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Inequalities for Upcrossings of Semimartingales via Skorohod Embedding

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0. Introduction

Let M be a martingale, and let $U(M,a,a+\varepsilon)$ be the number of upcrossings made by M from below a to above $a+\varepsilon$. The classical upcrossing inequality of Doob states that if M is uniformly integrable,

$$EU(M,a,a+\varepsilon) \leqq \frac{1}{\varepsilon}(\|M_\infty\|_1+|a|). \tag{0.1}$$

From the definition of the local time of M, $L_t^a(M)$, it is easy to check that, if M is uniformly integrable,

$$E\,L_\infty^a(M) \leqq \|M_\infty\|_1 \tag{0.2}$$

In [1] the quantity $L^*(M)=\sup\limits_a L_\infty^a(M)$ was introduced, and it was shown that if M is a continuous martingale,

$$\|L^*(M)\|_1 \leqq c\,\|M^*\|_1, \tag{0.3}$$

where c is a universal constant, and $M^*=\sup\limits_t |M_t|$. Comparing these three inequalities, and recalling that, for a continuous martingale M,

$$L_\infty^a(M)=\lim_{\varepsilon\downarrow 0} \varepsilon\, U(M,a,a+\varepsilon), \tag{0.4}$$

it is natural to conjecture that $E(\sup\limits_a \varepsilon\, U(M,a,a+\varepsilon))$ is bounded by $c\,\|M^*\|_1$. In this paper this conjecture will be proved, and the result will be extended, in a suitable form, to general semimartingales. The principal result is the following inequality.

Theorem. *Let X be a semimartingale, with decomposition $X=X_0+M+A$, where M is a martingale and A is previsible and of finite variation. Then there exist universal constants, c_p such that for each $p \geqq 1$*

$$\|\sup_a \varepsilon U(X, a, a+\varepsilon)\|_p \leqq c_p \left\| M^* + \int_0^\infty |dA_s| \right\|_p. \tag{0.5}$$

As this result holds for discontinuous, as well as continuous, semimartingales, it also holds for processes in discrete time.

The convergence in (0.4) is in L^1, so (0.3) is an immediate consequence of (0.5); in some ways the natural approach to these inequalities would be to prove (0.5) for discrete time processes, and then deduce the general result by suitable limiting arguments. However, a direct proof of (0.5) seems hard, and the approach adopted here is to deduce (0.5) from (0.3) (and its generalisation to semimartingales proved in [2]) by using the probabilistic tricks of path decomposition and Skorohod embedding.

Consider first the case when X is a continuous martingale. By time change this reduces to proving (0.5) for X of the form B^T, where B is a Brownian motion, and T any stopping time. In fact, by Lemma 4.1 of [2] it is enough to prove that, for any $n \geqq 0$,

$$E(\sup_a \varepsilon U_{T_n}(B, a+\varepsilon)) \leqq c E(B^*_{T_n}), \tag{0.6}$$

where $T_n = \inf\{t: |B_t| = n\varepsilon\}$.

We may decompose the process B as follows. Let $S_1, S_2, \ldots$ be the successive hits by B on the grid $\varepsilon\mathbb{Z}$ – so that the process $B_i^{(\varepsilon)} = B_{S_i}$, $i \geqq 0$, is a simple symmetric random walk on $\varepsilon\mathbb{Z}$.

We may consider the process B^{T_n} as being built out of a simple symmetric random walk on $\varepsilon\mathbb{Z}$, and a collection of independent, identically distributed Brownian journeys to $+\varepsilon$ and $-\varepsilon$. It is therefore intuitively clear (and it will be proved in Sect. 2) that $B^{(\varepsilon)}$ is independent of the random variables $L^{B_{S_i}}_{S_{i+1}} - L^{B_{S_i}}_{S_i}$, for $i \geqq 1$.

Let $n \geqq 1$ be fixed, let $N = \min\{i: B_i^{(\varepsilon)} = n\varepsilon\}$, let R be the smallest value of r which maximises $U_{T_n}(B, r\varepsilon, (r+1)\varepsilon)$, and let V be the value of this maximum. As R and V depend only on the process $B^{(\varepsilon)}$, if $S_{J_1}, \ldots, S_{J_V}$ are the times which mark the beginnings of the V upcrossings made by B from $R\varepsilon$ to $(R+1)\varepsilon$, then V is independent of each of the random variables $h_i = L^{R\varepsilon}_{S_{J_i+1}} - L^{R\varepsilon}_{S_{J_i}}$ $1 \leqq i \leqq V$. Therefore, since $L_{T_n} \geqq L^{R\varepsilon}_{T_n} \geqq \sum_{i=1}^{V} h_i$, and $Eh_i = \varepsilon$,

$$c E B^*_{T_n} \geqq E \sum_{i=1}^{V} h_i = EV \cdot Eh_i = \varepsilon EV$$

Finally,

$$V = \max_r U_{T_n}(B, r\varepsilon, (r+1)\varepsilon) \geqq \sup_a U_{T_n}(B, a, a+2\varepsilon),$$

proving (0.6).

Now let M be a general (right-continuous) martingale in H^1. By Monroe's result [7] M may be embedded in a Brownian motion – there exists a Brownian motion B_t, and a time change τ_t such that $M_\cdot \sim B_{\tau_\cdot}$, and B^{τ_∞} is uniformly integrable. Let $T = \tau_\infty$: using (0.5) for B^T, for any $p > 1$

$$\begin{aligned}\|\sup_a U(M,a,a+\varepsilon)\|_p &\leqq \|\sup_a U_T(B,a,a+\varepsilon)\|_p \\ &\leqq c_p\|B_T^*\|_p \leqq \frac{pc_p}{p-1}\|B_T\|_p \\ &= \frac{pc_p}{p-1}\|M_T\|_p \leqq \frac{pc_p}{p-1}\|M_T^*\|_p,\end{aligned}$$

proving (0.5) for general M in this case. Unfortunately, this cannot work for $p=1$, since there exist martingales M for which $\|M^*\|_1<\infty$, but $\|B_T^*\|_1=\infty$.

This difficulty is overcome by restricting attention to increases in the local time of B inside the envelope $\{(x,t):|x|<M_t^*\}$. Using the results of [2], it is shown in Sect. 4 that this restricted local time is bounded by an expression which depends on $\|M^*\|_1$ and $\|B_T\|_1$, rather than $\|B_T^*\|_1$. This enables the basic upcrossing inequality for a discrete martingale embedded in a Brownian motion to be proved (Proposition 4.2).

In Sect. 5 this inequality is extended to semimartingales, using the embedding theorem of Monroe [8]: any semimartingale is the time-change of a Brownian motion.

1. Basic Notation

Let $(\Omega,\mathscr{F},\mathscr{F}_t,P)$ be a probability space and X be any measurable stochastic process. Let

$$X_t^*=\sup_{s\leqq t}|X_s|,\ X^*=X_\infty^*.$$

Let $U_t^*(X,a,b)$ denote the number of upcrossings made by X across the interval (a,b) in the time interval $[0,t]$, and let

$$U_t^*(X,\varepsilon)=\sup_a U_t(X,a,a+\varepsilon).$$

We will frequently make use of the pathwise inequalities

$$\sup_r U_t(X,r\varepsilon,(r+1)\varepsilon)\leqq U_t^*(X,\varepsilon) \tag{1.1 a}$$

$$U_t^*(X,\varepsilon)\leqq \sup_r U_t(X,r\cdot\tfrac{1}{2}\varepsilon,(r+1)\cdot\tfrac{1}{2}\varepsilon). \tag{1.1 b}$$

For any random variable $\|f\|_p=(E|f|^p)^{1/p}$ is the L^p-norm of f, for each $p\geqq 1$. Let M be a local martingale; for $p\geqq 1$ we define the L^p and H^p norms of M by

$$\|M\|_{H^p}=\|[M,M]_\infty^{\frac{1}{2}}\|_p$$

$$\|M\|_{L^p}=\sup\{\|M_T\|_p,\ T \text{ a finite stopping time}\}.$$

The H^p norm is well known, and is related to $\|M^*\|_p$ by the Burkholder-Gundy inequalities: there exist universal constants c_p, C_p, such that

$$c_p \|M\|_{H^p} \leqq \|M^*\|_p \leqq C_p \|M\|_{H^p}.$$

It follows that if $\|M\|_{H^p} < \infty$, then M is a martingale.

If $p > 1$, and $\|M\|_{L^p} < \infty$, then by the martingale convergence theorem $\|M\|_{L^p} = \|M_\infty\|_p$, and M is again a martingale. If $p = 1$, and M is a uniformly integrable martingale $\|M\|_{L^1} = \|M_\infty\|_1$, but in general it is only true that $\|M_\infty\|_1 \leqq \|M\|_{L^1}$. If M is a martingale, then as $M_T = \lim_{t\to\infty} M_{T\wedge t}$, and $E|M_{T\wedge t}| \leqq E|M_t|$, by Fatou's Lemma $E|M_T|^p \leqq \lim_{t\to\infty} E|M_t|^p$ for any finite T, so that

$$\|M\|_{L^p} = \sup_t \|M_t\|_p. \tag{1.2}$$

This shows that, if M is a martingale relative to two filtrations, $\|M\|_{L^p}$ does not depend on the filtration. (1.2) is not true in general for local martingales.

If M is a local martingale, and $T_n \uparrow +\infty$, then

$$\lim_{n\to\infty} \|M^{T_n}\|_{L^p} = \|M\|_{L^p}.$$

A semimartingale X is a process of the form $X = X_0 + M + A$, where M is a local martingale, A is a process of locally finite variation, and $M_0 = A_0 = 0$. The H^p norm of X is defined, for $p \geqq 1$, by

$$\|X\|_{H^p} = \inf_{X = X_0 + M + A} \left\| |X_0| + [M,M]_\infty^{\frac{1}{2}} + \int_0^\infty |dAs| \right\|_p,$$

(see [4]). If $\|X\|_{H^1} < \infty$, we shall say X is an H^1-semimartingale; X has a canonical decomposition $X = X_0 + N + B$, where N is a martingale in H^1, and B is a previsible process of integrable variation. Further, $\|X\|_{H^p} = \| |X_0| + [N,N]_\infty^{\frac{1}{2}} + \int |dBs| \|_p$ for $p \geqq 1$. Note also that, if X is in fact a martingale, the two definitions of $\|X\|_{H^p}$ agree.

If X is any continuous semimartingale, then X has a canonical decomposition $X = X_0 + M + A$, where M is a continuous local martingale, and A is continuous and of locally finite variation. Thus any continuous semimartingale is locally in H^1.

Throughout this paper c_p will denote a universal constant depending only on p, the precise value of which will change from line to line.

Using the Burkholder-Gundy inequalities, if $X = X_0 + M + A$ is the canonical decomposition of an H^1-semimartingale X, then

$$\|X\|_{H^p} \leqq c_p \left\| |X_0| + M^* + \int_0^\infty |dAs| \right\|_p \leqq c_p \left\| X^* + \int_0^\infty |dAs| \right\|_p \leqq c_p \|X\|_{H^p} \tag{1.3}$$

We will only be concerned with the local time of continuous semimartingales. For any $a \in \mathbb{R}$, the local time of X at a is defined by Tanaka's formula

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t 1_{(X_s > a)} dX_s + \tfrac{1}{2} L_t^a(X). \tag{1.4}$$

When it is clear which process is being referred to, we will omit the dependence on X and write L_t^a: we will also use the notation $L(a,t)$ or $L(a,t,X)$ when a,t are themselves complicated expressions. By [9] we may take a version L: $(a,t)\to L_t^a$ which is jointly right-continuous with left limits in a, and continuous in t. We set

$$L_t^* = \sup_a L_t^a, \quad L^* = L_\infty^*.$$

The following inequality was proved in [2]: for any continuous semimartingale X, there exist constants c_p such that

$$\|L^*(X)\|_p \leqq c_p \|X - X_0\|_{H^p}, \qquad p \geqq 1. \tag{1.5}$$

In the remainder of this section we shall recall some elementary consequences of Lemma 1.2 of [6], which is stated here in a simplified form. (Note the misprint in [6, 1.2(a)].)

Lemma 1.1. *Let A be a non-negative, increasing and previsible process, with $A_0 = 0$, and f be a non-negative integrable random variable. If for all stopping time S,*

$$E(A_\infty - A_S) \leqq E(f\, 1_{(S<\infty)}), \tag{1.6}$$

then, for all $p \geqq 1$,

$$\|A_\infty\|_p \leqq p\, \|f\|_p.$$

The following result was proved in [6] in the case when X is a uniformly integrable martingale.

Corollary 1.2. *Let X be a continuous semimartingale, with canonical decomposition $X = X_0 + M + A$. Then for any $p \geqq 1$*

$$\|L_\infty^0(X)\|_p \leqq \sup_T p \left\| X_T^+ + \int_0^\infty |dA_s| \right\|_p \tag{1.7}$$

Proof. Suppose first that M is in H^1. For any stopping time S

$$L_\infty^0 - L_S^0 = X_\infty^+ - X_S^+ - \int_S^\infty 1_{(X_t>0)}\, dM_t - \int_S^\infty 1_{(X_t>0)}\, dA_t,$$

so that

$$E(L_\infty^0 - L_S^0) = E\left(X_\infty^+ - X_S^+ - \int_S^\infty 1_{(X_t>0)}\, dA_t\right) \leqq E\, 1_{(S<\infty)} \left(X_\infty^+ + \int_0^\infty |dA_t|\right).$$

By Lemma 1.1, therefore, we have

$$\|L_\infty^0\|_p \leqq p \left\| X_\infty^+ + \int_0^\infty |dA_s| \right\|_p.$$

The result now follows, since any continuous local martingale is locally in H^1.

In particular, if M is a local martingale,

$$\|L_\infty^0(M)\|_p \leqq p\,\|M\|_{L^p}, \qquad p \geqq 1. \tag{1.8}$$

This inequality is false, in general, for $p<1$ - see [10].

2. Skeletons of Brownian Motion

Let X_t be any continuous process. Let $\varepsilon>0$ be fixed: we define the *skeleton of X on the grid $\varepsilon\mathbb{Z}$*, denoted $X^{(\varepsilon)}$, as follows. Set

$$\begin{aligned} S_0(X) &= \inf\{s \geqq 0 \colon X_s \in \varepsilon\mathbb{Z}\} \\ S_{n+1}(X) &= \inf\{s \geqq 0 \colon |X_s - X_{S_n(X)}| = \varepsilon\} \\ X_n^{(\varepsilon)} &= X_{S_n(X)}. \end{aligned}$$

Thus $X^{(\varepsilon)}$ is a discrete time process, taking its values on $\varepsilon\mathbb{Z}$. In contexts where it is clear which process is being referred to, $S_n(X)$ will be shortened to S_n.

If X is a Brownian motion, it is intuitively clear that, conditional on whether $X_{S_{n+1}} - X_{S_n}$ is equal to $+\varepsilon$ or $-\varepsilon$, the path $X_{S_n+t} - X_{S_n}$, $0 \leqq t \leqq S_{n+1} - S_n$, is independent of the process $X^{(\varepsilon)}$. We shall call these parts of the path of X journeys from 0 to $\pm\varepsilon$, and will decompose X into $X^{(\varepsilon)}$ and two sequences of independent identically distributed random variables taking values in the set of journeys to $+\varepsilon$, and $-\varepsilon$.

Let J^+ be the space of journeys from 0 to $+\varepsilon$. More precisely, J^+ is the set of left-continuous functions $f\colon \mathbb{R}^+ \to \mathbb{R} \cup \{\partial\}$ with the properties

(i) $f(0)=0$

(ii) if $\zeta(f) = \inf\{s\colon f(s)=\varepsilon\}$ then

(a) $f(t)=\partial$ for $t>\zeta(f)$

(b) $f(t) \in (-\varepsilon, \varepsilon]$ for $0 \leqq t \leqq \zeta(f)$

(c) $f(t)$ is continuous for $0 \leqq t \leqq \zeta(f)$

(d) $f(\zeta(f)) = \varepsilon$.

Similarly, let J^- be the space of journeys from 0 to $-\varepsilon$, and let $J = J^+ \cup J^-$.

Let B be a Brownian motion, with $B_0=0$, $S_n = S_n(B)$, and

$$\xi_n(t) = \begin{cases} B_{S_{n-1}+t} - B_{S_{n-1}} & 0 \leqq t \leqq S_n - S_{n-1} \\ \partial & t > S_n - S_{n-1} \end{cases}$$

Thus $\xi_1, \xi_2, \ldots$ is a sequence of J-valued random variables, and by the Strong Markov property of B the ξ_i are independent and identically distributed. Let μ be their common probability distribution on J.

We now split the sequence (ξ_i) into two sequences, with values in J^+ and J^-. Let

$$N_r^{\pm} = \inf\left\{m\colon \sum_{i=1}^m 1_{J^\pm}(\xi_i) = r\right\}, \quad \xi_r^+ = \xi_{N_r^+}, \quad \xi_r^- = \xi_{N_r^-}.$$

The sequence (ξ_i^+) is therefore the sequence (ξ_i) with the values in J^- omitted.

Lemma 2.1. (i) (ξ_i^+), (ξ_i^-) *are sequences of independent identically distributed random variables, with law given by*

$$\mu^\pm(A)=P(\xi_i^\pm\in A)=2\mu(A\cap J^\pm) \quad \textit{for } A\subset J$$

(ii) $B^{(\varepsilon)}$, (ξ_i^+) *and* (ξ_i^-) *are mutually independent.*

Proof. Since (ξ_i) is a sequence of independent identically distributed random variables, it is a classical result that

$$\begin{aligned}
&P(\xi_1^+\in A_1,\dots,\xi_n^+\in A_n,\xi_1^-\in C_1,\dots,\xi_m^-\in C_m,1_{J^+}(\xi_1)=e_1,\dots,1_{J^+}(\xi_k)=e_k)\\
&=\prod_{i=1}^n P(\xi_1\in A_i|\xi_1\in J^+)\cdot\prod_{j=1}^m P(\xi_1^-\in B_j|\xi_1^-\in J^-)\cdot 2^{-k}\\
&=\prod_{i=1}^n \mu^+(A_i)\cdot\prod_{j=1}^m \mu^-(B_j)\cdot 2^{-k}.
\end{aligned}$$

This implies (i), and also that the sequences (ξ_i^+), (ξ_i^-), $(1_{J^+}(\xi_i))$ are independent. Since $B_n^{(\varepsilon)}=\varepsilon\left(2\sum_{i=1}^n 1_{J^+}(\xi_i)-n\right)$, (ii) follows.

This lemma gives a decomposition of B into the independent components $B^{(\varepsilon)}$, (ξ_i^+), (ξ_i^-). This decomposition may be reversed, so that given a simple symmetric random walk Y on $\varepsilon\mathbb{Z}$, and sequences of i.i.d.r.v. $\xi_i^\pm$ in $J^\pm$, with laws $\mu^\pm$, a Brownian motion W_t may be constructed, with $W^{(\varepsilon)}=Y$.

Lemma 2.1 is the independence result used in the sketch proof of (0.6) given in the introduction. In Sect. 4 a more complicated version of the same result will be needed:

Theorem 2.2. *Let B be a Brownian motion on the probability space $(\Omega,\mathscr{F},\mathscr{F}_t,P)$, where $\mathscr{F}$ contains a random variable with continuous distribution independent of $\mathscr{F}_\infty$. Let C_n be a discrete time process, with $C_n\in\mathscr{F}_{S_n(B)}$. Then there exists a filtration $(\mathscr{G}_t)$ and a process W_t such that*

(i) *W is a Brownian motion/$(\mathscr{G}_t)$,*

(ii) $W^{(\varepsilon)}=B^{(\varepsilon)}$,

(iii) $C_n\in\mathscr{G}_{S_n(W)}$.

Remark. The conditions on $\mathscr{F}_\infty$ and $\mathscr{F}$ are simply to ensure that a large enough supply of random variables independent of $\mathscr{F}_\infty$ can be found.

Proof. Let (ξ_i^+), (ξ_i^-) be independent identically distributed random variables, with distributions μ^+ and μ^-, and jointly independent of $\mathscr{F}_\infty$. Let W be the Brownian motion obtained from $B^{(\varepsilon)}$, (ξ_i^+), (ξ_i^-) by reversing the decomposition of Lemma 2.1, and let $\mathscr{H}_t$ be the natural filtration of W. Set

$$\mathscr{G}_t=\sigma(W_s,C_n 1_{[S_n,\infty)}(s),n>0,s\leqq t).$$

Thus W_t and $\mathscr{G}_t$ satisfy (ii) and (iii), and it remains to verify that W remains a Brownian motion when $(\mathscr{H}_t)$ is enlarged to $(\mathscr{G}_t)$. For this it is necessary and sufficient that, for each $t \geqq 0$, $\mathscr{H}_\infty$ and $\mathscr{G}_t$ should be conditionally independent given $\mathscr{H}_t$ - see [3] or [4].

For some elementary properties of conditional independence see, for example [4]. $\mathscr{H}_\infty$ and $\mathscr{G}_t$ are conditionally independent given $\mathscr{H}_t$ if and only if for each $t \geqq 0$

$$E(g \mid \mathscr{H}_\infty) \in \mathscr{H}_t \quad \text{for all } g \in b\mathscr{G}_t. \tag{2.1}$$

Let $S_n = S_n(W)$: we begin by proving (2.1) for S_n. Let $n \geqq 0$ be fixed, $\mathscr{E}_n = \sigma(W_{S_{n+t}} - W_{S_n}, t \geqq 0)$, and $Y_n = \sum_{i=1}^{n} 1_{(B_i^{(\varepsilon)} > B_{i-1}^{(\varepsilon)})}$. Then

$$\mathscr{G}_{S_n} = \sigma(\xi_i^+, i \leqq Y_n, \xi_i^-, i \leqq n - Y_n, B_r^{(\varepsilon)}, C_r, r \leqq n),$$
$$\mathscr{E}_n = \sigma(\xi_i^+, i > Y_n, \xi_i^-, i > n - Y_n, B_{n+r}^{(\varepsilon)} - B_n^{(\varepsilon)}, r \geqq 0),$$

so that $\mathscr{G}_{S_n}$ and $\mathscr{E}_n$ are independent. Hence, since $\mathscr{H}_\infty = \mathscr{H}_{S_n} \vee \mathscr{E}_n$, it follows that $\mathscr{H}_\infty$ and $\mathscr{G}_{S_n}$ are conditionally independent given $\mathscr{H}_{S_n}$.

Therefore, if $\phi \in b\mathscr{H}_t$, $\psi \in b\mathscr{G}_{S_n}$,

$$E(\phi \psi 1_{(S_n \leqq t)} \mid \mathscr{H}_\infty) = \phi 1_{(S_n \leqq t)} E(\psi \mid \mathscr{H}_\infty) = \phi 1_{(S_n \leqq t)} E(\psi \mid \mathscr{H}_{S_n}) \in \mathscr{H}_t. \tag{2.2}$$

By the monotone class Lemma it now follows from (2.2) that $E(g \mid \mathscr{H}_\infty) \in \mathscr{H}_t$ for any g of the form $g = g_n 1_{(S_n \leqq t)}$, where $g_n \in b(\mathscr{H}_t \vee \mathscr{G}_{S_n})$. However,

$$\sigma(g_n 1_{(S_n \leqq t)}, g_n \in b(\mathscr{H}_t \vee \mathscr{G}_{S_n}), n \geqq 0) = \mathscr{G}_t,$$

and, applying the monotone class Lemma again, (2.1) follows.

The following Lemma, which will be needed later, relates the L^p norms of B and $B^{(\varepsilon)}$.

Lemma 2.3. *Let M be a stopping time/$(\mathscr{F}_{S_n})$. Then*

$$\|(B^{(\varepsilon)})^M\|_{L^p} \leqq \|B^{S_M}\|_{L^p} \leqq \varepsilon + \|(B^{(\varepsilon)})^M\|_{L^p}.$$

Proof. For $n \geqq 0$, $S_{n \wedge M}$ is a stopping time/$(\mathscr{F}_t)$, so that the left hand inequality is immediate. If $t \geqq 0$, let $N = \min\{n \geqq 0: t \leqq S_n\}$: then $|B_{t \wedge S_M} - B_{S_{N \wedge M}}| \leqq \varepsilon$, proving the right hand inequality.

3. A Lower Bound for the Upcrossings of a Continuous Martingale

The inequality presented here is a complement to Theorem 5.3, and, together with (0.5), provides a two-sided bound on $U^*(M, \varepsilon)$, in the case when M is a continuous martingale.

Theorem 3.1. *Let M be a continuous martingale with $M_0 = 0$. There exist universal constants c_p, such that*

$$c_p \|M^*\|_p \leqq \varepsilon + \|\varepsilon U^*(M, \varepsilon)\|_p, \qquad 0 < p < \infty. \tag{3.1}$$

Proof. It is enough to prove this for $M = B^T$, where B is a Brownian motion and T is any stopping time: the general result then follows by time-change. Let

$Z=B^{(\varepsilon)}$ be the ε-skeleton of B, and $S_n=S_n(B)$, $n\geqq 0$, be the associated stopping times. Let $N=\inf\{n\colon S_n\geqq T\}$; we have $B_T^*\leqq B_{S_N}^*\leqq B_T^*+\varepsilon$.

Now $\langle Z\rangle_N=[Z,Z]_N=N\varepsilon^2$, and therefore, by the Burkholder-Gundy inequalities for $p\geqq 1$, and the inequality $\|X^*\|_p\leqq C_p\,\|\langle X\rangle_\infty^{\frac{1}{2}}\|_p$ for $p<1$ (see, for example [6]), it follows that for $0<p<\infty$ $\|Z_N^*\|_p\leqq C_p\,\|(\varepsilon^2 N)^{\frac{1}{2}}\|_p$.

For $a\in\varepsilon\mathbb{Z}$ let

$$U^a=\sum_{r=0}^{N-1} 1_{(Z_r=a,\ Z_{r+1}=a+\varepsilon)},\quad U^*=\sup_a U^a,\quad D^a=\sum_{r=0}^{N-1} 1_{(Z_r=a\ Z_{r+1}=a-\varepsilon)},$$

and

$$H^a=\sum_{r=0}^{N-1} 1_{(Z_r=a)}=U^a+D^a.$$

We have $\sum_a H^a=N$, and $\varepsilon(\sum_a U^a-\sum_a D^a)=Z_N$, and so

$$\tfrac{1}{2}(N\varepsilon^2+\varepsilon Z_N)=\varepsilon^2\sum_a U^a\leqq\varepsilon^2(2\cdot U^*Z_N^*/\varepsilon)=2\varepsilon U^*Z_N^*.$$

Now for $0<p<\infty$ there exist constants c_p such that for any f, g, $\|f-g\|_p\geqq c_p\|f\|_p-\|g\|_p$ (for $p\geqq 1$, $c_p=1$), and therefore, using Hölders inequality

$$\begin{aligned}&\|4\varepsilon U^*\|_p\,\|Z_N^*\|_p\\&\quad\geqq\|4\varepsilon U^*Z_N^*\|_{p/2}\geqq\|N\varepsilon^2-\varepsilon Z_N^*\|_{p/2}\geqq c_p\,\|N\varepsilon^2\|_{p/2}-\|\varepsilon Z_N^*\|_{p/2}\\&\quad\geqq c_p\,\|(N\varepsilon^2)^{\frac{1}{2}}\|_p^2-\varepsilon\,\|Z_N^*\|_p\geqq c_p\,\|Z_N^*\|_p^2-\varepsilon\,\|Z_N^*\|_p.\end{aligned}$$

Dividing by $\|Z_N^*\|_p$ we obtain the inequality $\|4\varepsilon U^*\|_p\geqq c_p\,\|Z_N^*\|_p-\varepsilon$, from which (3.1) follows immediately by applying the pathwise inequalities $U^*\leqq 1+U_T^*(B,\varepsilon)$, $B_T^*\leqq Z_N^*$.

Remarks. 1. Except for the minor difficulties caused by working with upcrossings rather than the occupation measures H^a, the proof above is essentially identical with the proof in [1] of the inequality $\|L^*(M)\|_p\geqq c_p\,\|M^*\|_p$ for a continuous martingale M with $M_0=0$.

2. It is not possible to remove the initial ε on the right hand side of (3.1): for example let $T=\inf\{s\geqq 0\colon |B_s|=\frac{1}{2}\varepsilon\}$ and $M=B^T$ - then $U^*(M,\varepsilon)=0$ while $M^*=\frac{1}{2}\varepsilon$.

3. This inequality does not extend to general discontinuous martingales - it is enough to consider the martingale $M_t=\phi 1_{(t\geqq 1)}$, where $P(\phi=1)=P(\phi=-1)=\frac{1}{2}$.

4. An Upcrossing Inequality from Local Time

Let $X=X_0+M+A$ be a continuous semimartingale, and C_t be a non-negative, right continuous adapted increasing process. Let

$$Y_t=(X_t\wedge C_t)\vee(-C_t); \tag{3.1}$$

we begin by obtaining a bound on $\|L^*(Y)\|_p$ for $p \geqq 1$. If $T_a = \inf\{t: C_t > a\}$ for $a \geqq 0$, and $T_a = \inf\{t: -C_t \leqq a\}$ for $a < 0$ then it is easily verified that

$$L_t^a(Y) = L_t^a(X) - L_{t \wedge T_a}^a(X), \tag{4.1}$$

so that the local time of Y is just the local time of X inside the envelope $\{(a,t): -C_t \leqq a < C_t\}$.

Theorem 4.1. *There exist universal constants c_p such that for $p \geqq 1$*

$$\|L^*(Y)\|_p \leqq c_p \left(\|M\|_{L^p} + \left\| \int_0^\infty |dA_s| \right\|_p + \|X^* \wedge C_\infty\|_p \right) \tag{4.1}$$

Remarks. 1. As $L^*(Y) \leqq L^*(X)$, by (1.5)

$$\|L^*(Y)\|_p \leqq c_p \left(\|M^*\|_p + \left\| \int_0^\infty |dA_s| \right\|_p \right), \quad \text{for } p \geqq 1. \tag{4.2}$$

This shows that, by restricting the region over which the supremum in $\sup_a L_\infty^a(X)$ is taken, $\|M^*\|_p$ may be replaced by the smaller term $\|M\|_{L^p}$. This is only of interest for p equal to 1, or close to 1, for otherwise, since $\|M^*\|_p \leqq \frac{p}{p-1} \|M\|_{L^p}$, the two terms are of similar size.

2. Setting $C_t \equiv x > 0$ we obtain:

$$\|\sup_{|a| \leqq x} L_\infty^a(X)\|_p \leqq c_p \left(\|M\|_{L^p} + \left\| \int_0^\infty |dA_s| \right\|_p + \|X^* \wedge x\|_p \right) \tag{4.3}$$

3. This inequality does not hold in general for $p < 1$. For, if it did, letting $x \downarrow 0$ in (4.3) we would obtain, for any continuous martingale M, the inequality $\|L_\infty^0(M)\|_p \leqq c_p \|M\|_{L^p}$, which, as was remarked in Sect. 1, is known to be false.

Proof. It is enough to prove (4.1) for X in H^1. For, if X is any semimartingale, let (T_n) be an increasing sequence of stopping times such that X^{T_n} is in H^1. Then, if (4.1) holds for each X^{T_n},

$$\begin{aligned} \|L_{T_n}^*(Y)\|_p &\leqq c \left(\|M^{T_n}\|_{L^p} + \left\| \int_0^{T_n} |dA_s| \right\|_p + \|X_{T_n}^* \wedge C_{T_n}\|_p \right) \\ &\leqq c \left(\|M\|_{L^p} + \left\| \int_0^\infty |dA_s| \right\|_p + \|X^* \wedge C_\infty\|_p \right), \end{aligned}$$

and so (4.1) holds for X also.

We may also suppose that C is continuous. For, if (4.1) does hold for continuous C, let $C_t^n = \int_{t-1/n}^t n C_s \, ds$, so that C^n is continuous, and $C^n \uparrow C$. If $Y^n = (X \wedge C^n) \vee (-C^n)$, then $L^*(Y^n) \uparrow L^*(Y)$, and (4.1) holds for C.

Let $Y = Y_0 + N + B$ be the canonical decomposition of Y. Then, since $\langle N \rangle \leqq 2 Y^* L^*(Y)$, for $p \geqq 1$

$$\|Y-Y_0\|_{H^p}=\left\|\langle N\rangle_\infty^{\frac{1}{2}}+\int_0^\infty |dB_s|\right\|_p \leqq c_p\,\|Y^*\|_p^{\frac{1}{2}}\,\|L^*(Y)\|_p^{\frac{1}{2}}+\left\|\int_0^\infty |dB_s|\right\|_p.$$

By (1.5), $\|L^*(Y)\|_p \leqq c_p\,\|Y-Y_0\|_{H^p}$, and therefore

$$\|L^*(Y)\|_p \leqq c_p\,\|Y^*\|_p^{\frac{1}{2}}\,\|L^*(Y)\|_p^{\frac{1}{2}}+C_p\left\|\int_0^\infty |dB_s|\right\|_p.$$

Now if $\lambda \leqq \alpha\lambda^{\frac{1}{2}}+\beta$ then $\lambda \leqq \alpha^2+\beta$, and so

$$\|L^*(Y)\|_p \leqq c_p\left(\|Y^*\|_p+\left\|\int_0^\infty |dB_s|\right\|_p\right). \tag{4.4}$$

By Tanaka's formula, as $Y_t=X_t-(X_t-C_t)^+ +(X_t+C_t)^-$,

$$\begin{aligned}
Y_t=Y_0&+\int_0^t dX_s-\int_0^t 1_{(X_s>C_s)}\,d(X_s-C_s)-\tfrac{1}{2}L_t^0(X-C)\\
&-\int_0^t 1_{(X_s\leqq -C_s)}\,d(X_s+C_s)+\tfrac{1}{2}L_t^0(X+C)\\
=Y_0&+\int_0^t 1_{(-C_s<X_s\leqq C_s)}\,dM_s+\int_0^t 1_{(-C_s<X_s\leqq C_s)}\,dA_s\\
&+\int_0^t (1_{(X_s>C_s)}-1_{(X_s\leqq -C_s)})\,dC_s-\tfrac{1}{2}L_t^0(X-C)+\tfrac{1}{2}L_t^0(X+C)
\end{aligned}$$

Hence

$$\int_0^\infty |dB_s| \leqq \int_0^\infty |dA_s|+C_\infty+\tfrac{1}{2}L_\infty^0(X+C)+\tfrac{1}{2}L_\infty^0(X-C).$$

By Lemma 1.2

$$\|L_\infty^0(X+C)\|_p \leqq p\left(\|M\|_{L^p}+\left\|\int_0^\infty |dA_s|+C_\infty\right\|_p\right),$$

and the same bound holds for $\|L_\infty^0(X-C)\|_p$. Therefore

$$\left\|\int_0^\infty |dB_s|\right\|_p \leqq c_p\left(\|M\|_{L^p}+\|C_\infty\|_p+\left\|\int_0^\infty |dA_s|\right\|_p\right),$$

and substituting this in (4.4) we obtain, since $Y^* \leqq C_\infty$

$$\|L^*(Y)\|_p \leqq c_p\left(\|M\|_{L^p}+\left\|\int_0^\infty |dA_s|\right\|_p+\|C_\infty\|_p\right). \tag{4.5}$$

Now if C_t is replaced by $C_t^n=C_t\wedge\left(X_t^*+\dfrac{1}{n}\right)$,

$$Y=(X\wedge C)\vee(-C)=(X\wedge C^n)\vee(-C^n),$$

and so the final term in (4.5) may be replaced by $\|C_\infty\wedge X^*\|_p+\dfrac{1}{n}$, for any $n\geqq 1$. (4.1) now follows.

The principal application of (4.1) is in the proof of the following inequality.

Proposition 4.2. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a probability space such that $\mathscr{F}$ supports a random variable independent of $\mathscr{F}_\infty$ with a continuous distribution, let B be a Brownian motion$/(\mathscr{F}_t)$, let $0=T_0 \leqq T_1 \leqq \ldots \leqq T_k$ be finite stopping times$/(\mathscr{F}_t)$, and let $X_i = B_{T_i}$, $0 \leqq i \leqq k$. Then there exists a universal constant c such that*

$$\|\varepsilon U^*(X,\varepsilon)\|_1 \leqq c(\varepsilon + \|X^*\|_1 + \|B^{T_k}\|_{L^1}) \tag{4.6}$$

Proof. Let $B^{(\varepsilon)}$ be the ε-skeleton of B, as defined in Sect. 1, let $S_n(B)$, $n \geqq 0$ be the associated stopping times, and let

$$C_n = \min\{r\varepsilon, r \in \mathbb{Z}\colon |X_i| \leqq r\varepsilon \quad \text{for all } i \text{ s.t. } T_i \leqq S_n\}.$$

Set $M = \min\{n\colon S_n \geqq T_k\}$, and let $C_n' \in \mathbb{R} \times \{0,1\}$ be defined by $C_n' = (C_n, 1_{(n<M)})$.

The first step in the proof is to obtain a bound on

$$\max_r U_M((B^{(\varepsilon)} \wedge C) \vee (-C), r\varepsilon, (r+1)\varepsilon). \tag{4.7}$$

Let R be the smallest value of r which maximises (4.7), let V be the value of this maximum, and let $N_1, \ldots, N_V$ be the initial times of the V upcrossings made by $(B^{(\varepsilon)} \wedge C) \vee (-C)$ from $R\varepsilon$ to $(R+1)\varepsilon$. Note that M, R, V, $N_1, \ldots, N_V$, are all $\mathscr{F}_\infty$ measurable.

Now applying Theorem 2.2 to B and C' we obtain a Brownian motion W on a filtration $(\mathscr{G}_t)$, with $W^{(\varepsilon)} = B^{(\varepsilon)}$. By the construction of W, the V upcrossings

$$W_{S_{N_i}(W)+t} - W_{S_{N_i}(W)}, 0 \leqq t \leqq S_{N_i+1}(W) - S_{N_i}(W), \quad i=1,\ldots,V,$$

are independent of $\mathscr{F}_\infty$. Hence $\alpha_i = L(R, S_{N_i+1}(W), W) - L(R, S_{N_i}(W), W)$, $i = 1, \ldots, V$ are also independent of $\mathscr{F}_\infty$.

Let $\tilde{C}_t = \sum_{n=0}^{\infty} C_n 1_{[S_n(W), S_{n+1}(W))}(t)$, and $Y = (W \wedge \tilde{C}) \vee (-\tilde{C})$. Writing $S = S_M(W)$, $L_S^*(Y) \geqq L_S^R(Y) \geqq \sum_{i=2}^{V} \alpha_i$, and therefore since V is $\mathscr{F}_\infty$ measurable, and $E\alpha_i = \varepsilon$, $EL_S^* \geqq \varepsilon E(V-1)$. By Theorem 4.1, and the definitions of W and C,

$$\varepsilon E(V-1) \leqq EL_S^*(Y) \leqq c(\|W^S\|_{L^1} + \|C_S \wedge W_S^*\|_1). \tag{4.8}$$

By Lemma 2.3, and the fact that $\|\cdot\|_{L^1}$ of a martingale does not depend on the filtration,

$$\|W^S\|_{L^1} \leqq \varepsilon + \|(W^{(\varepsilon)})^M\|_{L^1} = \varepsilon + \|(B^{(\varepsilon)})^M\|_{L^1} \leqq \varepsilon + \|B^{S_M(B)}\|_{L^1} \leqq 2\varepsilon + \|B^{T_k}\|_{L^1}. \tag{4.9}$$

Also,

$$\begin{aligned}\varepsilon U_k^*(X, 2\varepsilon) &\leqq \max_r \varepsilon U_k(X, r\varepsilon, (r+1)\varepsilon) \\ &\leqq \varepsilon \max_r [1 + U_M((B^{(\varepsilon)} \wedge C) \vee (-C), r\varepsilon, (r+1)\varepsilon)] \\ &= \varepsilon(1+V). \end{aligned} \tag{4.10}$$

Combining (4.10), (4.8) and (4.9), and noting that $C_S \leqq X^* + \varepsilon$, we obtain (4.6) for 2ε.

5. The Upcrossing Inequality for Semimartingales

Let $X_0, X_1, \ldots, X_k$ be any integrable discrete time process adapted to a filtration $\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_k$. We may decompose X into the sum of a martingale and a predictable process by the elementary Doob decomposition: let

$$\Delta X_r = X_r - X_{r-1}, \Delta A_r = E(\Delta X_r | \mathscr{F}_{r-1}), \Delta M_r = \Delta X_r - \Delta A_r,$$

and then, if $M_r = \sum_{i=1}^{r} \Delta M_i$, $A_r = \sum_{i=1}^{r} \Delta A_i$, $X = X_0 + M + A$ is the desired decomposition, which is evidently unique. We set

$$\|X\|_{H^p} = \left\| |X_0| + \left(\sum_{i=1}^{k} (\Delta M_i)^2\right)^{\frac{1}{2}} + \sum_{i=1}^{k} |\Delta A_i| \right\|_p. \tag{5.1}$$

Proposition 5.1. *Let $0 = X_0, X_1, \ldots, X_k$ be an integrable discrete time process. Let $\mathscr{F}_i$, $0 \leqq i \leqq k$ be the natural filtration of X, and let $X = M + A$ be the Doob decomposition of X. There exists a Brownian motion B_t on a filtration $(\mathscr{E}_t)$, and stopping times/$(\mathscr{E}_t)$ $0 = \tau_0 \leqq \tau_1 \leqq \ldots \leqq \tau_k$ such that $(X_0, \ldots, X_k)$ is equal in law to $(B_{\tau_0}, \ldots, B_{\tau_k})$. Further*

$$\|B^{\tau_k}\|_{L^1} \leqq \|M\|_{L^1} + 2 \left\| \sum_{i=1}^{k} |\Delta A_i| \right\|_1. \tag{5.2}$$

Remark. While the existence of the embedding is well known (see Monroe, [8]) the bound (5.2) appears to be new.

Proof. We modify slightly the standard Skorohod construction. (The procedure used here is less efficient than that used in [8], but does not lose very much, and is simpler to calculate with.) Let $X = M + A$ be the Doob decomposition of X, and

$$Y_{2r} = X_r \qquad 0 \leqq r \leqq k,$$
$$Y_{2r+1} = X_r + \Delta A_{r+1} \qquad 0 \leqq r \leqq k,$$
$$\mathscr{G}_{2r} = \mathscr{G}_{2r+1} = \mathscr{F}_r.$$

Note that $(\mathscr{G}_n)$ is the natural filtration of Y.

We now embed the process Y in a Brownian motion using the procedure of Monroe [8]. There exists a Brownian motion B_t, on a filtration $(\mathscr{E}_t)$, and stopping times $0 = T_0 \leqq T_1 \leqq \ldots \leqq T_{2k}$ such that

$$(Y_0, Y_1, \ldots, Y_{2k}) \sim (B_0, B_{T_1}, \ldots, B_{T_{2k}}).$$

By the construction of Y each jump is either a pure martingale jump, or one which is predictable: that is $\Delta Y_{2i+1} \in \mathscr{F}_{2i}$ $0 \leqq i \leqq k-1$, while $E(\Delta Y_{2i+2} | \mathscr{F}_{2i+1})$

$=0$, $0 \leqq i \leqq k-1$. For jumps of either of these types the stopping times T_i take particularly simple forms: we have

$$T_{2i+1}=\inf\{t \geqq T_{2i}: B_t - B_{T_{2i}} = f_i\},$$

where $f_i \in \sigma(B_{T_j}, j \leqq 2i)$ and

$$T_{2i+2}=\inf\{t \geqq T_{2i+1}: B_t - B_{T_{2i+1}} \in \{g_i, h_i\}\},$$

where g_i, h_i are finite and $\mathscr{E}_{T_{2i+1}}$ measurable, but in general depend on a random variable independent of B. (This second case is the "random 2-sided barrier" used in Skorohod's original construction.)

Let

$$V_t = \int_0^t \left(\sum_{i=0}^{k-1} 1_{(T_{2i}, T_{2i+1}]}(s) \right) dB_s,$$

$$U_t = \int_0^t \left(\sum_{i=0}^{k-1} 1_{(T_{2i+1}, T_{2i+2}]}(s) \right) dB_s,$$

Thus $B^{T_{2k}} = V^{T_{2k}} + U^{T_{2k}}$, and $\|B^{T_{2k}}\|_{L^1} \leqq \|V^{T_{2k}}\|_{L^1} + \|U^{T_{2k}}\|_{L^1}$. Also

$$(U_0, U_{T_2}, \ldots, U_{T_{2k}}) \sim (M_0, M_1, \ldots, M_k),$$

and

$$(V_0, V_{T_2}, \ldots, V_{T_{2k}}) \sim (A_0, A_1, \ldots, A_k).$$

The martingale $U^{T_{2k}}$ may be written $U^{T_{2k}} = \sum_{i=0}^{k-1} (U^{T_{2i+2}} - U^{T_{2i}})$, and, by the definition of T_{2i+2}, each of these martingales is uniformly integrable. Therefore $U^{T_{2k}}$ is also uniformly integrable, and

$$\|U^{T_{2k}}\|_{L^1} = \|U_{T_{2k}}\|_1 = \|M_k\|_1. \tag{5.3}$$

If $S=\inf\{t \geqq 0: B_t = a\}$, then it is easily verified that $\|B^S\|_{L^1} = 2|a|$. Therefore, for $0 \leqq i \leqq k-1$,

$$\|V^{T_{2i+2}} - V^{T_{2i}}\|_{L^1} = 2E|f_i| = 2E|V_{T_{2i+2}} - V_{T_{2i}}| = 2E|\Delta A_{i+1}|.$$

Hence $\|V^{T_{2k}}\|_{L^1} \leqq 2 \sum_{i=0}^{k-1} E|\Delta A_{i+1}|$, and combining this with (5.3) we obtain (5.2).

Corollary 5.2. *Let X, M, A and B be as in Theorem 5.1. Then*

$$\|B^{\tau_k}\|_{L^1} \leqq 2\|X\|_{H^1}.$$

The following Lemma is a special case of Proposition 12 of [5].

Lemma 5.3. *Let $X_0, X_1, \ldots, X_k$ be an integrable discrete time process, adapted to two filtrations $(\mathscr{F}_n)$, $(\mathscr{G}_n)$. If $\mathscr{G}_n \subset \mathscr{F}_n$ for each n, then*

$$\|X\|_{H^1(\mathscr{G}.)} \leqq 3\|X\|_{H^1(\mathscr{F}.)}.$$

Lemma 5.4. *Let X be a semimartingale on $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, and let $0 = T_0 \leqq T_1 \leqq \ldots \leqq T_k$ be stopping times. Let*

$$Y_i = X_{T_i}, \quad \mathscr{E}_i = \mathscr{F}_{T_i} \quad 0 \leqq i \leqq k.$$

There exists a universal constant c such that

$$\|Y\|_{H^1(\mathscr{E}.)} \leqq c \|X\|_{H^1(\mathscr{F}.)}. \tag{5.4}$$

Proof. If $\|X\|_{H^1} = \infty$ there is nothing to prove. Let $\|X\|_{H^1} < \infty$, and let $X = X_0 + M + A$ be the canonical decomposition of X. As $X^* \in L^1$, Y is integrable, and has a Doob decomposition (relative to $(\mathscr{E}_n)$) $Y = Y_0 + N + B$. Now

$$\begin{aligned} E \int_0^\infty |dA_s| &= \sup \left\{ E \int_0^\infty J_s dX_s,\ J \text{ previsible, } |J| = 1 \right\} \\ &\geqq \sup \left\{ E \sum_{i=1}^k J_i \Delta Y_i,\ J_i \in \mathscr{E}_{i-1}, |J| = 1 \right\} \\ &= E \sum_{i=1}^k |\Delta B_i|. \end{aligned}$$

By (1.3),

$$\|Y\|_{H^1(\mathscr{E}.)} \leqq cE \left(Y^* + \sum_{i=1}^k |\Delta B_i| \right) \leqq cE \left(X^* + \int_0^\infty |dA_s| \right) \leqq c \|X\|_{H^1(\mathscr{F}.)}.$$

Theorem 5.5. *Let X be a semimartingale. For each $p \geqq 1$ there exist universal constants c_p such that*

$$\|\varepsilon U^*(X, \varepsilon)\|_p \leqq \|X - X_0\|_{H^p}. \tag{5.5}$$

Proof. It is sufficient to prove the theorem for X with $X_0 = 0$. The proof consists of three steps: first the basic inequality

$$\|\varepsilon U^*(X, \varepsilon)\|_p \leqq c(\varepsilon + \|X\|_{H^1}) \tag{5.6}$$

is obtained, then (1.1) is used to extend (5.5) to $p \geqq 1$, and finally the ε on the right hand side of (5.5) is removed.

Let $X^{(n)}$ be the discrete time process $X_r^{(n)} = X_{r2^{-n}}$, $0 \leqq r \leqq 4^n$, and $\mathscr{F}_r^{(n)} = \mathscr{F}_{r2^{-n}}$. As X is right-continuous,

$$\|\varepsilon U^*(X, \varepsilon)\|_1 = \lim_{n \to \infty} \|\varepsilon U^*(X^{(n)}, \varepsilon)\|_1.$$

Let $k = 4^n$, and let B, and $T_0 \leqq T_1 \leqq \ldots \leqq T_k$ be the Brownian motion, and stopping times, obtained in Proposition 5.1 such that $(B_{T_0}, \ldots, B_{T_k}) \sim (X_0^{(n)}, \ldots, X_k^{(n)})$. By Proposition 4.2,

$$\|\varepsilon U^*(X^{(n)}, \varepsilon)\|_1 \leqq c(\varepsilon + \|(X^{(n)})^*\|_1 + \|B^{T_k}\|_{L^1}).$$

By Proposition 5.1, if $(\mathscr{X}_m^{(n)}, m \geqq 0)$ is the natural filtration of X,

$$\begin{aligned}\|B^{T_k}\|_{L^1} &\leqq 2\,\|X^{(n)}\|_{H^1(\mathscr{X}_m^{(n)})}\\ &\leqq c\,\|X^{(n)}\|_{H^1(\mathscr{F}_m^{(n)})} \qquad \text{by Lemma 5.3}\\ &\leqq c\,\|X\|_{H^1} \qquad \text{by Lemma 5.4.}\end{aligned}$$

Thus as $(X^{(n)})^* \leqq X^*$, and $\|X^*\|_1 \leqq c\|X\|_{H^1}$, (5.6) follows.

Now let $V_t(X) = U_{t-}^*(X, \varepsilon)$, so that V is previsible, and $V_t(X) \leqq U_t^*(X,\varepsilon) \leqq V_t(X)+\varepsilon$. Let $G(X) = [M,M]^{\frac{1}{2}} + \int_0^\infty |dA_s|$. Let S be any stopping time, $\tilde{X}_t = X_{S+t} - X_S$, $\tilde{M}_t = M_{S+t} - M_S$, $\tilde{A}_t = A_{S+t} - A_S$, $\tilde{\mathscr{F}}_t = \mathscr{F}_{S+t}$, and $Q = P|_{(S<\infty)}$. $\tilde{X}$ is a semimartingale/$(Q, \tilde{\mathscr{F}}.)$, with canonical decomposition $\tilde{X} = \tilde{M} + \tilde{A}$. We have

$$G(\tilde{X}) \leqq G(X), \qquad V_\infty(X) \leqq V_S(X) + V_\infty(\tilde{X}) + \varepsilon.$$

By (5.6), applied to $\tilde{X}$,

$$E^Q V_\infty(\tilde{X}) \leqq c E^Q(G(\tilde{X}) + \varepsilon),$$

and therefore

$$E(V_\infty(X) - V_S(X)) \leqq c E(G(X)+\varepsilon) 1_{(S<\infty)}.$$

Thus $V_t(X)$ and $G(X)+\varepsilon$ satisfy the conditions of Lemma 1.1, and so, for $p \geqq 1$, $\|V_\infty(X)\|_p \leqq cp\,\|G(X)+\varepsilon\|_p$. Hence

$$\|\varepsilon U^*(X,\varepsilon)\|_p \leqq cp\,\|G(X)+\varepsilon\|_p. \tag{5.7}$$

Now let $S = \inf\{t>0\colon U_t^*(X,\varepsilon) = 1\}$, and let $\tilde{X}$, $\tilde{M}$, $\tilde{A}$, $\tilde{\mathscr{F}}$, Q be as above. Note that

$$\|f\|_{Q,p} = (E^Q f^p)^{1/p} = \|f\, 1_{(S<\infty)}\|_p P(S<\infty)^{-1/p}.$$

Since $\varepsilon U_\infty^*(X,\varepsilon) \leqq \varepsilon U_\infty^*(\tilde{X},\varepsilon) + 1_{(S<\infty)}$, and $U_\infty^*(\tilde{X},\varepsilon) 1_{(S=\infty)} = 0$,

$$\begin{aligned}\|\varepsilon U_\infty^*(X,\varepsilon)\|_p &\leqq \|\varepsilon U_\infty^*(\tilde{X},\varepsilon)\|_p + \|\varepsilon 1_{(S<\infty)}\|_p\\ &= (P(S<\infty))^{1/p}\,\|\varepsilon U_\infty^*(\tilde{X},\varepsilon)\|_{Q,p} + (P(S<\infty))^{1/p}\\ &\leqq (P(S<\infty))^{1/p}(cp\,\|G(\tilde{X})+\varepsilon\|_{Q,p} + \varepsilon)\\ &\leqq (P(S<\infty))^{1/p}(cp\,\|G(X)\|_{Q,p} + (cp+1)\varepsilon)\\ &= cp\,\|X\|_{H^p} + (1+cp)\varepsilon(P(S<\infty))^{1/p}.\end{aligned}$$

As $\{S<\infty\} \subset \{X^* > \frac{1}{2}\varepsilon\}$, by Chebyshev's inequality

$$P(S<\infty) \leqq (\tfrac{1}{2}\varepsilon)^{-p} E(X^*)^p \leqq (\tfrac{1}{2}\varepsilon)^{-p}\,\|X\|_{H^p}^p,$$

and substituting for $P(S<\infty)$ we obtain (5.6).

Remark. Theorem 5.5 may be generalised to cover moderate convex functions F: for any semimartingale X in H^1

$$EF(\varepsilon U^*(X,\varepsilon)) \leqq c_F E\left[F([M,M]^{\frac{1}{2}} + \int_0^\infty |dA_s|)\right]. \tag{5.8}$$

The proof is only slightly more complicated than the case $F(x) = x^p$.

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