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The Central Limit Problem for, Infinite Products of, and Lévy Processes of Renewal Sequences

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Summary. Necessary and sufficient conditions are obtained for: (i) convergence of row products from a null triangular array of renewal sequences to a particular renewal sequence and (ii) convergence of an infinite product of renewal sequences to a non-trivial limit. These products correspond to intersections of regenerative phenomena of integers. Lévy processes of such regenerative phenomena are constructed.

1. Introduction

An *integer-valued renewal process* is a strictly increasing, integer-valued random walk starting at 0; the possibility that a terminal state ∞ may be reached in a finite number of steps is not excluded. Its (random) image, excluding ∞, is called a *regenerative phenomenon of integers.* To each integral-valued renewal process $(T_m\colon m=0,1,2,\dots)$ with $T_0=0$ there corresponds a *renewal sequence* $U=(u_n\colon n=0,1,2,\dots)$, where

$$u_n=P(T_m=n \text{ for some } m).$$

Products of renewal sequences correspond to intersections of independent regenerative phenomena of integers.

Let $f_n=P(T_1=n)$, $n=1,2,\dots,\infty$. Clearly, $f_\infty+\sum_{n=1}^{\infty} f_n=1$, which we also write as $\sum_{n=1}^{\infty+} f_n=1$; and any sequence $F=(f_n\colon n=1,2,\dots,\infty)$ of nonnegative numbers whose sum is one corresponds to a renewal process. Kendall [1967 and 1968] identified the infinitely divisible renewal sequences (for the product operation) and proved that those that are zero-free are the renewal sequences that can arise as limits of row products in null triangular arrays of renewal sequences.

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In Sect. 3 a criterion in terms of distributions $F=(f_n: n=1,2,\ldots,\infty)$ is given for convergence of such products to particular infinitely divisible renewal sequences. This criterion and an interesting construction of infinitely divisible regenerative phenomena of integers is what has been added in Sect. 3 to Kendall's results. In the process, new proofs of some of his results are obtained.

In Sect. 6, the construction of infinitely divisible regenerative phenomena of integers is extended to a construction of Lévy processes of regenerative phenomena of integers.

In Sect. 4, the connection between the vague topology on the set of distributions F and convergence questions for random subsets of the nonnegative integers is explored. Some of the topological considerations in Sect. 4 are important for Sects. 5 and 6.

In Sect. 5 necessary and sufficient conditions are given for an infinite product of renewal sequences to equal the trivial renewal sequence $(1,0,0,0,\ldots)$.

In Sect. 7 we remove the special role that 0 plays – as the first member of the range of the renewal process – by smoothing out regenerative phenomena of integers over all the integers to obtain what we shall call *homogenized regenerative phenomena of integers.* Kingman [1970a] uses the adjective *stationary* where I use *homogenized.* My choice is based on the consideration that I want *regenerative phenomenon* (unmodified) to denote a random set which, among other properties, has the property that it contains 0. Moreover, Kingman's stationary *regenerative phenomena* do not exactly correspond to my *homogenized regenerative phenomena;* for he allows infinite underlying measures whereas we will restict out attention to underlying probability measures.

Sections 8, 9, 10, and 11, all of which are about homogenized regenerative phenomena of integers, roughly parallel Sects. 3, 4, 5, and 6. However, some of the results in the latter sections are not obvious extensions of corresponding results in the earlier sections.

[Kendall, 1974] and [Mathéron, 1975] are good references for general theories of random sets.

2. Definitions and Preliminaries

Let $(\Omega,\mathcal{M},P)$ denote the underlying probability space. We have already introduced, for an integer-valued renewal process $(T_m: m=0,1,2,\ldots)$ where $T_0=0$, the corresponding renewal sequence U, and corresponding distribution F. Let

$$R(\omega)=\{n\neq\infty: T_m(\omega)=n \text{ for some } m\}$$

denote the corresponding regenerative phenomenon of integers, a term we sometimes abbreviate as *regenerative phenomenon,* just as we sometimes drop "integer-valued" and speak of *renewal processes.* The members of R are called *renewals;* so that F is the distribution of the waiting time between successive renewals and u_n equals the probability that n is a renewal, that is, $u_n=P(n\in R)$.

We use corresponding subscripts in the obvious manner; for example, U_2 is the renewal sequence for the renewal process $(T_{2,m}: m=0,1,2,\ldots)$ whose finite image is R_2, a random set to which 3 belongs with probability $u_{2,3}$.

The operation in which we are interested is $R_1 \cap R_2$ for independent R_1 and R_2. It is easy to prove that $R_1 \cap R_2$ is itself a regenerative phenomenon R characterized by $u_n = u_{1,n} u_{2,n}$ for all n or, more briefly, $U = U_1 U_2$.

Definition 1. A renewal sequence U and corresponding regenerative phenomenon of integers are called *infinitely divisible* if for each positive integer i there exists a renewal sequence U_i such that $U = U_i^i$.

Remark. We shall not carry over this use of *infinite divisibility* to the corresponding distribution and renewal pocess because we do not want to confuse the concept with the classical infinite divisibility of distribution functions and real-valued random variables.

Let $\mathscr{U}$ denote the space of all renewal sequences with the topology of pointwise convergence and $\mathscr{F}$ the space of distributions on $\{1, 2, \ldots, \infty\}$ with the topology of vague convergence, that is, the topology of pointwise convergence at each $n \neq \infty$. The relation between generating functions,

$$\left(\sum_{n=0}^{\infty} u_n z^n\right)\left(1-\sum_{n=1}^{\infty} f_n z^n\right)=1, \qquad |z|<1,$$

yields the following result.

Proposition 1. *The one-to-one correspondence $U \leftrightarrow F$ is a homeomorphism of $\mathscr{U}$ with $\mathscr{F}$. In particular, $\mathscr{U}$ is compact.*

According to Proposition 10 in Sect. 4 and Proposition 1 we have chosen the "right" topologies for $\mathscr{U}$ and $\mathscr{F}$.

We need names for certain renewal sequences. For positive integers d and nonnegative integers n, let ${}_d u_n = 1$ or $=0$ according as $n \equiv 0 \pmod{d}$ or not and let ${}_\infty u_n = 1$ or $=0$ according as $n=0$ or $n>0$. Clearly ${}_\infty U \cdot U = {}_\infty U$ and ${}_1 U \cdot U = U$ for all U.

Proposition 2. *The renewal sequence ${}_\infty U$ is infinitely divisible. If U is infinitely divisible, $U \neq {}_\infty U$, and U_i, $i=1,2,\ldots$, satisfy $U_i^i = U$, then, for some $d<\infty$, $U_i \to {}_d U$ as $i \to \infty$.*

Proof. (from Kendall [1967]). Clearly, $(u_{i,n}: i=1,2,\ldots)$ converges to, say, u_n^* where $u_n^* = 1$ or $=0$ according as $u_n > 0$ or $=0$. By Proposition 1, $U^* = (u_n^*: n \geqq 0)$ is a renewal sequence. Suppose $u_n^* = 1 = u_p^*$ where $p > n$. Since $(T_m^*: m = 0,1,2,\ldots)$ hits both n and p with probability one, it hits $n+p$ with probability 1 by starting over at p and it must be able to hit $p-n$ with probability one in order that it hit p after hitting n. So, $u_{p+n}^* = 1 = u_{p-n}^*$. Since the greatest common divisor, say, d, of p and n is a linear combination of p and n, $u_d^* = 1$. The reader can easily complete the proof. □

For two numbers a and b we use $a \wedge b$ to denote their minimum.

3. The Central Limit Problem

Let $(U_{j,k}: 1 \leqq j \leqq J_k,\ k=1,2,3,\ldots)$ be a *triangular array* of renewal sequences. Let

$$U_k = \prod_{j=1}^{J_k} U_{j,k}. \tag{1}$$

We are interested in studying the limiting behavior of U_k as $k \to \infty$ under certain assumptions on the triangular array.

Were the $U_{j,k}$ distribution functions on the real line, we would demand that $U_{j,k}$ approach the convolution identity as $k \to \infty$ uniformly in j. Were we to introduce the comparable condition in the present context, we would rule out the triangular array where all $U_{j,k}$ and all U_k equal, say, ${}_7U$. That would be somewhat unsatisfactory on aesthetic grounds and, in addition, we need more generality in this section so that we can use it in the proof of Theorem 5 in Sect. 5. Accordingly, we introduce the following definition.

Definition 2. Let d be a positive integer. A triangular array $(U_{j,k}: 1 \leqq j \leqq J_k,\ k=1,2,\ldots)$ is a *d-null* triangular array if $U_{j,k} \to {}_dU$ as $k \to \infty$ uniformly in j.

We shall consider d-null triangular arrays for various values of d. Thus, according to Proposition 2, at least the triangular arrays formed by roots of infinitely divisible renewal sequences other than ${}_\infty U$ come under our purview. Of course, we also apply the adjective "d-null" to, say, triangular arrays of distributons.

Proposition 3. *A triangular array $(F_{j,k}: 1 \leqq j \leqq J_k,\ k=1,2,\ldots)$ of distributions is d-null if and only if $f_{j,k,d} \to 1$ uniformly in j as $k \to \infty$.*

Proof. This result is an immediate consequence of Proposition 1. □

For d-null triangular arrays we, in Theorem 1, identify, via explicit formulas, those renewal sequences that can arise as limits of U_k, given by (1), as $k \to \infty$ and give necessary and sufficient conditions for convergence to a particular limit. In Theorem 2 we give a construction for such a limit and, as a consequence, we deduce that the class of such limits is the class of infinitely divisible renewal sequences. In our arguments we need the basic relation

$$u_p = \sum_{n=1}^{p} f_n u_{p-n}, \qquad p=1,2,\ldots. \tag{2}$$

We also need the notation

$$\mathscr{V} = \{(v_1, v_2, v_3, \ldots v_\infty): 0 \leqq v_r < \infty \ \text{ for all } r\}.$$

Typically we shall use V to denote a sequence belonging to $\mathscr{V}$ and it will be understood that v_r will denote the r^{th} coordinate of the sequence V. Let

$$\mathscr{V}^1 = \left\{ V \in \mathscr{V}: \sum_{r=1}^{\infty +} v_r < \infty \right\}.$$

Recall that "$\infty +$" indicates that the term with the subscript "∞" is one summand.

Theorem 1. *Let* $(U_{j,k}: 1 \leqq j \leqq J_k, k=1,2,\dots)$ *be a d-null triangular array of renewal sequences. In order that*

$$\prod_{j=1}^{J_k} U_{j,k}$$

converge to a renewal sequence other than ${}_{\infty}U$ *as* $k \to \infty$ *it is necessary and sufficient that there exists a* $V \in \mathscr{V}^1$ *such that*

$$\lim_{k\to\infty} \sum_{j=1}^{J_k} f_{j,k,qd} = v_{q-1}, \qquad q=2,3,\dots, \tag{3}$$

and

$$\lim_{k\to\infty} \sum_{j=1}^{J_k} (1-f_{j,k,d}) = \sum_{r=1}^{\infty+} v_r. \tag{4}$$

In the case of convergence the limiting renewal sequence U *is given by*

$$u_n = \begin{cases} \exp\left[-\sum_{r=1}^{\infty+} (q \wedge r)\, v_r\right] & \text{if } n=qd, q=0,1,2,\dots \\ 0 & \text{if } n \not\equiv 0 \pmod d. \end{cases} \tag{5}$$

In order that

$$\lim_{k\to\infty} \prod_{j=1}^{J_k} U_{j,k} = {}_{\infty}U$$

it is necessary and sufficient that

$$\lim_{k\to\infty} \sum_{j=1}^{J_k} (1-f_{j,k,d}) = \infty.$$

Remarks. By Fubini's Theorem and Fatou's Lemma,

$$\begin{aligned} \liminf_{k\to\infty} \sum_{j=1}^{J_k} (1-f_{j,k,d}) &\geqq \liminf_{k\to\infty} \sum_{q=2}^{\infty} \sum_{j=1}^{J_k} f_{j,k,qd} \\ &\geqq \sum_{q=2}^{\infty} \liminf_{k\to\infty} \sum_{j=1}^{J_k} f_{j,k,qd}; \end{aligned}$$

so, if (3) holds and $\sum(1-f_{j,k,d})$ converges as $k\to\infty$, then (4) holds for an appropriate $v_\infty \geqq 0$. Although d-nullity is assumed, no assumption about the relative magnitudes of $\sum f_{j,k,n}$ for $n \not\equiv 0 (\operatorname{mod} d)$ and $\sum f_{j,k,qd}$ for $q \geqq 2$ is made. Even so, $\sum f_{j,k,n}$ for $n \not\equiv 0$ does not enter into the theorem. The necessary and sufficient conditions in the theorem are given in terms of distributions. This is a desirable characteristic because the class of distributions is more easily characterized than is the class of renewal sequences.

Proof of Theorem 1. Part 1. All limits will be taken as $k \to \infty$ and j will be an understood index of summation and multiplication when none is indicated. Congruences are always modulo d.

Part 2. Clearly $\prod u_{j,k,n} \to 0$ if $n \not\equiv 0$. Consider a U such that $u_n = 0$ if $n \not\equiv 0$. In order that

$$\prod u_{j,k,qd} \to u_{qd}, \qquad q = 0, 1, 2, \ldots \tag{6}$$

it is necessary and sufficient that

$$\sum \log u_{j,k,qd} \to \log u_{qd}, \qquad q = 0, 1, 2, \ldots, \tag{7}$$

where $\log 0 = -\infty$. Since the triangular array is d-null, (7) and, therefore, (6) are each equivalent to

$$\sum (u_{j,k,qd} - 1) \to \log u_{qd}, \qquad q = 0, 1, 2, \ldots. \tag{8}$$

Part 3. Comparing (5) and (8) we see that we want to prove

$$\sum (u_{j,k,qd} - 1) \to -\sum_{r=1}^{\infty+} (q \wedge r)\, v_r, \qquad q = 0, 1, 2, \ldots \tag{9}$$

under the assumption that (3) and (4) hold. We proceed by induction on q. The case $q = 0$ is clear. Take $s \geqq 1$ and assume that (9) holds for $q < s$. By (2),

$$\begin{aligned}
\sum (u_{j,k,sd} - 1) &= \sum \left(\sum_{n=1}^{sd} f_{j,k,n}\, u_{j,k,sd-n} - 1 \right) \\
&= \sum (u_{j,k,(s-1)d} - 1) + \sum (f_{j,k,d} - 1)\, u_{j,k,(s-1)d} \\
&\quad + \sum \sum_{q=2}^{s} f_{j,k,qd}\, u_{j,k,(s-q)d} + \sum \sum_{\substack{n=1 \\ n \not\equiv 0}}^{sd} f_{j,k,n}\, u_{j,k,sd-n},
\end{aligned} \tag{10}$$

where, as is standard, $\sum_{q=2}^{1} (\ldots) = 0$. By d-nullity and the fact that $f_{j,k,n} \leqq 1 - f_{j,k,d}$ for $n \neq d$, the last sum is $o(\sum (1 - f_{j,k,d}))$. Since $u_{j,k,(s-q)d} \to 1$ uniformly in j,

$$\sum (f_{j,k,d} - 1)\, u_{j,k,(s-1)d} \to -\sum_{r=1}^{\infty+} v_r. \tag{11}$$

and

$$\sum \sum_{q=2}^{s} f_{j,k,qd}\, u_{j,k,(s-q)d} \to \sum_{q=2}^{s} v_{q-1}. \tag{12}$$

From (10), (11), (12), and the induction hypothesis we obtain

$$\sum (u_{j,k,sd} - 1) \to -\sum_{r=1}^{\infty+} (s \wedge r)\, v_r,$$

as desired.

Part 4. Suppose that $\sum (1 - f_{j,k,d}) \to \infty$.
Noting (8), we see that we want to prove

$$\sum (1 - u_{j,k,qd}) \to \infty, \qquad q = 1, 2, 3, \ldots. \tag{13}$$

From the basic relation (2) and the inequality

$$\sum_{\substack{n=1\\ n\neq d}}^{qd} f_{j,k,n} \leqq 1 - f_{j,k,d},$$

we obtain

$$\begin{aligned}\sum(1-u_{j,k,qd}) &= \sum(1-u_{j,k,(q-1)d}) + \sum(1-f_{j,k,d})\, u_{j,k,(q-1)d} \\ &\quad - \sum \sum_{\substack{n=1\\ n\neq d}}^{qd} f_{j,k,n} u_{j,k,qd-n} \\ &\geqq \sum(1-f_{j,k,d})\,[u_{j,k,(q-1)d} - \max\{u_{j,k,qd-n}\colon 1 \leqq n \leqq qd, n \neq d\}] \\ &\qquad + \sum(1-u_{j,k,(q-1)d}). \end{aligned} \tag{14}$$

We shall prove (13) by proving, with the aid of (14), that

$$\liminf \frac{\sum(1-u_{j,k,qd})}{\sum(1-f_{j,k,d})} \geqq 1, \qquad q=1,2,3,\ldots. \tag{15}$$

In case $q=1$, (15) follows from (14) and the fact that

$$u_{j,k,0} - \max\{u_{j,k,d-n}\colon 1 \leqq n < d\} \to 1$$

uniformly in j, a consequence of d-nullity. The induction is immediate from (14) and the fact that $u_{j,k,(q-1)d} \to 1$ uniformly in j.

Part 5. Assume that (8) holds. We want to prove that $\sum(1-f_{j,k,d}) \to \infty$ in case $u_{qd}=0$ for each $q>0$ and that there exists a sequence $V \in \mathscr{V}1$ satisfying (3), (4), and (5) otherwise.

Any subsequence of $\sum(1-f_{j,k,d})$ has a further subsequence that converges in $[0,\infty]$. If the limit of that further subsequence is ∞, then, by what we have already proven, (8) holds with $u_{qd}=0$ for each $q>0$. If the limit is finite, say w, then, since $f_{j,k,qd} \leqq 1 - f_{j,k,d}$ for $q=2,3,\ldots$, there exist finite numbers $v_{q-1}, 2 \leqq q < \infty$, and a further subsequence such that along that further subsequence $\sum f_{j,k,qd} \to v_{q-1}, q=2,3,\ldots$. Let $v_\infty = w - \sum_{r=1}^{\infty} v_r$. By the remark immediately following the theorem and what we have already proven, (5) holds. Since we are assuming that the full sequence in (8) converges in $[-\infty,0]$, we have finished the proof. □

As previously mentioned. Proposition 2 implies that Theorem 1 applies to the situation where the k^{th} row of the triangular array consists of k copies of the k^{th} root of a fixed infinitely divisible $U \neq {}_\infty U$. Accordingly, such a U satisfies (5) for some $V \in \mathscr{V}^1$ and d. Hence,

$$\log u_{(q-1)d} - \log u_{qd} = \sum_{r=q}^{\infty+} v_r, \qquad q=1,2,\ldots,$$

so

$$v_q = \log \frac{u_{(q-1)d}\, u_{(q+1)d}}{(u_{qd})^2}, \qquad q=1,2,\ldots. \tag{16}$$

Therefore, V can be recovered from U. So we (but the readeer should remember that only Theorems 1 and 2 and not any of the corollaries contain information not already obtained by Kendall [1967 and 1968]) have proven:

Corollary 1. *The relationship* (5) *establishes a one-to-one correspondence between a subset of* $\{(d, V): 1 \leqq d < \infty, V \in \mathscr{V}^1\}$ *and the set of infinitely divisible renewal sequences different from* ${}_\infty U$.

Using (16) we also obtain the following result.

Corollary 2. *If* $U \neq {}_\infty U$ *is infinitely divisible, then there exists a unique* d *such that* $u_n = 0$ *for* $n \not\equiv 0 \pmod d$ *and*

$$u_{(q-1)d}\, u_{(q+1)d} \geqq u_{qd}^2 \quad \text{for } q = 1, 2, 3, \ldots.$$

Corollaries 3 and 4, which follow Theorem 2, are improvements of Corollaries 1 and 2.

The construction we use in the forthcoming Theorem 2 is similar to one used by Takács [1956], Kendall [1968], Kingman [1970b], Mandelbrot [1972], and Shepp [1972] although they work in the context of real numbers.

Theorem 2. *Let* $V \in \mathscr{V}$. *For each* $n = 1, 2, \ldots$ *and* $r = 1, 2, \ldots, \infty$, *let*

$$B_{n,r} = \begin{cases} \{p\colon n \leqq p < n+r\} & \text{with probability } 1 - e^{-v_r} \\ \emptyset & \text{with probability } e^{-v_r}. \end{cases}$$

Do this independently for the various pairs (n, r). *Then*

$$\{0, 1, 2, \ldots\} - \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{\infty+} B_{n,r} \tag{17}$$

is a regenerative phenomenon of integers equal to ${}_\infty R$ *in case* $V \notin \mathscr{V}^1$ *and having renewal sequence* U *given by*

$$u_n = \exp\left[-\sum_{r=1}^{\infty+} (n \wedge r)\, v_r\right]$$

in case $V \in \mathscr{V}^1$.

Proof. The event that p belong to the set (17) is determined by the sets $B_{n,r}$ for $n \leqq p$. Given that p does so belong, the numbers larger than p that belong to (17) are determined by the sets $B_{n,r}$ for $n > p$. Hence, the random set (17) is indeed a regenerative phenomenon R. The probability u_p that $p \in R$ equals

$$\begin{aligned} &P\{B_{n,r} = \emptyset \text{ whenever } n \leqq p < n+r\} \\ &= \prod_{n=1}^{p} \prod_{r=p-n+1}^{\infty+} e^{-v_r} = e^{-\sum_{r=1}^{\infty+} (p \wedge r)\, v_r}. \end{aligned}$$

Corollary 3. *The subset of* $\{1, 2, 3 \ldots\} \times \mathscr{V}^1$ *mentioned in Corollary 1 is* $\{1, 2, 3 \ldots\} \times \mathscr{V}^1$. *Every limit of row products of renewal sequences from a d-null triangular array is infinitely divisible and every regenerative phenomenon constructed via Theorem 2 is infinitely divisible.*

Proof. From Theorem 2 every $(1, V)$ with $V \in \mathscr{V}^1$ corresponds to a renewal sequence other than ${}_\infty U$. Fix $V \in \mathscr{V}^1$ and let $(T_m: m=0,1,2,\ldots)$ be the corresponding random walk. Then $(dT_m: m=0,1,2,\ldots)$ is a random walk whose renewal sequence is obviously given by d and V in (5).

Let $(d, V) \in \{1,2,3\ldots\} \times \mathscr{V}^1$ and let U denote the corresponding renewal sequence. Let U_i denote the renewal sequence corresponding to $(d, i^{-1} V)$. By (5), $U = U_i^i$, so U is infinitely divisible. □

Corollary 4. *A bounded sequence $(w_0, w_1, w_2, \ldots)$ is an infinitely divisible renewal sequence if and only if for some d: $w_n = 0$ if $n \not\equiv 0 \pmod d$, $w_0 = 1$, $w_n \geqq 0$ for all n, and*

$$w_{(q-1)d}\, w_{(q+1)d} \geqq w_{qd}^2 \quad \text{for } q=1,2,3,\ldots.$$

Proof. The "only if" part follow from Corollary 2 for an infinitely divisible renewal sequence other than ${}_\infty U$ and trivially (any d) for ${}_\infty U$.

Suppose that for some d: $w_n = 0$ if $n \not\equiv 0 \pmod d$, $w_0 = 1$, $w_n \geqq 0$ for all n, and

$$w_{(q-1)d}\, w_{(q+1)d} \geqq w_{qd}^2 \quad \text{for } q=1,2,3,\ldots. \tag{18}$$

An easy induction on $q \geqq 1$ shows that $w_{qd} > 0$ if and only if $w_d > 0$. So, if $w_d = 0$ we have ${}_\infty U$ and if $w_d > 0$ we can define, as in (16),

$$v_q = \log \frac{w_{(q-1)d}\, w_{(q+1)d}}{w_{qd}^2}, \quad q=1,2,3,\ldots.$$

From (18) and the boundedness of $(w_0, w_1, w_2, \ldots)$ we see, for some $C \geqq 1$, $w_{(q-1)d}/w_{qd} \downarrow C$ as $q \to \infty$. Let $v_\infty = \log C$. Clearly $V = (v_1, v_2, \ldots, v_\infty) \in \mathscr{V}$. To see that $V \in \mathscr{V}^1$, we note that

$$\sum_{q=r}^{s} v_q = \log \frac{w_{(r-1)d}}{w_{rd}} - \log \frac{w_{sd}}{w_{(s+1)d}} \to \log \frac{w_{(r-1)d}}{w_{rd}} - v_\infty \tag{19}$$

as $s \to \infty$. By Corollary 3, (d, V) yields, via (5), an infinitely divisible renewal sequence $U \neq {}_\infty U$. That $U = (w_0, w_1, w_2, \ldots)$ follows from an easy induction proof that $w_{rd} = u_{rd}$, $r = 0, 1, \ldots$, a proof based on $w_0 = 1 = u_0$ and

$$\log w_{rd} = \log w_{(r-1)d} - \sum_{q=r}^{\infty+} v_q, \quad r \geqq 1,$$

a consequence of (19). □

Remark. A sequence $(x_0, x_1, x_2, \ldots)$ is called a *Kaluza sequence* if $x_0 = 1$, $x_q \geqq 0$ for all q, and

$$x_{q-1}\, x_{q+1} \geqq x_q^2 \quad \text{for } q \geqq 1.$$

As an illustration relevant to Theorems 1 and 2, consider the triangular array in which $J_k = k$ and

$$f_{j,k,n} = \begin{cases} \dfrac{2k}{n+1} \dbinom{n-1}{(n-1)/2} \left[(1-k^{-1})\, k^{-1}\right]^{(n+1)/2} & \text{if } n \equiv 1 \pmod 2 \\ 0 & \text{if } n \equiv 0 \pmod 2 \text{ or } n = \infty. \end{cases}$$

Here $F_{j,k}$ is known [Feller 1968, XI.3 and II.12(4)] to be the distribution of the first epoch at which the random walk whose steps are -1 with probability k^{-1} and $+1$ with probability $1-k^{-1}$ reaches $+1$; so, the corresponding regenerative phenomenon is the set of epochs at which the random walk reaches a new maximum. Since $f_{j,k,1}=1-k^{-1}$, the triangular array is 1-null and

$$\sum(1-f_{j,k,1})=1.$$

Also,

$$\sum f_{j,k,3}=(1-k^{-1})^2\to 1$$

and

$$\sum f_{j,k,n}\to 0 \qquad \text{for } n\neq 1,3.$$

Hence, the conditions (3) and (4) of Theorem 1 are satisfied with $V=(0,1,0,0,\ldots,0)$. By Theorem 2 the limiting regenerative phenomenon can be constructed as the set left uncovered by a certain random collection of sets each covering two consecutive positive integers. The positive interger 1 has probability e^{-1} of being uncovered, whereas each other positive integer has probability e^{-2} of being uncovered.

A regenerative phenomenon R is a.s. unbounded or a.s. bounded according as $f_\infty=0$ or $f_\infty>0$. Since $\sum_{n=0}^{\infty} u_n$ equals the expected cardinality of R, it equals f_∞^{-1} whether $<\infty$ or $=\infty$. These observations prove the following theorem.

Theorem 3. *An infinitely divisible regenerative phenomenon R of integers is a.s. bounded or a.s. unbounded according as $V\in\mathscr{V}$, corresponding to R via Theorem 2, satisfies*

$$\sum_{n=0}^{\infty} \exp\left[-\sum_{r=1}^{\infty+} (n\wedge r)\, v_r\right]<\infty$$

or not.

4. Random Subsets of Nonnegative Integers

Suppose the renewal process $(T_m\colon m=0,1,2,\ldots)$ is defined on the probability space $(\Omega,\mathscr{M},P)$. The mapping R from $(\Omega,\mathscr{M},P)$ to the class $\mathscr{S}$ of subsets of $\{0,1,2,\ldots\}$ induces a probability measure W on a collection of subclasses of $\mathscr{S}$. Even though they may be well-known to some readers, the forthcoming Propositions 4, 5, 6, and 7 are stated here for completeness. For sets S_1 and S_2 we use the notation $S_1\triangle S_2=(S_1-S_2)\cup(S_2-S_1)$.

Proposition 4. *The collection of classes*

$$\{S\in\mathscr{S}\colon (S\triangle S_0)\cap\{0,1,\ldots,n\}=\emptyset\}, \qquad S_0\in\mathscr{S}, 0\leqq n<\infty,$$

is a countable base for a separable topology on $\mathscr{S}$ (a topology that we henceforth assume $\mathscr{S}$ carries).

We omit the easy proof.

Proposition 5. *The function σ: on $\mathscr{S}\times\mathscr{S}$ defined by*

$$\sigma(S_1,S_2)=\sum_{n=0}^{\infty} 2^{-n}\mathbf{1}_{S_1\triangle S_2}(n),$$

where $\mathbf{1}_S$ denotes the indicator function of S, is a metric for the topological space $\mathscr{S}$.

We omit the easy proof.

Proposition 6. *Let $S_k\in\mathscr{S}$, $k=1,2,\ldots,\infty$. Then the following three statements are equivalent:*

(i) $S_k\to S_\infty$ *as* $k\to\infty$;

(ii) *for each* $n\in\{0,1,2\ldots\}$ *there exists* K *such that* $(S_k\triangle S_\infty)\cap\{0,1,\ldots,n\}=\emptyset$ *if* $k\geqq K$;

(iii) $\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty}S_i=S_\infty=\bigcup_{k=1}^{\infty}\bigcap_{i=k}^{\infty}S_i$.

We omit the easy proof.

Proposition 7. *The topological space $\mathscr{S}$ is compact.*

Proof. Sequential compactness is obvious and is sufficient to imply compactness. □

Let $\mathscr{W}$ denote the set of Borel probability measures on $\mathscr{S}$. Endow $\mathscr{W}$ with the *vague* topology, that is, the one for which $W_k\to W$ if and only if $\int_{\mathscr{S}} h\,dW_k\to\int_{\mathscr{S}} h\,dW$ for every continuous real-valued function h on $\mathscr{S}$.

Proposition 8. *The topological space $\mathscr{W}$ is compact, separable, and metrizable. Let $W_k\in\mathscr{W}$, $k=1,2,\ldots,\infty$. Then $W_k\to W_\infty$ as $k\to\infty$ if and only if*

$$\lim_{k\to\infty} W_k(\{S\in\mathscr{S}: (S\triangle S_0)\cap\{0,1,\ldots,n\}=\emptyset\})$$
$$=W_\infty(\{S\in\mathscr{S}: (S\triangle S_0)\cap\{0,1,\ldots,n\}=\emptyset\}) \tag{20}$$

for every $S_0\in\mathscr{S}$ and positive integer n.

Proof. The first assertion is an immediate consequence of [Parthasarathy, 1967, Theorems II.6.2 and II.6.4] and the fact that $\mathscr{S}$ is a compact, separable metric space.

Suppose that $W_k\to W_\infty$. Fix $S_0\in\mathscr{S}$ and a positive integer n. Let

$$h(S)=\begin{cases}1 & \text{if } (S\triangle S_0)\cap\{0,1,\ldots,n\}=\emptyset\\ 0 & \text{otherwise.}\end{cases}$$

Since h is clearly continuous, $\int h\,dW_k\to\int h\,dW_\infty$ and, thus, (20) holds.

On the other hand suppose that (20) holds for every S_0 and n. Let h be an arbitrary continuous function on $\mathscr{S}$. If I is an open interval, $h^{-1}(I)$ is an open subset of $\mathscr{S}$ and, hence, a countable union of sets of the form

$$\{S\in\mathscr{S}: (S\triangle S_0)\cap\{0,1,\ldots,n\}=\emptyset\}.$$

An easy approximation argument based on this fact yields $\int h\,dW_k\to\int h\,dW_\infty$. □

We now come to the proposition to which we alluded in the opening sentences of this section.

Proposition 9. *Let R be a regenerative phenomenon of integers. For $\mathscr{C}$ a Borel subclass of $\mathscr{S}$, $\{\omega: R(\omega)\in\mathscr{C}\}\in\mathscr{M}$, the σ-field of the underlying probability space. Moreover, W, defined by*

$$W(\mathscr{C})=P(\{\omega: R(\omega)\in\mathscr{C}\}),\, \mathscr{C} \text{ Borel,}$$

is a member of $\mathscr{W}$.

Proof. Fix $S_0\in\mathscr{S}$ and $0\leqq n<\infty$ and let

$$\mathscr{C}_0=\{S\in\mathscr{S}: (S\triangle S_0)\cap\{0,1,\dots,n\}=\emptyset\}.$$

Then

$$\{\omega: R(\omega)\in\mathscr{C}_0\}=\{\omega: T_m(\omega)\notin\{0,1,\dots,n\}-S_0 \text{ for every } m \text{ and each } p\in S_0\cap\{0,1,\dots,n\} \text{ equals } T_m(\omega) \text{ for some } m\}. \tag{21}$$

Hence, $\{\omega: R(\omega)\in\mathscr{C}_0\}\in\mathscr{M}$. Since every open subclass of $\mathscr{S}$ is the countable union of classes of the form $\mathscr{C}_0$ and the collection of classes $\mathscr{C}$ for which $\{\omega: R(\omega)\in\mathscr{C}\}\in\mathscr{M}$ is a σ-field, $\{\omega: R(\omega)\in\mathscr{C}\}\in\mathscr{M}$ for each Borel $\mathscr{C}$. The remainder of the proposition then follows. □

Consider two regenerative phenomena R_1 and R_2 possibly defined on different probability spaces and let W_1 and W_2 denote the corresponding members of $\mathscr{W}$. It is easy to see, by looking at events of the form (21), that $W_1=W_2$ if and only if $F_1=F_2$. Accordingly there is a natural injection of the space $\mathscr{F}$ of distributions, or, equivalently, of the space $\mathscr{U}$ of renewal sequences, into $\mathscr{W}$. We have already seen in Proposition 1 that $\mathscr{U}$ and $\mathscr{F}$ are naturally hemeomorphic to each other. The following proposition is the rationale for our previous choice of topology for $\mathscr{F}$.

Proposition 10. *The natural injection of $\mathscr{F}$ into $\mathscr{W}$ is a homeomorphism of $\mathscr{F}$ onto a compact subset of $\mathscr{W}$.*

Proof. Let $F_k\in\mathscr{F}$, $k=1,2,\dots,\infty$, and suppose that $F_k\to F_\infty$. Let W_k, $k=1,2,\dots,\infty$, be the corresponding members of $\mathscr{W}$ and fix $S_0\in\mathscr{S}$ and a positive integer n. Assume $0\in S_0$. Write

$$S_0\cap\{0,1,\dots,n\}=\{0,t_1,t_1+t_2,\dots,t_1+\dots+t_M\}$$

where each t_i is a positive integer. Let $s=n-(t_1+\dots+t_M)$ [$=n$ if $M=0$]. Then

$$\lim_{k\to\infty} W_k(\{S: (S\triangle S_0)\cap\{0,1,\dots,n\}=\emptyset\}) \tag{22}$$

$$=\lim_{k\to\infty}\left(\prod_{m=1}^{M} f_{k,t_m}\right)\sum_{p=s+1}^{\infty+} f_{k,p}$$

$$=\left(\prod_{m=1}^{M} f_{\infty,t_m}\right)\sum_{p=s+1}^{\infty+} f_{\infty,p}$$

$$=W_\infty(\{S: (S\triangle S_0)\cap\{0,1,\dots,n\}=\emptyset\}) \tag{23}$$

where it is understood that an empty product equals 1. If $0 \notin S_0$, then (22) and (23) have the value 0. By Proposition 8 we see that $W_k \to W_\infty$.

Now let $W \in \mathscr{W}$ and $W_k \in \mathscr{W}$, $k=1,2,\ldots,\infty$, and suppose that $W_k \to W$. Let $F_k \in \mathscr{F}$ correspond to W_k. By the compactness of $\mathscr{F}$, we know that every subsequence of $(F_k: k=1,2,\ldots)$ has a furfther subsequence that converges. By what we proved in the preceding paragraph the limit of the further sequence corresponds to W under the natural injection from $\mathscr{F}$ into $\mathscr{W}$. Hence, $F_k \to F$ where F corresponds to W.

Of course, the image in $\mathscr{W}$ of $\mathscr{F}$ under what has now been shown to be a homeomorphism inherits compactness from $\mathscr{F}$. □

From the compactness in Proposition 9 we conclude that the limit of a sequence of regenerative phenomena is a regenerative phenomenon. Here is the precise statement.

Theorem 4. *Let $(R_k: k=1,2,\ldots)$ be a sequence of regenerative phenomena of integers defined on a common probability space. Suppose, for some random subset Q of $\{0,1,2\ldots\}$, that, as $k\to\infty$, $R_k \to Q$ in probability. Then Q is a regenerative phenomenon of integers.*

5. Infinite Products of Renewal Sequences

In this section, Theorem 5 gives necessary and sufficient conditions for the intersection of an infinite sequence of independent regenerative phenomena to equal $\{0\}$, as opposed to a more interesting random set, a random set which, by Theorem 4, is necessarily a regenerative phenomenon. From Propositions 6, 10, and 1 we see that such an infinite intersection is a limit that is a regenerative phenomenon whose renewal sequence is the infinite product of the renewal sequences corresponding to the independent regenerative phenomena.

Theorem 5. *Let $(U_k: k=1,2,\ldots)$ be a sequence of renewal sequences and let ${}_\infty U = (1,0,0,\ldots)$. Then*

$$\prod_{k=1}^{\infty} U_k = {}_\infty U$$

if and only if: (i) $f_{k,\infty}=1$ *for some* k *or* (ii) $\{k: \max\{f_{k,n}: n \leqq N\} \leqq \frac{1}{2}\}$ *is an infinite set for every* $N<\infty$ *or* (iii) *for some* $d<\infty$

$$\sum_{k=1}^{\infty} (1-f_{k,d})\, \mathbf{1}_{\{i:\, f_{i,d}>\frac{1}{2}\}}(k) = \infty. \tag{24}$$

Proof. By Proposition 1, $\prod_{k=1}^{\infty} U_k$ is a renewal sequence U.

Suppose that none of (i), (ii), and (iii) hold. Choose N so that $\mathscr{K} = \{k: \max\{f_{k,n}: n \leqq N\} \leqq \frac{1}{2}\}$ is a finite set and choose

$$p = N!\prod_{k\in\mathscr{K}} \min\{n: f_{k,n}>0\}.$$

We want to show $u_p>0$. Clearly

$$\prod_{k\in\mathscr{K}} u_{k,p}>0. \tag{25}$$

For $k\notin\mathscr{K}$ there exists a unique divisor $n(k)$ of p such that $f_{k,n(k)}>\frac{1}{2}$. Since

$$u_{k,p}\geqq f_{k,n(k)}^{p/n(k)}\geqq f_{k,n(k)}^{p},$$
$$1-u_{k,p}\leqq 1-f_{k,n(k)}^{p}\leqq p[1-f_{k,n(k)}].$$

Since (iii) does not hold,

$$\sum_{k\notin\mathscr{K}}(1-u_{k,p})\leqq\sum_{k\notin\mathscr{K}}\sum_{d=1}^{N} p[1-f_{k,d}]\,\mathbf{1}_{\{i:\,f_{i,d}>\frac{1}{2}\}}(k)<\infty. \tag{26}$$

By (25) and (26), $u_p>0$ and, hence $U\neq_\infty U$.

Suppose that (i) holds, that is, that $U_k=_\infty U$ for some k. Then, clearly, $U=_\infty U$.

Suppose that (ii) holds. Fix N. For k such that $\max\{f_{k,n}: n\leqq N\}\leqq\frac{1}{2}$ we shall prove that $u_{k,n}\leqq 1-2^{-n}$ for $0<n\leqq N$ and, hence, that $u_n=0$ for $0<n\leqq N$. We then let $N\to\infty$ to obtain $U=_\infty U$. Suppose $\max\{f_{k,n}: n\leqq N\}\leqq\frac{1}{2}$. Clearly, $u_{k,1}=f_{k,1}\leqq\frac{1}{2}$. Take $n\leqq N$ and assume that $u_{k\cdot q}\leqq 1-2^{-q}$ for $q<n$. Then, by (2),

$$\begin{aligned}u_{k,n}&=f_{k,n}+\sum_{q=1}^{n-1}u_{k,q}f_{k,n-k}\\&\leqq f_{k,n}+(1-2^{-(n-1)})(1-f_{k,n})\\&=1-2^{-(n-1)}(1-f_{k,n})\leqq 1-2^{-n},\end{aligned}$$

as desired.

Suppose that (iii) holds. Fix d so that (24) holds and let

$$\mathscr{I}=\{i: f_{i,d}>\tfrac{1}{2}\}.$$

We shall show that $\prod_{k\in\mathscr{I}} U_k=_\infty U$. Let

$$\alpha=\limsup_{\substack{k\to\infty\\ k\in\mathscr{I}}}(1-f_{k,d}).$$

Continuing the supposition that (iii) holds, we also suppose that $\alpha>0$. The argument for the case where (ii) holds applies here also to yield $u_{k,n}\leqq 1-2^{-n}$ for $n<d$, $k\in\mathscr{I}$. For infinitely many $k\in\mathscr{I}$

$$\begin{aligned}u_{k,d}&=f_{k,d}+\sum_{n=1}^{d-1}u_{k,n}f_{k,d-n}\\&\leqq f_{k,d}+(1-2^{(d-1)})(1-f_{k,d})\\&=1-2^{-(d-1)}(1-f_{k,d})\leqq 1-\alpha 2^{-d}.\end{aligned}$$

Now an induction argument similar to the one used for the case where (ii) holds shows, for any $k\in\mathscr{I}$ satisfying $u_{k,d}\leqq 1-\alpha 2^{-d}$, that $u_{k,n}\leqq 1-\alpha 2^{-n}$ for $n>d$. Since our inequalities hold for infinitely many k, $\prod_{k\in\mathscr{I}} U_k=_\infty U$ as desired.

Finally we consider the case $\alpha=0$ as we continue the supposition that (iii) holds. Write $\mathscr{I}=\{g(1), g(2), \ldots\}$ where $g(1)<g(2)<\ldots$. Let $J_1, J_2, \ldots$ be positive integers and let

$$F_{j,k}=F_{g(J_1+\ldots+J_{k-1}+j)}, \quad 1 \leqq j \leqq J_k, 1 \leqq k.$$

Since $\alpha=0$, the triangular array $(F_{j,k}: 1 \leqq j \leqq J_k, k=1,2,\ldots)$ is d-null (compare Proposition 3). By compactness, a subsequence of

$$\prod_{j=1}^{J_k} U_{j,k}$$

converges. If $U \neq {}_\infty U$, then the limit of this subsequence does not equal ${}_\infty U$. By Theorem 1,

$$\sum_{j=1}^{J_k} (1-f_{j,k,d})$$

converges as k approaches ∞ through the subsequence. Thus, we can force a contradiction by initially choosing $J_1, J_2, \ldots$ so that

$$\lim_{k \to \infty} \sum_{j=1}^{J_k} (1-f_{j,k,d})=\infty. \quad \square$$

6. Lévy Processes of Regenerative Phenomena

Let U be an infinitely divisible renewal sequence other than ${}_\infty U$. As we have seen there is a unique corresponding $V \in \mathscr{V}^1$ and a unique corresponding $d \in \{1,2,3,\ldots\}$. For $t \in [0,\infty)$, not necessarily an integer, let $U(t)$ denote the renewal sequence determined by $V(t)=tV$. In particular, $U(1)=U$ and $V(1)=V$. It is clear that, for $t_2>t_1$, $U(t_2)=U(t_1)\,U(t_2-t_1)$. The next theorem shows how to extend the construction of a single regenerative phenomenon in Theorem 2 to a natural construction, on one probability space, of regenerative phenomena $R(t)$ corresponding to the renewal sequences $U(t)$.

Theorem 6, Part I. *Let $V \in \mathscr{V}^1$ and let*

$$\{\Lambda_{n,r}: n=1,2,\ldots, r=1,2,\ldots,\infty\}$$

be an independant family of random variables such that the distribution function of $\Lambda_{n,r}$ is $\lambda \mapsto 1-e^{-\lambda v_r}$, $0 \leqq \lambda < \infty$, with the convention that $\Lambda_{n,r}=\infty$ a.s. if $v_r=0$. For each (n,r), let

$$B_{n,r}(t)=\begin{cases} \emptyset & \text{if } t<\Lambda_{n,r} \\ \{p: n \leqq p < n+r\} & \text{if } t \geqq \Lambda_{n,r}. \end{cases}$$

Then

$$R(t)=\{0,1,2,\ldots\}-\bigcup_{n=1}^{\infty} \bigcup_{r=1}^{\infty+} B_{n,r}(t), \qquad t \in [0,\infty),$$

is a stochastic process of regenerative phenomena of integers with the following properties:

(i) *for each* $t, R(t)$ *corresponds to the renewal sequence* $U(t)$ *determined by*

$$u_n(t)=\exp\left[-t\sum_{r=1}^{\infty+}(n\wedge r)\,v_r\right];$$

(ii) *a.s.* $R(t)\uparrow R(0)=\{0,1,2\ldots\}$ *as* $t\downarrow 0$;
(iii) *a.s.* $R(t)\downarrow\{0\}$ *as* $t\to\infty$ *provided* $V\neq(0,0,\ldots 0)$;
(iv) *a.s.* R *is right continuous and has left limits on* $[0,\infty)$;
(v) *at each* t, R *is a.s. continuous.*

Proof. Theorem 2 implies that $R(t)$, for each t, is a regenerative phenomenon and that (i) holds. By construction, $t\mapsto R(t)$ is monotone for every point in the sample space. Hence, by Proposition 6, R has left and right limits on $[0,\infty)$ and $R(t)$ converges as $t\to\infty$. Clearly, $U(t)\to_1 U$ as $t\downarrow 0$ and, in case $V\neq(0,0,\ldots,0)$, $U(t)\to_\infty U$ as $t\to\infty$. That (ii) and (iii) hold follows from Propositions 1 and 10 and the fact that we already know that $R(t)$ converges as $t\downarrow 0$ and as $t\to\infty$. Similarly, the continuity of U and existence of one-sided limits of R yields (v).

Since $V\in\mathscr{V}^1$, it is almost surely true that, for each positive λ and each positive integer n, $\Lambda_{p,r}\leqq\lambda$ for only finitely many pairs (p,r), $p\leqq n$. Fix a point in the probability space for which this is the case and fix n. Then fix t_0 and choose $t_1>t_0$ such that $\Lambda_{p,r}\notin(t_0,t_1]$ for every (p,r) for which $p\leqq n$. By Proposition 5 the distance between $R(t)$ and $R(t_0)$ is less than or equal to 2^{-n} for $t\in[t_0,t_1]$. Hence, $t\mapsto R(t)$ is right continuous. □

We could prove the strong Markov property and quasi-left-continuity in the context of Part I of Theorem 6; but we also want stationary, independent increments in analogy with real-valued processes formed out of classical infinitely divisible distributions. There is a problem: if, for instance, one knows both $R(2)$ and $R(3)$, there is no unique natural set to choose as the one which intersected with $R(2)$ gives $R(3)$. Accordingly, we shall begin with a construction that includes increments in a natural manner and which, as a consequence, gives rise to a richer family of σ-fields than is suggested by Part I of Theorem 6. It is in this more elaborate context that we shall prove the strong Markov property and quasi-left-continuity.

For $n=1,2,\ldots,r=1,2,\ldots,\infty$, let $\Omega_{n,r}$ be the class of subsets of $(0,\infty)$ having no accumulation points in $[0,\infty)$. Let $\mathscr{G}_{n,r}$ denote the Borel field generated by events of the form $\{\omega_{n,r}\in\Omega_{n,r}\colon \operatorname{card}(\omega_{n,r}\cap[0,t])=z\}$, $0\leqq t<\infty$, $z=0,1,2,\ldots$. Let $P_{n,r}$ be the probability measure induced on $\mathscr{G}_{n,r}$ by the condition: the smallest member of $\omega_{n,r}$ and the difference between successive members of $\omega_{n,r}$ are to be independent, exponentially distributed random variables with mean $1/v_r$ (If $v_r=0$, $P_{n,r}(\{\emptyset\})=1$, where $\emptyset$ is to be viewed as a member of $\Omega_{n,r}$ and thus $\{\emptyset\}\subset\Omega_{n,r}$.)

As the next step in constructing all the apparatus connected with a Hunt process we form the probability space

$$\Omega=\mathscr{S}\times\left(\mathop{\times}_{n=1}^{\infty}\mathop{\times}_{r=1}^{\infty+}\Omega_{n,r}\right).$$

We adopt a standard notation for a member ω of Ω:

$$\omega=(S;\ \omega_{n,r}\colon n=1,2,\dots,r=1,2,\dots,\infty), \tag{27}$$

where $S\in\mathscr{S}$ and $\omega_{n,r}\in\Omega_{n,r}$ for each (n,r). We introduce a probability measure for each initial state in $\mathscr{S}$:

$$P^S=\mathscr{E}_S\times\left(\mathop{\times}_{n=1}^{\infty}\mathop{\times}_{r=1}^{\infty+}P_{n,r}\right),\qquad S\in\mathscr{S},$$

where $\mathscr{E}_S$ denotes the probability measure supported by $\{S\}$. Let $\mathscr{G}_0$ denote the Borel σ-field of subsets of $\mathscr{S}$ and let $\mathscr{M}$ be the completion, with respect to the family $\{P^S\colon S\in\mathscr{S}\}$, of

$$\mathscr{G}_0\times\left(\mathop{\times}_{n=1}^{\infty}\mathop{\times}_{r=1}^{\infty+}\mathscr{G}_{n,r}\right).$$

For $0\leqq t\leqq\infty$, let $\mathscr{M}_t$ be the completion, with respect to the family $\{P^S\colon S\in\mathscr{S}\}$, of the σ-field generated by events of the form

$$\{\omega\colon S\in\mathscr{C},\ \operatorname{card}(\omega_{n,r}\cap[0,s])=z_{n,r}\ \text{ for }\ n=1,2,\dots,r=1,2,\dots,\infty\},$$

where $s\leqq t$, $\mathscr{C}\in\mathscr{G}_0$, and each $z_{n,r}$ is a nonnegative integer. In particular, $\mathscr{M}_\infty=\mathscr{M}$.

In order to obtain the full apparatus needed for a Hunt process we still need to define the main ingredient which is $R(t,\omega)$ for $(t,\omega)\in[0,\infty]\times\Omega$ and shift operators θ_t. Since we also want to discuss increments into the future, we actually need to define a whole family of incremental processes – one for each starting time. For ω given by (27), $t_0\in[0,\infty)$ and $t\in[t_0,\infty]$, let

$$B_{n,r}(t,\omega;\ t_0)=\begin{cases}\emptyset & \text{if } (t_0,t]\cap\omega_{n,r}=\emptyset\\ \{p\colon n\leqq p<n+r\} & \text{otherwise,}\end{cases}$$

$$R(t,\omega;\ t_0)=\{0,1,2,\dots\}-\bigcup_{n=1}^{\infty}\bigcup_{r=1}^{\infty+}B_{n,r}(t,\omega;\ t_0),$$

$$R(t,\omega)=S\cap R(t,\omega;\ 0),$$

and

$$\theta_t(\omega)=(R(t,\omega);\ \theta_t(\omega_{n,r})\colon n=1,2,\dots,r=1,2,\dots,\infty)$$

where $\theta_t(\omega_{n,r})=\{t_1: t+t_1\in\omega_{n,r}\}$.

Theorem 6, Part II. *Let Ω, $\mathscr{M}$, $\mathscr{M}_t$ for $0\leqq t\leqq\infty$, P^S for $S\in\mathscr{S}$, $\mathscr{G}_0$, $R(t,\omega;\ t_0)$ for $(t,\omega;\ t_0)\in[0,\infty]\times\Omega\times[0,\infty)$ with $t\geqq t_0$, and $R(t,\omega)$ for $(t,\omega)\in[0,\infty]\times\Omega$ be as* *defined in the preceding three paragraphs. Then, in the terminology of Blumenthal and Getoor* [1968, I.9],

$$(\Omega, \mathscr{M}, \mathscr{M}_t, R(t,\cdot), \theta_t, P^S)$$

is a Hunt process with state space $(\mathscr{S}, \mathscr{G}_0)$. *Under* $P^{\{0,1,2,\ldots\}}$, *each* $R(t,\cdot)$ *is, for* $0 \leqq t < \infty$, *a regenerative phenomenon of integers which can be identified with* $R(t)$ *of Part I of Theorem 6 via the definition*

$$\Lambda_{n,r}(\omega) = \min\{s \colon s \in \omega_{n,r}\}.$$

Under any P^S, *each* $R(t,\cdot;\, t_0)$ *is a regenerative phenomenon of integers corresponding to* $U(t-t_0)$. *For* $t_0 < t_1 < t_2 < \ldots < t_M$:

$$R(t_M, \omega;\, t_0) = \bigcap_{i=1}^{M} R(t_i, \omega;\, t_{i-1}), \qquad \omega \in \Omega,$$

$$R(t_M, \omega) = S \cap R(t_0, \omega;\, 0) \cap \bigcap_{i=1}^{M} R(t_i, \omega;\, t_{i-1}), \qquad \omega \in \Omega,$$

and the family

$$\{R(t_i, \cdot\,;\, t_{i-1}) \colon 1 \leqq i \leqq M\}$$

of regenerative phenomena of integers is an independent family under any P^S.

Proof. We shall prove three of the properties that a process must have in order to be a Hunt process - namely, that $\mathscr{M}_t = \mathscr{M}_{t+}$, the strong Markov property, and the quasi-left-continuity. All other features of the theorem either are immediate consequences of the definitions and the "lack of memory" property of the exponential distributions used in defining the measures $P_{n,r}$ or are contained in Part I of Theorem 6.

Each transition operator for the process $(t,\omega) \mapsto R(t,\omega)$ (obviously Markov with respect to the σ-fields $\mathscr{M}_t$) takes continuous functions to continuous functions since if S_1 and S_2 are two starting states for which $(S_1 \triangle S_2) \cap \{0,1,\ldots,n\} = \emptyset$, then

$$\begin{aligned}&[R(t,(S_1;\, \omega_{n,r}\colon n=1,2,\ldots, r=1,2,\ldots,\infty))\\ &\triangle R(t,(S_2;\, \omega_{n,r}\colon n=1,2,\ldots, r=1,2,\ldots,\infty))] \cap \{0,1,\ldots,n\} \equiv \emptyset.\end{aligned}$$

Hence [Blumemthal and Getoor, 1968, I(8.11) and subsequent remarks], $(\Omega, \mathscr{M}, \mathscr{M}_{t+}, R(t,\cdot), \theta_t, P^S)$ is a strong Markov process. Since, also, $R(\cdot,\omega)$ is right continuous a.s., Blumenthal and Getoor's [1968, pp. 49-50] argument applies to give quasi-left-continuity.

We cannot conclude immediately that $\mathscr{M}_t = \mathscr{M}_{t+}$ since the σ-fields $\mathscr{M}_t$ are not the completions of the σ-fields determined by the process $(t,\omega) \mapsto R(t,\omega)$. We get around this difficulty by constructing another strong Markov process Y such that the completions of the σ-fields determined by Y are $\mathscr{M}_t$. Then [Blumenthal and Getoor, 1968, I(8.12)], it will follow that $\mathscr{M}_t = \mathscr{M}_{t+}$. As a state space we choose a countable cartesian product of copies of $\{0,1\}$:

$$\mathop{\times}_{n=1}^{\infty} \mathop{\times}_{r=1}^{\infty+} \{0,1\}.$$

It is metrizable with, say, the distance between $x=(x_{n,r})$ and $y=(y_{n,r})$ being given by

$$\sum_{n=1}^{\infty}\sum_{r=1}^{\infty} 2^{-n-r}|x_{n,r}-y_{n,r}|+\sum_{n=1}^{\infty} 2^{-n}|x_{n,\infty}-y_{n,\infty}|.$$

Let

$$\beta\colon \{0,1,2,\ldots\}\times\{0,1,2,\ldots,\infty\}\to\{0,1,2,\ldots\}$$

be a bijection. Define the component $Y_{n,r}(t,\omega)$ of $Y(t,\omega)$ by

$$\begin{aligned}&Y_{n,r}(t,(S,\omega_{n,r}\colon n=1,2,\ldots,r=1,2,\ldots,\infty))\\ &=\begin{cases}0 & \text{if } \mathbf{1}_S(\beta(n,r))+\operatorname{card}(\omega_{n,r}\cap[0,t]) \text{ is even}\\ 1 & \text{otherwise.}\end{cases}\end{aligned}$$

It is easy to see that each transition operator for the process $(t,\omega)\mapsto Y(t,\omega)$ takes continuous functions to continuous functions; so [Blumenthal and Getoor, 1968, I(8.11) and subsequent remarks] $(\Omega,\mathcal{M},\mathcal{M}_{t+},Y(t,\cdot),\theta_t,P^x)$, where P^x, for $x=(x_{n,r})$, equals

$$P^{\{\beta(n,r)\colon x_{n,r}=1\}},$$

is strong Markov. $\square$

Remark. Had we not been able to prove $\mathcal{M}_t=\mathcal{M}_{t+}$ we would have obtained a Hunt process by replacing $\mathcal{M}_t$ by $\mathcal{M}_{t+}$.

On the basis of Theorem 6 (both parts) we are certainly entitled to use the term *Lévy process of regenerative phenomena* for the process $(t,\omega)\mapsto R(t,\omega)$.

The Lévy process $R(t)$ determined by $v_r=1/r^2$, $r=1,2,\ldots$, and $v_\infty=0$ is an interesting example. By Theorem 3, $R(t)$ is unbounded for $t\leqq 1$ and bounded for $t>1$. Instead, use

$$v_r=\frac{1}{r^2}+\frac{2}{r^2\log r},\qquad r=2,3,\ldots,$$

to obtain an $R(t)$ that is unbounded for $t<1$ and bounded for $t\geqq 1$. Of course, neither of these examples contradicts the fact that, with probability one, R is continuous at 1.

One can imagine a number of interesting questions about the sample functions of Lévy processes of regenerative phenomena.

7. Homogenized Regenerative Phenomena

Definition 3. A *homogenized regenerative phenomenon of integers* is a random subset R^H of the set of all integers with the properties: (i) $P(n\in R^H)$ does not depend on n; (ii) conditioned on $n\in R^H$, events of the form $\{p_i\in R^H\colon i=1,2,\ldots,\alpha\}$ where each $p_i>n$ are independent of events of the form $\{q_j\in R^H\colon j=1,2,\ldots,\beta\}$ where each $q_j<n$; and (iii) $P(n+p\in R^H|n\in R^H)$, $p\geqq 0$, does not depend on n.

The regenerative phenomena dicussed in the preceding parts of this paper have properties (ii) and (iii) of Definition 3. Of course they fail to have property (i) and not just because negative integers were not considerd. The number 0 played, in Sects. 2 through 5, a special role: $P(0\in R)=1$ for all (non-homogenized) regenerative phenomena. This special role of 0 disappears for a homogenized regenerative phenomenon, but, nevertheless, there is, unless the homogenized regenerative phenomenon is the empty set with probability one, a corresponding renewal sequence U: $u_p=P(n+p\in R^H|n\in R^H)$, $p\geqq 0$. The question arises: to which U's and F's does there correspond a homogenized regenerative phenomenon? Here is the widely known answer.

Lemma 1. *The distributions F that correspond to homogenized regenerative phenomena R^H of integers are exactly those with finite expectation, that is, those for which $f_\infty=0$ and $\sum_{n=1}^{\infty} nf_n<\infty$. For such an F,*

$$P(n\in R^H)=P(R^H\neq\emptyset)\left(\sum_{n=1}^{\infty} nf_n\right)^{-1}. \tag{28}$$

Proof. Following Feller [1968, XV.2(k)] we introduce the transition matrix

$$\begin{bmatrix} f_1 & f_2 & f_3 & f_4 & \dots & f_\infty \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{29}$$

for a state space $\{E_0, E_1, E_2, \dots, E_\infty\}$.

Suppose that F has finite expectation. Then the expected time to return to E_0, starting at E_0, is $\sum_{n=1}^{\infty} nf_n<\infty$. Hence, there is an invariant probability measure which assigns the reciprocal of this expectation to the state E_0. Use it to obtain a stationary Markov chain. Let R^H equal the (random) set of epochs at which this chain is in the state E_0. Clearly, R^H is a homogenized regenerative phenomenon for which $P(p+n\in R^H|n\in R^H)=u_p$ for $p\geqq 0$. (Of course, $U=(u_0, u_1, u_2, \dots)$ is the renewal sequence corresponding to F.)

On the other hand, suppose that R^H is a homogenized regenerative phenomenon. Define a process with state space $\{E_0, E_1, E_2, \dots, E_\infty\}$ to be in state E_m at epoch n if $j+n\notin R^H$ for $0\leqq j<m$ and, in case $m<\infty$, $m+n\in R^H$. Exclude the trivial case $P(R^H=\emptyset)=1$ for which there is no corresponding $F\in\mathscr{F}$ or $U\in\mathscr{U}$. The process we have defined is a stationary Markov chain with a transition matrix of the form (29). The probability that, at any particular epoch, the chain is in a state that communicates (in both directions) with E_0 is $P(R^H\neq\emptyset)$. The expected time to return to E_0 starting at E_0 is $\sum_{n=1}^{\infty} nf_n$ if $f_\infty=0$ and $+\infty$ if $f_\infty>0$. Since the Markov chain is stationary, this expected time is finite. Since

R^H equals the set of epochs at which the chain is in the state E_0, (28) follows from well-known facts about Markov chains. □

One renewal sequence corresponds to many stochastically distinct homogenized regenerative phenomena. The renewal sequence determines $P(n+p\in R^H \mid n\in R^H)$ but not $P(n\in R^H)$. The reason for this is that with positive probability R^H may equal $\emptyset$.

For any $F\in\mathscr{F}$ and corresponding $U\in\mathscr{U}$ we set $\rho(F)=\rho(U)$ equal to the reciprocal of the expectation of F:

$$\rho(F)=\rho(U)=\begin{cases} 0 & \text{if } f_\infty>0 \text{ or } \sum_{n=1}^{\infty} nf_n=\infty. \\ \left(\sum_{n=1}^{\infty} nf_n\right)^{-1} & \text{otherwise.}\end{cases} \tag{30}$$

Let ${}_\Delta U$ be a symbol that should be considered as an ideal renewal sequence corresponding to an ideal distribution ${}_\Delta F$: the meanings of both ${}_\Delta U$ and ${}_\Delta F$ will be nurtured by subsequent developments.

Theorem 7. *Let ρ be defined by* (30) *and*

$$\mathscr{F}^H=\{(0,{}_\Delta F)\}\cup\{(\gamma,F)\colon 0<\gamma\leqq 1, F\in\mathscr{F}, \rho(F)>0\}.$$

For any homogenized regenerative phenomenon R^H of integers there exists a unique $(\gamma,F)\in\mathscr{F}^H$ such that: (i) $P(R^H=\emptyset)=1-\gamma$; (ii) $P(n\in R^H)=\gamma\rho(F)$; *and* (iii) *if* $\gamma>0$, $P(n+p\in R^H \mid n\in R^H)=u_{|p|}$. *Conversely, corresponding to any $(\gamma,F)\in\mathscr{F}^H$ there is a homogenized regenerative phenomenon R^H, unique up to stochastic equivalence, such that* (i), (ii), *and* (iii) *hold.*

Proof. The theorem is an immediate consequence of Lemma 1, Definition 3, and, to obtain $|p|$ in (iii), the basic formula for conditional probability: (ii) of Definition 3 is needed to establish the uniqueness up to stochastic equivalence. □

Let

$$\mathscr{U}^H=\{(0,{}_\Delta U)\}\cup\{(\gamma,U)\colon 0<\gamma\leqq 1, U\in\mathscr{U}, \rho(U)>0\}.$$

There is an obvious natural one-to-one correspondence between $\mathscr{F}^H$ and $\mathscr{U}^H$.

For $F\in\mathscr{F}$ and corresponding $U\in\mathscr{U}$, $U\neq{}_\infty U$, we set, using GCD for greatest common divisor,

$$\delta(F)=\delta(U)=\mathrm{GCD}\{n\colon u_n>0\}, \tag{31}$$

which we know, by basic renewal theory, to equal

$$\mathrm{GCD}\{n<\infty\colon f_n>0\}.$$

It is also well-known [Feller, 1968, XV] that

$$\lim_{q\to\infty} u_{q\delta(U)}=\rho(U)\,\delta(U). \tag{32}$$

Let R_1^H and R_2^H be two independent homogenized regenerative phenomena corresponding to (γ_1,U_1) and (γ_2,U_2). Let (γ,U) correspond to $R_1^H\cap R_2^H$. From

(ii) of Theorem 7 we see that $\gamma=0$ if either $\gamma_1=0$ or $\gamma_2=0$. Suppose $\gamma_1>0$ and $\gamma_2>0$ so that U_1 and U_2 are members of $\mathscr{U}$. From (ii) and (iii), respectively, of Theorem 7 we obtain

$$\gamma\rho(U)=\gamma_1\gamma_2\rho(U_1)\rho(U_2) \tag{33}$$

and $U=U_1U_2$. Clearly, $\delta(U)$ equals the least common multiple of $\delta(U_1)$ and $\delta(U_2)$. From (32) and (33), we obtain

$$\rho(U)\,\delta(U)=\rho(U_1)\,\rho(U_2)\,\delta(U_1)\,\delta(U_2) \tag{34}$$

and then $\gamma=\gamma_1\gamma_2/\mathrm{GCD}\{\delta(U_1),\delta(U_2)\}$. We define $(\gamma_1, U_1)(\gamma_2, U_2)$ to equal (γ, U). Thus,

$$(\gamma_1, U_1)(\gamma_2, U_2)=\begin{cases}(0, {}_{\Delta}U) & \text{if } \gamma_1\gamma_2=0\\ \left(\dfrac{\gamma_1\gamma_2}{\mathrm{GCD}\{\delta(U_1),\delta(U_2)\}}, U_1U_2\right) & \text{if } \gamma_1\gamma_2>0.\end{cases} \tag{35}$$

Also of some interest is the formula

$$\rho(U)=\rho(U_1)\,\rho(U_2)\,\mathrm{GCD}\{\delta(U_1),\delta(U_2)\} \tag{36}$$

which easily follows from (34).

Definition 4. A member (γ, U) of $\mathscr{U}^H$ and corresponding homogenized regenerative phenomenon of integers are called *infinitely divisible* if for each positive integer i there exists a $(\gamma_i, U_i)\in\mathscr{U}^H$ such that $(\gamma, U)=(\gamma_i, U_i)^i$.

Of the renewal sequences ${}_1U, {}_2U, \ldots {}_{\infty}U$ that played a central role for regenerarive phenomena only ${}_1U$ will now play such a role. Consider, for instance, $(\gamma, {}_2U)$ and suppose $(\gamma, {}_2U)=(\gamma_3, U_3)^3$ for some γ_3 and U_3. Since ${}_2U=U_3^3$, $\delta(U_3)=2$. From (35), we obtain $\gamma=\gamma_3^3/4$, which, since $\gamma_3\leqq 1$, implies $\gamma\leqq 1/4$. So if $\gamma>1/4$, $(\gamma, {}_2U)$ has no cube root. Similarly, $(\gamma, {}_dU)$ is not infinitely divisible in $\mathscr{U}^H$ for any $d>1$ and $\gamma>0$.

By examining Theorem 7 we are led to specifying a topology on $\mathscr{F}^H$ by specifying the following necessary and sufficient conditions for convergence of sequences: $(\gamma_k, F_k)\to(\gamma, F)$ as $k\to\infty$ if and only if

$$\begin{Bmatrix}\gamma_k\rho(F_k) & \text{if} & \gamma_k>0\\ 0 & \text{if} & \gamma_k=0\end{Bmatrix}\to\begin{Bmatrix}\gamma\rho(F) & \text{if} & \gamma>0\\ 0 & \text{if} & \gamma=0\end{Bmatrix} \tag{37}$$

and, in case $\gamma>0$, $f_{k,n}\to f_n$ for each n as $k\to\infty$. (If $\gamma>0$ and (37) holds, then $\gamma_k>0$ for large k so that $f_{k,n}$ is defined for large k.) We make $\mathscr{U}^H$ homeomorphic to $\mathscr{F}^H$: $(\gamma_k, U_k)\to(\gamma, U)$ if and only if (compare (30))

$$\begin{Bmatrix}\gamma_k\rho(U_k) & \text{if} & \gamma_k>0\\ 0 & \text{if} & \gamma_k=0\end{Bmatrix}\to\begin{Bmatrix}\gamma\rho(U) & \text{if} & \gamma>0\\ 0 & \text{if} & \gamma=0\end{Bmatrix} \tag{38}$$

and, in case $\gamma>0$, $u_{k,n}\to u_n$ for each n.

The fact that $\{(\gamma, F)\in\mathscr{F}^H\colon \gamma=1\}$ is *not* a closed subset of $\mathscr{F}^H$ is the reason we have chosen not to exclude, via an altered definition, the case $\gamma<1$. The following two examples illuminate this consideration.

Let $f_{k,k}=1-f_{k,1}=1/k$ and $\gamma_k=1$. Then $(\gamma_k, F_k)\to(0.5, {}_1F)$ where ${}_1F=(1,0,0,\ldots)$. Consider a fixed, finite set S of integers and homogenized regenerative phenomena R_k^H corresponding to F_k. For large k, the probability is close to 0.5 that $R_k^H \cap S=\emptyset$ and close to 0.5 that $R_k^H \cap S=S$.

Now let $f_{k,k}=1-f_{k,1}=1/\sqrt{k}$ and $\gamma_k=1$. Then $(\gamma_k, F_k)\to(0, {}_\Delta F)$. For large k, the probability is close to 1 that $R_k^H \cap S=\emptyset$.

Proposition 11. *The topological spaces $\mathscr{F}^H$ and $\mathscr{U}^H$ are compact and homeomorphic to each other under the natural one-to-one correspondence between them.*

Proof. That $\mathscr{F}^H$ and $\mathscr{U}^H$ are homeomorphic follows from the definitions and the homeomorphism between $\mathscr{F}$ and $\mathscr{U}$ (compare Proposition 1.). It remains to prove compactness.

An arbitrary sequence in $\mathscr{F}^H$ has a subsequence $((\gamma_k, F_k)\colon k=1,2,\ldots)$ such that, for some $C\in[0,1]$ and $F\in\mathscr{F}$,

$$\left.\begin{cases}\gamma_k\,\rho(F_k) & \text{if } \gamma_k>0\\ 0 & \text{if } \gamma_k=0\end{cases}\right\}\to C$$

and, in case $\gamma_k>0$ for all large k, $f_{k,n}\to f_n$ for each $n<\infty$ $\left(\text{where at this point we only know } \sum_{n=1}^{\infty} f_n\leqq 1\right)$. If $C=0$, then $(\gamma_k, F_k)\to(0, {}_\Delta F)$. Suppose $C>0$. Then $\rho(F_k)\geqq C/2$ for all large k and, hence, for such k,

$$\sum_{n=p}^{\infty} f_{k,n}\leqq p^{-1}\sum_{n=1}^{\infty} n f_{k,n}\leqq 2(p\,C)^{-1}\to 0$$

as $p\to\infty$ uniformly in k. Therefore $\sum_{n=1}^{\infty} f_n=1$. Let $\gamma=C/\rho(F)$ which, by the following argument based on Fatous's Lemma, is no larger than 1;

$$\begin{aligned} C/\rho(F) &= \lim_{k\to\infty} \gamma_k\,\rho(F_k)/\rho(F)\\ &\leqq [\limsup_{k\to\infty} \gamma_k]\,[\limsup_{k\to\infty} \rho(F_k)/\rho(F)]\\ &\leqq 1\cdot 1=1. \quad \square \end{aligned}$$

The renewal sequences ${}_dU$ for $d>1$ play no special role in the following proposition even though it is an analogue of Proposition 2.

Proposition 12. *The member $(0, {}_\Delta U)$ of $\mathscr{U}^H$ is infinitely divisible. If (γ, U) is infinitely divisible, $\gamma\neq 0$, and (γ_i, U_i), $i=1,2,\ldots$, satisfy $(\gamma_i, U_i)^i=(\gamma, U)$, then $(\gamma_i, U_i)\to(1, {}_1U)$ as $i\to\infty$ and $\delta(U)$, defined by* (31), *equals 1.*

Proof. By (35) and the proof and statement of Proposition 2, $\delta(U)=\delta(U_i)$ for all i and $U_i\to_{\delta(U)} U$ in $\mathscr{U}$. From (35),

$$\gamma_i^i=\gamma\,\delta(U)^{i-1}, \qquad i=1,2,\ldots, \tag{39}$$

an impossibility for $\delta(U)>1$, since then the right side goes to ∞ as $i\to\infty$ while the left side is bounded by 1. Insert $\delta(U)=1$ in (39) to obtain $\gamma_i\to 1$ as $i\to\infty$. Examining the sentence containing (38), we see that we can complete the proof by showing $\rho(U_i)\to\rho({}_1U)$, which, we know, equals 1. From (36), $\rho(U_i)=\rho(U)^{1/i}\to 1$. □

8. Another Central Limit Problem

We now turn to the central limit problem for intersections of independent homogenized regenerative phenomena; that is, for products of members of $\mathscr{U}^H$. The simple nature of Proposition 12 indicates that nullity should involves less complications for homogenized regenerative phenomena than it does for regenerative phenomena.

Definition 5. A triangular array $((\gamma_{j,k}, U_{j,k})\colon 1\leqq j\leqq J_k, k=1,2,\dots)$ of members of $\mathscr{U}^H$ is a *null* triangular array if $(\gamma_{j,k}, U_{j,k})\to(1, {}_1U)$ as $k\to\infty$ uniformly in j.

We, of course, also the use the term *null* in connection triangular arrays of members of $\mathscr{F}^H$. From the sentence containing (37) we immediately obtain the following result.

Proposition 13. *A triangular array* $((\gamma_{j,k} F_{j,k})\colon 1\leqq j\leqq J_k, k=1,2,\dots)$ *of members of* $\mathscr{F}^H$ *is null if and only if* $\gamma_{j,k}\,\rho(F_{j,k})\to 1$ *as* $k\to\infty$ *uniformly in* j, *where* ρ *is defined in* (30). *In particular,* $\delta(F_{j,k})$, *defined by* (31), *equals* 1 *for all* (j,k) *with* k *sufficiently large provided the array is null.*

For the present context, when we use (5) we want a positive limit as $q\to\infty$ and we want to incorporate the parameter γ. Accordingly, we let

$$\mathscr{V}^H=\left\{(\gamma, V)\colon 0<\gamma\leqq 1,\ V\in\mathscr{V}^1,\ v_\infty=0,\ \sum_{r=1}^{\infty} r v_r<\infty\right\}.$$

Theorem 8. *Let* $((\gamma_{j,k}, U_{j,k})\colon 1\leqq j\leqq J_k, k=1,2,\dots)$ *be a null triangular array of members of* $\mathscr{U}^H$. *Let* ρ *be defined by* (30). *In order that*

$$\prod_{j=1}^{J_k}(\gamma_{j,k}, U_{j,k})$$

converge to a member of $\mathscr{U}^H$ *other than* $(0, {}_\Delta U)$ *as* $k\to\infty$ *it is necessary and sufficient that there exists* $(\gamma, V)\in\mathscr{V}^H$ *such that*

$$\lim_{k\to\infty}\sum_{j=1}^{J_k} f_{j,k,n}=v_{n-1}, \qquad n=2,3,\dots, \tag{40}$$

and

$$\lim_{k\to\infty}\sum_{j=1}^{J_k}\left[1-\gamma_{j,k}\,\rho(F_{j,k})\right]=\sum_{r=1}^{\infty} r v_r-\log\gamma. \tag{41}$$

In the case of convergence the limiting member of $\mathscr{U}^H$ *is* (γ, U) *where* U *is given by*

$$u_n=\exp\left[-\sum_{r=1}^{\infty}(n\wedge r)\,v_r\right], \qquad n=0,1,2,\dots. \tag{42}$$

In order that

$$\lim_{k\to\infty} \prod_{j=1}^{J_k} (\gamma_{j,k}, U_{j,k}) = (0, {}_{\Delta}U)$$

it is necessary and sufficient that

$$\lim_{k\to\infty} \sum_{j=1}^{J_k} [1-\gamma_{j,k}\,\rho(F_{j,k})] = \infty.$$

Remark. Suppose (40) holds for some $V \in \mathscr{V}$ amd $\sum [1-\gamma_{j,k}\,\rho(F_{j,k})]$ converges as $k\to\infty$. By compactness and the theorem a subsequence of $\prod(\gamma_{j,k}, U_{j,k})$ converges to a limit $\neq (0, {}_{\Delta}U)$. Applying the theorem again we conclude that (41) holds for some γ and that $(\gamma, V) \in \mathscr{V}^H$.

Proof of Theorem 8. Part 1. All unindicated limits will be taken as $k\to\infty$ and j will be an understood index of summation and multiplication throughout.

Part 2. From (35), (36), and Proposition 13 we obtain

$$\prod(\gamma_{j,k}, U_{j,k}) = \left(\prod \gamma_{j,k}, \prod U_{j,k}\right) \tag{43}$$

and

$$\rho\left(\prod U_{j,k}\right) = \prod \rho(U_{j,k}) \tag{44}$$

for sufficiently large k. [Let $(\gamma_k, U_k) = \prod(\gamma_{j,k}, U_{j,k})$. One way we shall use (43) and (44) is to obtain $\gamma_k\,\rho(U_k) = \prod \gamma_{j,k}\,\rho(U_{j,k})$. That this is true is, however, really a more basic property: in the light of (ii) of Theorem 7 we defined multiplication of members of $\mathscr{U}^H$ so that it would be true.]

Part 3. From

$$\begin{aligned}&\sum -\log[\gamma_{j,k}\,\rho(F_{j,k})]\\ &= \sum\{[1-\gamma_{j,k}\,\rho(F_{j,k})]\,[1+o(1-\gamma_{j,k}\,\rho(F_{j,k}))]\},\end{aligned}$$

the fact that $\gamma_{j,k}\,\rho(F_{j,k}) \leqq 1$ for each (j,k), and the fact (compare Proposition 13) that $\gamma_{j,k}\,\rho(F_{j,k}) \to 1$ uniformly in j, we conclude that

$$\lim \sum -\log[\gamma_{j,k}\,\rho(F_{j,k})] = \lim \sum [1-\gamma_{j,k}\,\rho(F_{j,k})] \tag{45}$$

in the sense that if either limit equals $+\infty$, then so does the other and if either limit does not exist, then neither does the other.

Part 4. Suppose that

$$\prod(\gamma_{j,k}, U_{j,k}) \to (0, {}_{\Delta}U).$$

By (43), (44), and the sentence containing (38), $\prod \gamma_{j,k}\,\rho(F_{j,k}) \to 0$. This and (45) imply that $\sum[1-\gamma_{j,k}\,\rho(F_{j,k})] \to \infty$, as desired.

Part 5. Suppose that $(\gamma, U) \in \mathscr{U}^H$, $\gamma > 0$, and

$$\prod(\gamma_{j,k}, U_{j,k}) \to (\gamma, U).$$

By (43), (44) and the sentence containing (38),

$$\prod U_{j,k} \to U \text{ (in } \mathscr{U}) \tag{46}$$

and

$$\prod \gamma_{j,k}\, \rho(F_{j,k}) \to \gamma\, \rho(U). \tag{47}$$

From (46) and Theorem 1 we conclude that there exists $V \in \mathscr{V}$ such that (40) holds, $\delta(U)=1$, and

$$u_n = \exp\left[-\sum_{r=1}^{\infty +} (n \wedge r)\, v_r\right], \qquad n=0,1,2,\ldots. \tag{48}$$

Then from (32) and (48) we obtain

$$\rho(U) = \exp\left[-\sum_{r=1}^{\infty +} r v_r\right] > 0 \tag{49}$$

so that $(\gamma, V) \in \mathscr{V}^H$ and (42) holds. To obtain (41) we take logarithms in (47) and use (45) and (49).

Part 6. Assume now that either (40) and (41) hold for some $(\gamma, V) \in \mathscr{V}^H$ or $\sum [1-\gamma_{j,k}\, \rho(F_{j,k})] \to \infty$. The limit of a convergent subsequence of $\prod (\gamma_{j,k}, U_{j,k})$ either is determined by the right sides of (40) and (41) or equals $(0, {}_\Delta U)$. Hence, all the convergent subsequences have the same limit. Since $\mathscr{U}^H$ is compact, the original sequence converges. □

Corollary 5. *The relationship* (42) *establishes a one-to-one correspondence between* $\mathscr{V}^H$ *and the set of infinitely divisible members of* $\mathscr{U}^H$ *different from* $(0, {}_\Delta U)$.

Proof. Let $(\gamma, V) \in \mathscr{V}^H$. By Corollary 3, U given by (42) is a member of $\mathscr{U}$. By (42) and (32), $(\gamma, U) \in \mathscr{U}^H$. The i^{th} root of (γ, U) is $(\gamma^{1/i}, U_i)$ where U_i is given by (42) with $i^{-1} V$ in lieu of V.

Now suppose $(\gamma, U) \in \mathscr{U}^H$ is infinitely divisible. Since Proposition 12 holds, the argument leading to Corollary 1 applies here also to yield a unique corresponding (γ, V) with $V \in \mathscr{V}^1$. Then (32) implies that $(\gamma, V) \in \mathscr{V}^H$. □

The following analogue of Theorem 2 has a proof that is similar to the proof of that theorem and which we shall omit. Mandelbrot [1972] and Shepp [1972] have similar constructions in the context of real numbers.

Theorem 9. *Let* $V \in \mathscr{V}$ *and* $\gamma \in (0,1]$. *For each* $n=\ldots,-1,0,1,\ldots$ *and* $r=1,2,\ldots,\infty$, *let*

$$B_{n,r} = \begin{cases} \{p\colon n \leq p < n+r\} & \textit{with probability } 1-e^{-v_r} \\ \emptyset & \textit{with probability } e^{-v_r}. \end{cases}$$

Do this independently for the various pairs (n,r) *and independently of that let*

$$B_\Delta = \begin{cases} \{\ldots,-1,0,1,\ldots\} & \textit{with probability } 1-\gamma \\ \emptyset & \textit{with probability } \gamma. \end{cases}$$

Then

$$\{\ldots,-1,0,1,\ldots\}-B_{\Delta}-\bigcup_{n=-\infty}^{\infty}\bigcup_{r=1}^{\infty+}B_{n,r}$$

is an infinitely divisible homogenized regenerative phenomenon of integers equal to $\emptyset$ *with probability one in case* $(\gamma, V)\notin\mathscr{V}^H$ *and corresponding to* (γ, U) *with* U *given by* (42) *in case* $(\gamma, V)\in\mathscr{V}^H$.

In the light of Theorem 9 one may, for some purposes, want to introduce the notation $v_{\Delta}=-\log\gamma$.

As an example consider the null triangular array with $J_k=k$, $f_{j,k,k}=1/k^2=1-f_{j,k,1}$, and $\gamma_{j,k}=1$. By Theorem 8, the row products converge to $(1/e, {}_1U)$ and the corresponding $(\gamma, V)\in\mathscr{V}^H$ is $(1/e,(0,0,\ldots))$. Fix a finite set S. For k sufficiently large the corresponding interection of independent homogenized regenerative phenomena has probability close to $1/e$ of containing S and probability close to $1-1/e$ of being disjoint from S.

For another example let $J_k=k$, $f_{j,k,3}=1/k=1-f_{j,k,1}$, and $\gamma_{j,k}=1$. By Theorem 8 we again have convergence - to a limit determined by $(1, V)\in\mathscr{V}^H$, where $V=(0,1,0,0,\ldots,0)$. By Theorem 9 the limiting homogenized regenerative phenomenon can be constructed as the residual set obtained by covering each pair of consecutive integers with probability $1-1/e$. (Of course, some points get covered twice.)

9. Random Subsets of Integers

To obtain a section analogous to Sect. 4 we want to use $\{-n,\ldots,-1,0,1,\ldots n\}$ where $\{0,1,\ldots,n\}$ is used there. Our topology on $\mathscr{F}^H$ was designed with (ii) and (iii) of Theorem 7 in mind. For that reason, the entire Sect. 4 can, in a straightforward manner, be modified to fit the context of homogenized regenerative phenomena.

10. Infinite Intersections of Homogenized Regenerative Phenomena

Theorem 10. *Let* $((\gamma_k, U_k)\colon k=1,2,\ldots)$ *be an infinite sequence of members of* $\mathscr{U}^H$. *Then*

$$\prod_{k=1}^{\infty}(\gamma_k, U_k)=(0, {}_{\Delta}U)$$

if and only if $\gamma_k=0$ *for some* k *or*

$$\sum_{k=1}^{\infty}[1-\gamma_k\,\rho(F_k)]=\infty.$$

We omit the easy proof (that does not rely on Sect. 8).

Suppose that

$$\prod_{k=1}^{\infty}(1, U_k)=(\gamma, U)$$

for some $\gamma>0$. If, say, $\delta(F_1)=\delta(F_2)>1$, where δ is defined by (31), then $\gamma<1$. We now argue that this is typical of the ways in which $\gamma<1$ is possible. From

Theorem 10 and the obvious inequality $\delta(F_k)\,\rho(F_k)\leqq 1$, we see that $\delta(F_k)$ equals 1 for all but finitely many k's. If, for some $k_1 \neq k_2$,

$$\mathrm{GCD}\{\delta(F_{k_1}), \delta(F_{k_2})\} > 1,$$

then $1>\gamma>0$. If this is not the case for any distinct k_1 and k_2, then the event that the intersection of corresponding independent homogenized regenerative phenomena be empty is a tail event; so, since we have assumed $\gamma>0$, we obtain $\gamma=1$.

It should be remembered that when Theorem 10 is converted to the language of homogenized regenerative phenomena, there is no question of a.s. convergence - it necessarily occurs. The question is: is the limit trivial? Contrast this with Theorem 8 where convergence in distribution is the context and it need not occur.

11. Lévy Processes of Homogenized Regenerative Phenomena

All aspects of Sect. 6 carry over to the present context in an obvious manner except that, since there is no example of a homogenized regenerative phenomenon that is bounded without being empty, there are no analogues of the examples at the end of Sect. 6.

Here are some of the major changes needed in Sect. 6 for the present context. There is a Lévy process $R^H(t)$ of homogenized regenerative phenomena of integers for each $(\gamma, V)\in\mathscr{V}^H$. The corresponding $(\gamma(t), U(t))\in\mathscr{U}^H$ is given by $\gamma(t)=\gamma^t$ and

$$u_n(t)=\exp\left[-t\sum_{r=1}^{\infty}(n\wedge r)\,v_r\right].$$

In the present context, necessrily $\delta(U(t))=1$ (compare (31)), whereas in Theorem 6 that was implicitly assumed as a convenience. As $t\to\infty$, $R^H(t)\to\emptyset$ unless $\gamma=1$ and $V=(0,0,\ldots)$. We generalize the construction of Theorem 9, so we use $B_{n,r}(t)$ for nonpositive n as well as for positive n and also $B_\Delta(t)$. Accordingly, for an analogue of Theorem 6 we need a number v_Δ: it should equal $-\log\gamma$.

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