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The Lifetime of Conditioned Brownian Motion

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1. Introduction

In this paper we show that Brownian h-paths, $h>0$ harmonic, have finite lifetimes in plane domains of finite area. The analogous result for bounded domains in higher dimensions is false - we give an example of a bounded domain in $\mathbb{R}^3$ and a positive harmonic function for which the h-paths have infinite lifetime almost surely. This difference in behavior is related to the scaling property of Brownian motion.

In what follows, D will be a domain in $\mathbb{R}^n$ which has a Green function G. Denote by Δ the minimal Martin boundary of D [7]. Let Ω be the space of all continuous (in Martin topology) functions ω: $[0,\infty)\to D\cup\Delta$ with the property that $\omega(s)\in\Delta$ implies $\omega(t)\in\Delta$ for $t\geqq s$. Let $X(t,\omega)=\omega(t)$ be the coordinate process. Denote by $(\Omega,\mathscr{F},\mathscr{F}_t,X(t),\theta_t,P_x)$ the standard Brownian motion started at $x\in D$. If $p(t,x,y)$ is the transition density of Brownian motion killed on exiting D and $h>0$ is harmonic in D, define

$$p^h(t,x,y)=h(x)^{-1}\,p(t,x,y)\,h(y)$$

and let P_x^h denote the measure on Ω induced by p^h. These are the h-paths of Doob [3].

Theorem 1. *If D is a domain in $\mathbb{R}^2$, $h>0$ is harmonic in D, and $\tau_D=\inf\{t>0\colon X(t)\notin D\}$, then*

$$E_x^h(\tau_D)\leqq c\,|D| \tag{1.1}$$

where c is an absolute constant and $|D|$ is the area of D.

Statement (1.1) implies that if h is positive and harmonic on D, a plane domain of finite area, then

$$\lim_{t\to\infty} E_x[h(X(t));\,t<\tau_D]=0. \tag{1.2}$$

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The difficulty in proving this directly is that the supermartingale $h(X(t))I(t<\tau_D)$ need not be uniformly integrable. If it were, then the constant times, t, could be replaced by any sequence T_n of stopping times increasing to infinity. However, consider the following example: Let D be the unit disc in $\mathbb{R}^2$ and h the Poisson kernel with pole at $(1,0)$. Let $D_n=\{z\in D\colon h(z)>n\}$. ($D_n$ is the interior of a circle tangent to D at $(1,0)$.) Let T_n be the hitting time of D_n by $X(t)$. Then $h(X(t\wedge T_n\wedge\tau_D))$ is a uniformly integrable martingale, hence

$$h(0)=E_0(h(X(T_n\wedge\tau_D)))=E_0(h(X_{T_n});\ T_n<\tau_D).$$

Thus (1.2) does not hold with t replaced by T_n.

Another instructive example to consider is Littlewood's crocodile [5, p. 268]. The lifetimes of h-paths are finite even when conditioned to go to the bad end of the crocodile. Nothing is gained in terms of delaying the process by throwing up obstacles; that is, in $\mathbb{R}^2$.

Finally, let us mention some alternative formulations and consequences of our result. The well-known relationship between Green functions and occupation times yields:

Corollary 1. *Let $D\subseteq\mathbb{R}^2$ be a domain having a Green function $G(x,y)$. Then there is an absolute constant c such that, for any function $h\geqq 0$ harmonic on D,*

$$\int_D G(x,y)\,h(y)\,dy\leqq c\,|D|\,h(x).$$

In particular, if $h=1$,

$$\int_D G(x,y)\,dy\leqq c\,|D|.$$

Doob [3] shows that if Brownian motion is conditioned using the Green function of D with pole *inside* D then the conditioned paths have finite lifetimes. Combining this with Theorem 1 and the Riesz decomposition theorem produces:

Corollary 2. *If $h>0$ is superharmonic on D then $P_x^h(\tau_D<\infty)=1$ for any $x\in D$, whenever D is a plane domain of finite area.*

Our result also gives a refinement of a result of Lamb [6]. He shows that if $h\geqq 0$ is harmonic on a domain D then h may be decomposed uniquely as $h=h_1+h_2+h_3$ where $h_1(X(t\wedge\tau_D))$ is a nonnegative uniformly integrable martingale, $h_2(X(t\wedge\tau_D))$ is a nonnegative martingale with limit 0, and $h_3(X(t\wedge\tau_D))$ is a nonnegative supermartingale whose expectation approaches 0; moreover h_2-paths have infinite lifetimes. Thus our result shows that h_2 vanishes for plane domains of finite area.

In Sect. 2 we give the proof of Theorem 1. In Sect. 3 we construct a bounded region D in $\mathbb{R}^3$ and a positive harmonic function h such that $P_x^h(\tau_D=\infty)=1$ for all x in D.

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2. Proof of Theorem 1

We begin by establishing two lemmas.

Lemma 2.1. *Let D be an open subset of $\mathbb{R}^2$. There exists an absolute constant c such that $E_x(\tau_D) \leqq c|D|$, for all $x \in D$.*

Proof. Fix x in D. We may assume $|D|=1$ since if $a>0$ and $aD=\{ay: y \in D\}$ then $|aD|=a^2 D$ and $E_{ax}(\tau_{aD})=a^2 E_x(\tau_D)$. Thus it suffices to show that $E_x(\tau_D)$ is less than a constant independent of D. Setting $B=B(x,1)=\{y: |x-y|<1\}$, we have

$$\begin{aligned} E_x(\tau_D) &= E_x(\tau_D;\, \tau_D<\tau_B)+E_x(\tau_D;\, \tau_B \leqq \tau_D) \\ &\leqq E_x\tau_B + E_x(E_{X(\tau_B)}(\tau_D);\, \tau_B \leqq \tau_D) \\ &\leqq E_x\tau_B + P_x(\tau_B \leqq \tau_D) \sup_{y \in D} E_y(\tau_D). \end{aligned}$$

Once we have established $P_x(\tau_B \leqq \tau_T) \leqq p<1$ with p independent of D and x, we will have

$$E_x(\tau_D) \leqq \frac{c}{1-p}, \qquad \text{with } c=E_x\tau_B=1/2$$

as desired. Take $r<1$, put $K=D^c \cap B(x,r)$ and observe that $|K| \geqq \pi r^2-1$. Then if λ is the restriction of Lebesgue measure to K and G is the Green function for $B(x,1)$ set $G^\lambda(z)=\int G(z,y)\,\lambda(dy)$. Then

$$G^\lambda(x)=\int G(x,y)\,\lambda(dy) \leqq 2\pi \int_0^1 \log\left(\frac{1}{r}\right) r\,dr=\frac{\pi}{2}.$$

If $\omega_z(K)$ is the harmonic measure of K relative to $B(x,1)$ then $\omega_z(K) \geqq \frac{2}{\pi} G^\lambda(z)$. Hence

$$\omega_z(K) \geqq \frac{2}{\pi} G^\lambda(x) \geqq \frac{2}{\pi} \log\left(\frac{1}{r}\right)|K| \geqq \frac{2}{\pi} \log\left(\frac{1}{r}\right)(\pi r^2-1).$$

Choosing any value of $r<1$ such that the last expression is positive completes the proof. (For a different proof see [2, p. 148].)

Lemma 2.2. *Let h be a nonzero minimal harmonic function on D. Then for any $x \in D$, $P_x^h(h(X(\tau_D-))=\infty)=1$.*

Proof. We show first that $h(X(\tau_D-))=0$, P_x - a.s. If this were not the case then it is not difficult to show, using the fact that two-dimensional Brownian motion does not hit points, that there exist disjoint Borel subsets A_1 and A_2 of the boundary of D, such that

$$P_x(h(X(\tau_D-))\, I(X(\tau_D) \in A_i)>0)>0, \qquad i=1,2$$

Let $g_i(x)=E_x(h(X(\tau_D-));\ X(\tau_D) \in A_i)$. Then $g_i(x) \leqq h(x)$, $X \in D$; but since h is minimal, we then must have $g_i(x)=c_i h(x)$, for constants c_i, which is clearly impossible.

Next let $D(n)$ be a sequence of open subsets of D, whose interiors increase to D, and whose closures are compact subsets of D. For $M>0$ put $T_M = \inf\{t \geqq 0: h(X(t)) \geqq M\}$. Then $h(X(T_M \wedge \tau_{D(n)}))$ is a uniformly integrable martingale in n. Letting n approach infinity we obtain

$$h(x) = E_x h(X(T_M \wedge \tau_D)) = E_x(h(X(T_M)); T_M < \tau_D).$$

This means that $P_x^h(T_M < \tau_D) = 1$, and $h(X(t \wedge \tau_D))$ is unbounded with P_x^h - probability 1.

Finally, $h(X(t \wedge \tau_D))$ has an infinite limit at $\tau_D -$ since $\frac{1}{h(X(t))} I(t < \tau_D))$ is a nonnegative h-supermartingale.

Let us now return to the proof of Theorem 1.

We assume without loss of generality that h is minimal. Define

$$\begin{aligned} D_n &= \{x \in D: 2^{n-1} < h(x) < 2^{n+1}\}, \qquad n = 0, \pm 1, \pm 2, \ldots \\ C_n &= \{x \in D: h(x) = 2^n\}, \qquad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

so that $D = \bigcup_{n=-\infty}^{\infty} D_n$ and $\sum_{n=-\infty}^{\infty} |D_n| \leqq 2|D|$. Consider the stopping times

$$\begin{aligned} R(n) &= \tau_{D_n} = \inf\{t > 0: X(t) \notin D_n\} \\ S(n, 0) &= \inf\{t > 0: X(t) \in C_n\} \\ T(n, 1) &= R(n) \circ \theta_{S(n,0)} + S(n, 0) \end{aligned}$$

and if $S(n, 0), \ldots, S(n, k-1)$, $T(n, 1), \ldots, T(n, k-1)$ have been defined, set

$$T(n, k) = \begin{cases} R(n) \circ \theta_{S(n,k-1)} + S(n, k-1), & S(n, k-1) < \infty, \\ \infty, & S(n, k-1) = \infty. \end{cases}$$

$$S(n, k) = \begin{cases} S(n, 0) \circ \theta_{T(n,k)} + T(n, k), & T(n, k) < \infty \\ \infty, & T(n, k) = \infty. \end{cases}$$

Finally set

$$N(n) = \inf\{k \geqq 0: S(n, k) = \infty\} = \inf\{k \geqq 0: T(n, k+1) = \infty\},$$

and

$$L(n, k) = T(n, k) - S(n, k-1) = R(n) \circ \theta_{S(n,k-1)}.$$

In order to estimate $E_x^h(L(n, k-i); N(n) = k)$, $i = 1, 2, \ldots, k-1$, we establish the following observations, each for $n = 0, 1, 2, \ldots$:

(2.1) $$P_x^h(S(n, 0) < \infty) = 1/2, \qquad x \in C_{n+1},$$

(2.2) $$P_x^h(S(n, 0) < \infty) = 1, \qquad x \in C_{n-1},$$

(2.3) $$P_x^h(X(R(n)) \in C_{n-1}) = 1/3, \qquad x \in C_n,$$

(2.4) $$P_x^h(X(R(n)) \in C_{n+1}) = 2/3, \qquad x \in C_n,$$

(2.5) $$E_x^h(R(n) \mid X(R(n)) \in C_{n-1}) \leqq 3 E_x^h(R(n)), \qquad \text{for } x \in C_n,$$

and

(2.6) $$E_x^h(R(n)|X(R(n))\in C_{n+1})\leqq\tfrac{3}{2}E_x^h(R(n)),\qquad x\in C_n.$$

For (2.1) we have

$$P_x^h(S(n,0)<\infty)=\frac{2^n}{h(x)}=\frac{1}{2}\quad \text{for } x\in C_{n+1}.$$

Statement (2.2) follows at once from Lemma 2.2. For (2.3) we have

$$P_x^h(X(R(n))\in C_{n-1})=\frac{2^{n+1}}{3}\left(\frac{1}{h(x)}-\frac{1}{2^{n+1}}\right)=\frac{1}{3}$$

for $x\in C_n$. Of course (2.4) follows from (2.3).

Turning to (2.5), we have

$$\begin{aligned}&E_x^h(R(n)|X(R(n))\in C_{n-1})\\&=\int_0^\infty P_x^h(R(n)>\lambda|X(R_n)\in C_{n-1})\,d\lambda\\&=P_x^h(X(R(n))\in C_{n-1})^{-1}\int_0^\infty P_x^h(R(n)>\lambda,X(R(n))\in C_{n-1})\,d\lambda\\&\leqq 3\int_0^\infty P_x^h(R(n)>\lambda)\,d\lambda\qquad\text{(by (2.3))}\\&=3E_x^h(R(n)).\end{aligned}$$

Similarly (2.6) follows from (2.4). Now, for $i=1,2,\dots,k-1$ we have

$$\begin{aligned}&E_x^h(L(n,k-i);\,N(n)=k)\\&\quad=E_x^h(E_x^h(L(n,k-i);\,N(n)=k|X(T(n,k-i)))\,I(T(n,k-i)<\infty))\\&\quad=E_x^h(E_x^h(L(n,k-i);\,N(n)\geqq k-i|X(T(n,k-i))\,P_x^h(N(n)\\&\quad=k|X(T(n,k-i)))\,I(T(n,k-i)<\infty))\end{aligned}$$

by the strong Markov property. Using (2.1)-(2.4)

$$\begin{aligned}P_x^h(N(n)=k|X(T(n,k-i)))\,I(T(n,k-i)<\infty)\\=\begin{cases}\frac{1}{2}(\frac{2}{3})^i, & X(T(n,k-i))\in C_{n-1},\\ \frac{1}{4}(\frac{2}{3})^i, & X(T(n,k-i)))\in C_{n+1},\end{cases}\end{aligned}$$

Also,

$$\begin{aligned}&E_x^h(E_x^h(L(n,k-i),N(n)\geqq k-i|X(T(n,k-i)))\\&\quad\cdot I(X(T(n,k-i))\in C_{n-1},T(n,k-i)<\infty))\\&\quad=E_x^h(E_x^h(L(n,k-i);\,N(n)\geqq k-i|X(T(n,k-i))\in C_{n-1},T(n,k-i)<\infty)\\&\qquad\cdot I(X(T(n,k-i)\in C_{n-1},T(n,k-i)<\infty))\\&\quad=E_x^h(E_x^h(R(n)\circ S(n,k-i-1)|X(T(n,k-i))\in C_{n-1},T(n,k-i)<\infty)\end{aligned}$$

$$\cdot I(X(T(n,k-i))\in C_{n-1}, N(n)\geqq k-i))$$
$$= E_x^h(E_x^h(E^h_{X(S(n,k-i-1))}(R(n))|X(R(n)\in C_{n-1})|$$
$$\cdot X(T(n,k-i))\in C_{n-1}, T(n,k-i)<\infty)$$
$$\cdot I(X(T(n,k-i))\in C_{n-1}, N(n)\geqq k-i))$$
$$\leqq \sup_{x\in C_n} E_x^h(R(n)|X(R(n))\in C_{n-1})\, P_x^h(N(n)\geqq k-i, X(T(n,k-i)\in C_{n-1})$$
$$\leqq 3(\sup_{x\in C_n} E_x^h(R(n)))(\tfrac{2}{3})^{k-i-1}\tfrac{1}{3}, \quad \text{by (2.5)},$$
$$= (\tfrac{2}{3})^{k-i-1}\sup_{x\in C_n} E_x^h(R(n)).$$

In an entirely similar manner we obtain

$$E_x^h(E_x^h(L(n,k-i);\, N(n)\geqq k-i|X(T(n,k-i)))$$
$$\cdot I(X(T(n,k-i))\in C_{n+1}, T(n,k-i)<\infty))$$
$$\leqq (\tfrac{2}{3})^{k-i}\sup_{x\in C_n} E_x^h(R(n)).$$

Since

$$\tau_D \leqq \sum_{n=-\infty}^{\infty}\sum_{k=1}^{N(n)} L(n,k),$$

$$E_x^h(\tau_D)\leqq \sum_{n=-\infty}^{\infty} E_x^h \sum_{k=1}^{N(n)} L(n,k) \tag{2.7}$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty} E_x^h\left(\sum_{i=1}^{k} L(n,i);\, N(n)=k\right)$$
$$\leqq \left(\sum_{k=1}^{\infty} k(\tfrac{2}{3})^{k-1}\right)\sum_{n=-\infty}^{\infty}\sup_{x\in C_n} E_x^h(R(n))$$
$$= 9\sum_{n=-\infty}^{\infty}\sup_{x\in C_n} E_x^h R(n).$$

Finally, for $x\in C_n$

$$E_x^h(R(n)) = \int_0^\infty P_x^h(R(n))>\lambda)\,d\lambda$$
$$= h(x)^{-1}\int_0^\infty E_x(h(X(\lambda));\, R(n)>\lambda)\,d\lambda$$
$$\leqq 2\int_0^\infty P_x(R(n)>\lambda)\,d\lambda$$
$$= 2E_x R(n)\leqq 2c|D_n| \quad \text{by Lemma 2.1.}$$

Thus,

$$E_x^h(\tau_D)\leqq 18c\sum_{n=-\infty}^{\infty}|D_n|\leqq 36c|D|$$

and the proof is complete.

3. An Example

The purpose of this section is to construct an example of a bounded domain D in $\mathbb{R}^3$ and a harmonic function h such that $\tau_D=\infty$ with P_x^h-probability 1 for any x in D. It is necessary to recall some facts concerning the last exit decomposition of Brownian motion (see [4, 8, or 9].) Let K be a Borel subset of D and μ the last exit time from K of Brownian motion $X(t)$ conditioned with some harmonic function h. Then the process $X(\mu+t)$ has continuous paths on $[0,\infty)$ and is strong Markov on $(0,\infty)$ with respect to an appropriate sequence of σ-fields; moreover, the semigroup on $(0,\infty)$ is that of $X(t)$ conditioned not to hit K.

We will also need the following simple result.

Lemma 3.1. *Let B denote the ball in $\mathbb{R}^3$ centered at 0 with radius 1, and let $0<\delta<\frac{1}{4}$ be given. There exist absolute constants c_1 and c_2 such that*

$$P_x^g(\tau_B>c_1)>c_2, \qquad |x|<\delta,$$

for any positive harmonic function g.

Proof. By Harnack's inequality we may choose c_2, so small that $\frac{g(y)}{g(x)}\geqq 2c_2$ for $|y|<2\delta$, $|x|<2\delta$. Define $\tau_{2\delta}$ by $\tau_{2\delta}=\inf\{t: |x(t)-x(0)|>2\delta\}$. We may choose c_1 so small that $P_x(\tau_{2\delta}<c_1)>\frac{1}{2}$ for $|x|<\delta$. Then for all x

$$\begin{aligned} P_x^g(\tau_B>c_1) &= g(x)^{-1}E_x(g(X(c_1));\ \tau_B>c_1) \\ &\geqq g(x)^{-1}E_x(g(X(c_1));\ \tau_{2\delta}>c_1) \\ &\geqq 2c_2 P_x(\tau_{2\delta}>c_1), \qquad >c_2, \end{aligned}$$

and the proof is complete.

Note that the result holds for B a ball of radius η if δ is replaced by $\delta\eta$ and c_1 by $\eta^2 c_1$.

Consider for D the following region in $\mathbb{R}^3$: For $n=2,3,\ldots$ place n^6 nonintersecting balls of radius $\frac{1}{2}n^{-3}$ in a square array with centers on $\{(x,y,n^{-2});\ 0\leqq x\leqq 3,\ 0<y\leqq 3\}$. Thus we have infinitely many levels with balls on level n arranged in n^3 rows (of n^3 balls each) parallel to the y axis. Order the balls on the n-th level lexicographically according to the (x,y) coordinates of their centers and denote them by B_n^k, $k=1,2,\ldots,n^6$. Now on each level connect each ball to the next in order by a thin cylindrical tube. On all levels but the first connect similarly the first ball to the last ball on the level directly above and on all levels connect the last ball to the first ball on the level directly below. Use smooth tubes of finite total volume. For definiteness let us make the following additional assumptions about the connecting tubes: each tube intersects its balls in circular caps subtending solid angle $\varepsilon<\frac{\pi}{2}$, and the two caps in each ball (except the first) are antipodal. That completes the description of D.

Some subsequence of the sequence of centers of the balls comprising D is fundamental in the sense of Martin, and hence converges to a certain Martin boundary point ξ. The corresponding harmonic function $K(\cdot,\xi)$ may or may not be minimal, but in any case there is some minimal function h corresponding to a point in the support of the canonical representing measure of $K(\cdot, \xi)$. The function h vanishes identically on the Euclidean boundary of D. Therefore, by Lemma 2.2, the h-paths must thread their way through the entire maze of D. Fix x_0 in B_2^1. Since h is minimal $P_{x_0}^h(\tau_D=\infty)$ must be either 0 or 1. We show it is 1.

Define $T(n, k)$ to be the time between the last exit, $\mu(n, k)$, from the tube joining B_n^{k-1} to B_n^k and the first exit from B_n^k. Let $L(n)$ be the total time spent in the layer determined by $z=n^{-2}$. We need only show that there are constants λ_1 and λ_2 such that

$$P_{x_0}^h(L(n)>\lambda_1)>\lambda_2. \tag{3.1}$$

Indeed it then follows that

$$P_{x_0}^h(L(n)>\lambda_1 \text{ infinitely often})>0$$

and hence $P_{x_0}^h(\tau_D=\infty)>0$, which implies the desired conclusion.

Fix n and let V denote the random vector of last exit positions $X(\mu(n, k))$, $k=1, 2, \ldots, n^6$. Choose $0<\delta<\frac{1}{4}$ and let β_n^k denote the ball of radius $\frac{1}{2}n^{-3}\delta$ concentric with B_n^k. We will show that there is an absolute constant c_3 such that

$$P_{x_0}^h(X(\mu(n, k)+t\wedge T(n, k)) \text{ hits } \beta_n^k | \mathrm{V}) \geqq c_3, \quad \text{a.s.} \tag{3.2}$$

However, let us first show how (3.1) follows. Conditioned on V the $T(n, k)$ are independent random variables. (This follows from the conditional independence of past and future at a last exit time. See [4 or 9].) Let c_1 be as in Lemma 3.1 and let A_k be the event in (3.2). Then

$$\begin{aligned} P_{x_0}^h(T(n, k)>n^{-6}c_1 | V) &\geqq P_{x_0}^h(T(n, k)>n^{-6}c_1, A_k | V) \\ &\geqq E_{x_0}^h(P_{X(\sigma)}^h(\tau_{B_n^k}>n^{-6}c_1 | V);\, A_k | V), \end{aligned}$$

where $X(\sigma)$ denotes the hitting position on β_n^k. The conditional probability inside may be collapsed to a single conditioning with a new harmonic function on B_n^k. More specifically, note that for $B \subseteq \operatorname{supp} P_x^h X(\mu(n, k+1))^{-1}$

$$k(B, x)=P_x^h(X(\mu(n, k+1)) \in B)$$

is h-harmonic in B_n^k (use Dynkin's formula, for example) and $k(\cdot, x) \ll k(\cdot, y_0)$ where y_0 is any fixed point in B_n^k (use maximum principle on the harmonic function $k(B, \cdot)h(\cdot)$). Then $k_z(x)=\lim\limits_{B \to \{z\}} \dfrac{k(B, x)}{k(B, y_0)}$, B shrinking nicely to z, defines an h-harmonic function on B_n^k as the convergence is uniform on compact subsets of B_n^k. Using now the above stated conditional independence

$$\begin{aligned} &E_{x_0}^h(P_{X(\sigma)}^h(\tau_{B_n^k}>n^{-6}c_1 | V); A_k | V)(\omega) \\ &\quad = E_{x_0}^h(P_{X(\sigma)}^h(\tau_{B_n^k}>n^{-6}c_1 | X(\mu(n, k+1))=z); A_k | V)(\omega) \end{aligned}$$

where $z=z(\omega)=X(\mu(n,k+1)(\omega),(\omega))$. Furthermore,

$$E_{x_0}^h(P_{X(\sigma)}^h(\tau_{B_n^k}>n^{-6}c_1)\,k_z(X(\tau_{B_n^k}))\,k_z(X(\sigma))^{-1};A_k|V)(\omega)$$
$$=E_{x_0}^h(P_{X(\sigma)}^{hk_z}(\tau_{B_n^k}>n^{-6}c_1);A_k|V)(\omega).$$

It then follows from Lemma 3.1 and (3.2) that

$$P_{x_0}^h(T(n,k)>n^{-6}c_1|V)\geqq c_2c_3, \quad \text{a.s.}$$

Let N denote the number of the $T(n,k)$ in the triangular arrays of random variables, independent in each row, which are greater than $n^{-6}c_1$. It follows from the weak law of large numbers (see e.g. [1]), that there is a constant c_4 such that $P\left(N>\frac{n^6}{2}c_2c_3\middle|V\right)>c_4$. Thus

$$P\left(\sum_{k=1}^{n^6}T(n,k)>\tfrac{1}{2}c_1c_2c_3\middle|V\right)>c_4.$$

Since $L(n)\geqq\sum_{k=1}^{n^6}T(n,k)$, (3.1) follows with $\lambda_1=\frac{1}{2}c_1c_2c_3$ and $\lambda_2=c_4$.

There remains only to show (3.2). Let C denote the intersection with ∂B_n^k of the outgoing tube. Let $Y(t)$ denote the process $X(\mu(n,k)+(t\wedge T(n,k)))$. Then there is a nonnegative function g, harmonic in B_n^k with boundary values supported in C, such that the semigroup of $Y(t)$ on $(0,\infty)$ is the same as that of $X(t\wedge\tau_{B_n^k})$ conditioned by g. With these remarks in mind it is easy to see that (3.2) follows from

Lemma 3.2. *Let* $0<\delta<\frac{1}{4}$ *and* B *and* B_δ *be the balls of radius* 1 *and* δ *respectively, centered at the origin. For* $0<\varepsilon<\frac{1}{2}$ *define* I_ε *as the intersection with* ∂B *of the ball of radius* ε *centered at* $(0,1,0)$, *and* J_ε *as the intersection with* B *of a similar ball centered at* $(0,-1,0)$. *Let g be any nonnegative function harmonic on* B *with boundary values supported in* I_ε. *Finally, let* $Y(t)$ *be a process defined on a probability space* $(\Omega,\mathscr{F},P)$ *and having the following properties:*

(3.3) $Y(t)$ *has continuous paths on* $[0,\infty)$.

$$P(Y(0)\in J_{\varepsilon/2})=1. \tag{3.4}$$

(3.5) $Y(t)$ *is strong Markov on* $(0,\infty)$.

(3.6) *The semigroup of* $Y(t)$ *on* $(0,\infty)$ *is the same as that of Brownian g-paths.*

Then there is a constant c_4, *depending only on* ε *and* δ, *such that*

$$P(Y(t)\text{ hits }B_\delta)\geqq c_4.$$

Proof. Let $P(z,w)$ denote the Poisson kernel for the annular region $B\setminus B_\delta$ with pole at w and let $Q(z,w)$ denote the Poisson kernel for B with pole at w. Then there is an absolute constant c_5 such that

$$\frac{P(z,w)}{Q(z,w)}\leqq c_5<1 \qquad \text{for } z\in J_\varepsilon,\ w\in I_\varepsilon. \tag{3.7}$$

To see this, first note that it is enough to show that

$$\lim_{x\uparrow 1} \frac{P((0,-x,0),w)}{Q((0,-x,0),w)}<1$$

for each w in $I_{2\varepsilon}$. Fix such a w and let $q(x)$ and $p(x)$ stand for the denominator and numerator above and ω_x for harmonic measure relative to $B\backslash B_\delta$ at the point $(0,-x,0)$. Then

$$p(x)=q(x)-\int_{2B_\delta} Q(z,w)\,\omega_x(dz).$$

Since $Q(z,w)$ is bounded below by some positive absolute constant on B_δ it suffices to show that $\lim_{x\uparrow 1} \frac{\omega_x(B_\delta)}{q(x)}>0$. This follows at once from the explicit expressions

$$\omega_x(B_\delta)=\frac{\delta(1-x)}{x(1-\delta)},$$

and

$$q(x)=c(1-x^2)\,|w-(0,-x,0)|^{-3}.$$

Now it follows from (3.3) and (3.4) that there is a strictly positive stopping time τ such that $Y(\tau)$ belongs to J_ε with probability 1. Fix z in J_ε. By (3.6) we have

$$P_z(Y(t)\text{ hits }B_\delta)=\frac{u(z)}{g(z)}$$

where u is a function harmonic in $B\backslash B_\delta$ with boundary values g on B_δ and 0 on ∂B. Now

$$g(z)=\int_{I_\varepsilon} g(w)\,Q(z,w)\,dw$$

and

$$u(z)=g(z)-\int_{I_\varepsilon} g(w)\,P(z,w)\,dw.$$

But

$$\int_{I_\varepsilon} g(w)\,P(z,w)\,dw \leqq c_5 \int_{I_\varepsilon} g(w)\,Q(z,w)\,dw$$

by (3.7). Thus

$$\frac{u(z)}{g(z)}\geqq 1-c_5>0.$$

Finally, by (3.5) we have

$$P(Y(t)\text{ hits }B_\delta)=EP_{Y(\tau)}(Y(t)\text{ hits }B_\delta)\geqq 1-c_5$$

and the proof is complete.

References

1. Chow, Y.S., Teicher, H.: Probability Theory. Berlin-Heidelberg-New York: Springer 1978
2. Chung, K.L.: Lectures from Brownian motion to Markov processes. Berlin-Heidelberg-New York: Springer 1982
3. Doob, J.L.: Conditional Brownian motion and the boundary limits of harmonic functions. Bull. Soc. Math. France **85**, 431–458 (1957)
4. Getoor, R.K., Sharpe, M.J.: Last Exit Decompositions. Indiana Math. J. **23**, 377–404 (1973)
5. Ito, K., McKean, H.P.: Diffusion Processes and their Sample Paths. Berlin-Heidelberg-New York: Springer 1974
6. Lamb, C.W.: A note on harmonic functions and martingales. Annals Math. Stat. **42**, No. 6, 2044–2049 (1971)
7. Martin, R.S.: Minimal positive harmonic functions. Trans. Amer. Math. Soc. **49**, 137–172 (1941)
8. Meyer, P.A., Smythe, R.T., Walsh, J.B.: Birth and Death of Markov processes. Proc. Sixth Berkeley Sympos. Math. Statist. and Probability Vol. III, 295–305, Univ. Calif. (1972)
9. Pittenger, A.O., Shih, C.T.: Coterminal families and the strong Markov property. Trans. Amer. Math. Soc. **182**, 1–42 (1973)

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