we first show that for each $x \in \mathbb{R}$ at which $0<M(x)<1$

$$y = y(r) = \xi + x\eta \to e_L \quad \text{as} \quad r \to S.$$

Indeed, if $y(r) > \delta > e_L$ for $r=t_n \to S$, then

$$1-M_{t_n}(x)=\int_x^{\infty} m_{t_n}(u)\,du<\left\{\int_{\delta}^{\infty} e^{-t_n x}F(x)\,dx \Big/ \int_{-\infty}^{\delta} e^{-t_n x}F(x)\,dx\right\}\to 0 \quad \text{as} \quad t_n\to S$$

which contradicts the weak convergence. Similarly, if $y<\delta<e_L$ for $r=t_n\to S$ then $M_{t_n}(x)=0$ which again leads to a contradiction. Therefore, for each r there exists $B(r)$, with $B(r)\nearrow\infty$ as $r\to S$ such that $m_r(x)$ is log-*concave* for $x<B(r)$. Now applying Lemma 2.2, we obtain the desired result:

$$m_r(x)\to m(x)$$

at all points, x, at which m is positive and continuous. Therefore (12) follows upon setting $x=0$ in (10). □

Corollary 2. If (i) *and* (iiia) *of Theorem* 2 *hold,* $S>0$, *and also*
(ii B) $\eta(r)$ *satisfies Condition* B; *then* (13) *holds.*

Proof. Just as for Corollary 1, since (ii B) again ensures that $M=N(0,1)$. □

If F has a density we have the following result.

Theorem 3. *If F has density f and if*

(i) $\omega(r)$ *satisfies* (2);

(ii) $\eta(r)$ *satisfies Condition* B;

and either (iii) (11) *holds*

or (iiia) f is log-*concave in an interval* $[e_L, \Delta)$, $\Delta>e_L$

then
$$f(\xi(r))\sim\omega(r)\exp\{r\xi(r)\}/\{\eta(r)\sqrt{2\pi}\}, \qquad r\to S. \tag{14}$$

Proof. The proof follows from those given for Corollaries 1 and 2 by setting

$$m_r(x)=\eta\cdot\exp\{-r[\xi+x\eta]\}\cdot f(\xi+x\eta)/\omega(r),$$

and proceedings as in those proofs. □

Before turning to applications and examples, we record some remarks concerning the above results.

Remark 1. Theorems 1, 2 and 3 can readily be rephrased in terms of asymptotic expansions at the right end-point,

$$e_R=\sup\{x\colon F(x)<1\}\leqq\infty,$$

by simply checking that the conditions (2) and B hold as $r\to R$ and that either (11) holds as $r\to R$ or that $1-F(x)$ is log-*concave* for $x\in(\Delta, e_R)$, $\Delta<e_R$.

Remark 2. If $\eta(r)$ is regularly varying at $S=\infty$, with parameter $\rho>-1$, then Condition B follows immediately.

Remark 3. A referee has pointed out the following sufficient condition for Condition B. Let

$$\nu(r)=(d/dr)\,\eta^2(r)\equiv(\eta^2(r))'.$$

Then a sufficient condition for (8), when $S=\infty$, is

$$v(r)/\eta^3(r)\to 0 \quad \text{as} \quad r\to\infty. \tag{15}$$

Indeed (15) implies $(1/\eta)'\to 0$ as $r\to\infty$ and writing $\lambda=1/\eta$ we have

$$\begin{aligned}\eta(r+\theta/\eta(r))/\eta(r) &= \lambda(r)/[\lambda(r)+\theta\lambda(r)\lambda'(\theta^*)]\\ &= [1+\theta\lambda'(\theta^*)]^{-1}\to 1 \quad \text{as} \quad r\to\infty;\end{aligned}$$

where $\theta^*=\theta^*(\theta,r)$ satisfies $r<\theta^*<r+\theta/\eta(r)$.

Remark 4. One of the practical drawbacks of Theorems 1, 2, and 3 is that the asymptotic expansion is given in terms of $F(\xi(r))$ or $f(\xi(r))$ as $r\to S$. In some situations it may be difficult to invert $\xi(r)$ in order to get the expansion for $F(x)$ or $f(x)$ as $x\to e_L$.

Remark 5. For the case $S=\infty$, and f (a density of F) being a *convolution kernel* Hirschman and Widder (1955) give the expansion (14). They show, in fact, that the conditions (15), (p. 113) and (11) (p. 64) hold generally for ω satisfying (5) as long as $c\neq 0$ or the infinite product does not reduce to a finite one. It is interesting to note that they also show that f must be log-*concave* if it is a convolution kernel (viz. our Theorem 3).

Remark 6. The Abelian theorem together with the remark on p. 409 of Balkema et al. (1979) provides an Abelian analogue of the relation (13). Their result is that if

(i) $F(x)\sim\exp\int_0^x s(y)\,dy$ as $x\to\infty$,

(ii) $s'(x)$ exists and is negative, and

(iii) $c^2(x)=|s'(x)|$ satisfies Condition B,

then

$$\omega(s(r))\sim[2\pi r s(r)]^{1/2}\,\mathrm{e}^{-rs(r)}, \qquad r\to\infty. \tag{16}$$

Relations, such as (16), but in terms of $c(r)$, can also be determined (Balkema (1982, private communication)) quite straightforwardly. Note that (i) and (ii) imply that F is asymptotically equivalent to a log-*concave* function.

3. Applications and Examples

One of the problems that motivated the previous investigation was the determination of the domain of attraction of the minimum of an independent and identically distributed sequence of variables with distribution F. We have the following result and a similar one can be derived for the maximum.

Theorem 4. *Suppose $\{X_i\}$ is a sequence of independent random variables, identically distributed with distribution F. Let*

$$W_n=\min(X_1,\ldots,X_n).$$

If

(i) *the conditions of Theorem* 1 *or of Theorem* 2 *hold,*

(ii) $S>0$ *and* $r\eta(r)\to\infty$ *as* $r\to S$,

(iii) M *has a density* m *positive and continuous at* 0;

then

$$(W_n-a_n)/b_n \Rightarrow \Lambda \qquad (\text{see (4)}) \tag{17}$$

where a_n *and* b_n *may be chosen as follows:*

$$a_n=\xi(r_n), \qquad b_n=1/r_n \tag{18}$$

and r_n *solves*

$$m(0)\cdot\omega(r)\cdot\exp\{r\xi(r)\}/\{r\eta(r)\}=1/n.$$

Proof. Let, for $j\geqq 0$

$$F_j(x)=\int_{-\infty}^{x} F_{j-1}(y)\,dy, \qquad F_0=F.$$

Since $S>0$, we deduce

$$\omega_j(r)=\int_{-\infty}^{\infty} e^{-rx}F_j(dx)=r^{-j}\omega(r), \qquad 0<r<S.$$

Defining $m_{j,r}(x)$ in terms of F_j and ω_j just as m_r was defined in terms of F and ω (see (10)), we have that the corresponding transform is given by

$$\mu_{j,r}(s)=[1+s/\{r\eta(r)\}]^{-j}\mu_r(s).$$

By hypothesis (i) and (ii) above, $\mu_{j,r}(r)\to\mu(s)$ as $r\to S$. Moreover

(a) $\{\mu_{j,r}(-is)\}$ is dominated if $\{\mu_r(-is)\}$ is – for the Theorem 1 conditions;

or

(b) F_j is log-*concave on* (e_L,Δ) if F_{j-1} is, $j\geqq 1$ – for the Theorem 2 conditions.

In either case we may therefore conclude that, for each integer $j\geqq 0$ and as $r\to S$,

$$F_j(\xi(r))\sim m(0)\cdot r^{-j}\cdot\omega(r)\cdot\exp\{r\xi(r)\}/\{r\eta(r)\}. \tag{19}$$

Thus we may show that

$$F(x)F_2(x)/\{F_1(x)\}^2\to 1 \quad \text{as} \quad x\to e_L. \tag{20}$$

By a by now well known result, (20) ensures the existence of $\{a_n\}$ and $\{b_n\}$ such that (17) holds (see de Haan (1970)). In fact, the form (18) for the sequence of normalizing constants also follows from de Haan's results (p. 90). □

Although Theorem 4 is not intended as a characterization of the Λ-domain of attraction of the minimum, we *conjecture* that the condition $r\eta(r)\to\infty$ as $r\to S$ is indeed necessary. The main use of the theorem will, of course, be for those circumstances in which ω is known but F is not available in simple form.

The obvious corollary to Theorem 4 follows.

Corollary 4. *If the conditions of Corollary 1 or Corollary 2 hold, then F belongs to the Λ-domain of attraction (with respect to the minimum).*

Proof. The conditions and results of Corollaries 1 and 2 together with Lemma 2.3, ensure that all the conditions of Theorem 4 are met. □

Examples which are also convolution kernel transforms of course include the infinite convolution of exponentials discussed by Feigin and Yashchin (1982). Here

$$-\log\omega(r)=\sum_{j=1}^{\infty}\log(1+r/\lambda_j),\qquad \Sigma\lambda_j^{-1}<\infty$$

and since $|\mu_r(-is)|^{-2}$ has a positive coefficient Taylor expansion and converges to $\exp\{s^2\}$ we know that for some R_0

$$|\mu_r(-is)|^2\leqq(1+s^4/4)^{-1}\qquad r>R_0,$$

and therefore (11) is satisfied. See Hirschman and Widder (1955) for further details.

Another example is the positive stable distributions F_α for $0<\alpha<1$. Here

$$-\log\omega_\alpha(r)=r^\alpha\qquad r>0,\qquad 0<\alpha<1,$$

and Condition B follows from the regular variation of $\eta^2(r)=\alpha(1-\alpha)r^{\alpha-2}$. In order to apply Theorem 1 we now proceed to check (11).

Since

$$\mu_r(-is)=[1-is/r\eta]^{-1}\exp\{-r^\alpha([1-is/r\eta]^\alpha-1)-is\xi/\eta\}$$

we have

$$|\mu_r(-is)|=[1+s^2/r^2\eta^2]^{-1/2}\exp\{-r^\alpha f(s,\alpha)\}.$$

where

$$f(s,\alpha)=[1+s^2/r^2\eta^2]^{\alpha/2}\cos[\alpha\,\mathrm{artan}(s/r\eta)]-1.$$

By considering large and small values of $|s/r\eta|$ separately it is not hard to determine positive constants $V(\alpha)$, $B(\alpha)$ and $W(\alpha)$ such that

$$f(s,\alpha)\geqq V(\alpha)(s^2/r^2\eta^2),\qquad |s/r\eta|\leqq B(\alpha),$$
$$f(s,\alpha)\geqq W(\alpha)|s/r\eta|^\alpha,\qquad |s/r\eta|\geqq B(\alpha).$$

Thus we have

$$|\mu_r(-is)| \leqq \exp\{-[V/(\lambda+1)]s^2\} + \exp\{-[W(\lambda+1)^{-\alpha/2}]|s|^\alpha\}$$
$$\text{for } |s| < \infty \quad \text{and} \quad r > 1,$$

where $\lambda = \alpha(1-\alpha)$. We therefore conclude that (11) holds and hence

$$F_\alpha(\alpha r^{\alpha-1}) \sim (2\pi\lambda)^{-1/2} r^{-\alpha/2} \exp\{-(1-\alpha)r^\alpha\}, \qquad r \to \infty$$

or

$$F_\alpha(t) \sim \{2\pi(1-\alpha)\alpha^{\beta+1}t^{-\beta}\}^{-1/2} \exp\{-(1-\alpha)\alpha^\beta t^{-\beta}\}, \qquad t \to 0,$$

where $\beta = \alpha(1-\alpha)$. This example was also considered by Berg (1960) and the same result of course follows from the known asymptotic formulae for stable densities (see Skorohod (1954)).

An *illustrative* example to which Theorems 2 and 4 are applicable but for which $\eta(r)$ does not satisfy Condition B is given by the following. Taking, for some $\lambda > 0$,

$$F(x) = \begin{cases} e^{\lambda x}, & x \leqq 0 \\ 1 & x \geqq 0 \end{cases}$$

we see that

$$\omega(r) = \frac{\lambda}{\lambda - r} \qquad -\infty < r < \lambda \equiv S,$$

and

$$\xi(r) = \frac{-1}{\lambda - r}, \qquad \eta(r) = \frac{1}{\lambda - r}.$$

We first note that

$$\frac{\eta(r + \theta/\eta(r))}{\eta(r)} = \frac{1}{1-\theta}$$

so that Condition B does not hold. However $r\eta(r) \to \infty$ as $r \to \lambda$. Moreover,

$$\mu_r(s) = \left[1 + \frac{s(\lambda - r)}{r}\right]^{-1} e^{-s}(\lambda - r)/(\lambda - r - s(\lambda - r))$$
$$= \left[1 + \frac{s(\lambda - r)}{r}\right]^{-1} e^{-s}\left(\frac{1}{1-s}\right) \to e^{-s}\left(\frac{1}{1-s}\right), \qquad r \to \lambda.$$

Since F is log-*concave* (actually log-*linear*) on $(-\infty, 0]$ and the limit distribution has a density continuous at 0 ($m(0) = e^{-1}$) we may apply Theorem 4 to conclude that F belongs to the Gumbel (Λ) domain of attraction (in the sense of minimum). This result is of course directly available from F itself.

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Appendix

Proof of Lemma 2.1. (i) For $R<r<S$ assume (2) holds and

$$\omega^{(\alpha)}(r)=\int_{-\infty}^{\infty}(-1)^{\alpha}x^{\alpha}e^{-rx}F(dx)<\infty.$$

Then

$$\delta^{-1}[\omega^{(\alpha)}(r+\delta)-\omega^{(\alpha)}(r)]=\int_{-\infty}^{\infty}(-1)^{\alpha}x^{\alpha}e^{-rx}\delta^{-1}[e^{-\delta x}-1]F(dx).$$

Now $\delta^{-1}(e^{-\delta x}-1)\to -x$ as $\delta\to 0$ and for any $\varepsilon>0$

$$|x^{\alpha}|\,|\delta^{-1}[e^{-\delta x}-1]|<|x|^{\alpha+1}e^{|\delta x|}<e^{|\varepsilon x|} \quad \text{for } \delta<\delta_0(\varepsilon).$$

So by dominated convergence we obtain

$$\omega^{(\alpha+1)}(r)=\int_{-\infty}^{\infty}(-1)^{\alpha+1}x^{\alpha+1}e^{-rx}F(dx) \tag{A1}$$

and the result Lemma 2.1 (i) follows by induction.

(ii) Follows from an application of integration by parts (see Feller (1971, p. 150)).

(iii) The existence of $\xi(r)$ and $\eta^2(r)$ follows from (i). Moreover,

$$\eta^2(r)=\frac{\omega''(r)}{\omega(r)}-\left(\frac{\omega'(r)}{\omega(r)}\right)^2=E(X_r^2)-(EX_r)^2=\operatorname{var}(X_r)$$

where

$$P(X_r\in dx)=e^{-rx}F(dx)/\omega(r).$$

(iv) Suppose first that $e_L>-\infty$. Then $S=\infty$ and by (i) and (ii)

$$\begin{aligned}\xi(r)+\frac{1}{r}&=\int_0^{\infty}(x+e_L)F(x+e_L)e^{-rx}dx\Big/\int_0^{\infty}F(x+e_L)e^{-rx}dx\\&=\int_0^{\infty}xF(x+e_L)e^{-rx}dx\Big/\int_0^{\infty}F(x+e_L)e^{-rx}dx+e_L\to e_L \quad \text{as} \quad r\to\infty\end{aligned}$$

by the result of Doetch (1970, p. 221).

For the case $e_L=-\infty$ note that for $c\in\mathbb{R}$

$$\xi(r)=\left[\int_{-\infty}^{-c}+\int_{-c}^{\infty}\right](xe^{-rx}F(dx))\Big/\left[\int_{-\infty}^{-c}+\int_{-c}^{\infty}\right](e^{-rx}F(dx)). \tag{A2}$$

If $S<\infty$ then the integrals on $(-c,\infty)$ tend to constants as $r\to S$ and therefore since for $c>0$,

$$\int_{-\infty}^{-c}xe^{-rx}F(dx)<-c\int_{-\infty}^{-c}e^{-rx}F(dx)$$

we have for any $c>0$,

$$\limsup_{r\to S} \xi(r) < -c. \tag{A3}$$

If $S=\infty$, then taking $c=0$ in (A2) gives

$$\begin{aligned}\limsup \xi(r) &= \limsup \left\{ \int_{-\infty}^{0} x e^{-rx} F(dx) \Big/ \int_{-\infty}^{0} e^{-rx} F(dx) \right\} \\ &= \limsup \left\{ \int_{-\infty}^{0} x e^{-rx} F(x)\,dx \Big/ \int_{-\infty}^{0} e^{-rx} F(x)\,dx \right\} \\ &\leqq \limsup \left[\int_{-A}^{0} (x+A) e^{-rx} F(x)\,dx \Big/ \int_{-A}^{0} e^{-rx} F(x)\,dx \right] - A \\ &= -A \end{aligned} \tag{A4}$$

for any $A>0$, again by application of Doetch's result. Hence, via (A3) or (A4) we conclude

$$\xi(r) \to -\infty$$

as required.

Proof of Lemma 2.2. The proof is by contradiction.

Suppose at $x_0 \in \mathbb{R}$ m is positive and continuous, and for some $\varepsilon>0$

$$\limsup u_n(x_0) > m(x_0)(1+2\varepsilon). \tag{A5}$$

Then for a subsequence $\{k\}$

$$u_k(x_0) > m(x_0)(1+\varepsilon), \quad \text{all } k. \tag{A6}$$

Let

$$w_k = \sup\{x<x_0 : u_k(x) = m(x)\}$$

where $\sup(\emptyset) = -\infty$ by definition.

(i) We show that $w_k \to x_0$.

Suppose not. Then there exists $\delta_1>0$ and a sub-subsequence $\{l\}$ such that

$$w_l < x_0 - \delta_1, \quad \text{all } l. \tag{A7}$$

Also there exists $\delta_2>0$ such that m is continuous on $[x_0-\delta_2, x_0]$ and

$$m_0 = \max_{x\in[x_0-\delta_2, x_0]} m(x) \leqq m(x_0)(1+\varepsilon/2).$$

Let $\delta = \min(\delta_1, \delta_2)$. Then we have, by log-*concavity* (for $x_0<B_l$),

$$\begin{aligned}\int_{x_0-\delta}^{x_0} u_l(x)\,dx &> u_l(x_0) \int_{x_0-\delta}^{x_0} [u_l(x_0)/u_l(x_0-\delta)]^{(x-x_0)/\delta}\,dx \\ &= \delta u_l(x_0)[1-u_l(x_0-\delta)/u_l(x_0)] \cdot [-\log\{u_l(x_0-\delta)/u_l(x_0)\}] \\ &\geqq \delta u_l(x_0) \geqq \delta(1+\varepsilon) m(x_0) \geqq \frac{1+\varepsilon}{1+\frac{\varepsilon}{2}} \delta m_0\end{aligned}$$

$$\geqq \frac{1+\varepsilon}{1+\frac{\varepsilon}{2}}[M(x_0)-M(x_0-\delta)].$$

The last inequality contradicts the hypothesis

$$\int_{x_0-\delta}^{x_0} u_l(x)\,dx \to M(x_0)-M(x_0-\delta)>0.$$

(ii) Hence we assume that $w_k \to x_0$ and consider the line joining

$$(w_k, \log u_k(w_k)) \quad \text{and} \quad (x_0, \log u_k(x_0)).$$

The slope of this line is h_k and

$$h_k > [\log m(x_0) - \log m(w_k) + \log(1+\varepsilon)]/(x_0 - w_k) \to \infty, \qquad k \to \infty.$$

Thus by the log-*concavity* of u_k (for k large enough, i.e., when $B_k > x_0$) we have for $x < w_k$,

$$\log u_k(x) \leqq \log u_k(w_k) - (w_k - x)\,h_k = \log m(w_k) - (w_k - x)\,h_k.$$

Since $w_k \to x_0$ we conclude that for any $-\infty < a \leqq x \leqq b < x_0$

$$u_k(x) \to 0$$

and so by dominated convergence, for all $a < b < x_0$,

$$\int_a^b u_k(x)\,dx \to 0$$

which contradicts the weak convergence assumption given the positivity and continuity of m at x_0. Therefore

$$\limsup u_n(x_0) \leqq m(x_0)$$

and similarly we may show that

$$\liminf u_n(x_0) \geqq m(x_0)$$

and the proof is complete. □

Proof of Lemma 2.3. For the case $S=\infty$ (i) and (ii) follows directly from Theorem 2 and Theorem 5 of Bloom (1976) on setting $\phi(x) = 1/\eta(x)$ (in his notation).

Similarly for $S<\infty$, (i) follows from Bloom's Theorem 2 by using the same proof. As for (ii), since $S<\infty$ implies $e_L = -\infty$ we have that $\xi(r) \to -\infty$ by Lemma 2.1 (iv). If $\eta(r) \to \eta(S) < \infty$ as $r \to S$ then for $a < r < S$

$$\xi(r) > \xi(a) - \int_a^S \eta^2(u)\,du = K(a) > -\infty$$

which contradicts $\xi(r) \to -\infty$ as $r \to S$. Hence $\eta(r) \to \infty$ as $r \to S$. □

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