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On Berry-Esseen Type Bounds for *m*-Dependent Random Variables Valued in Certain Banach Spaces

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1. Introduction

Throughout this paper F will denote a separable Banach space. We shall assume that F satisfies the following condition:

"The norm $\| \ \|$ of F, as a function $F-\{0\}\to\mathbb{R}$, is three times continuously Fréchet-differentiable, and its differentials satisfy $\sup\{\|D_x^1\|, \|D_x^2\|, \|D_x^3\|: \|x\|=1\}=R<+\infty$ where D_x^i denotes the differential of order i of $\|.\|$." (1.1)

Let (Ω, Σ, P) be a fixed probability space. An F-valued random variable X is a Bochner measurable map $\Omega\to F$. We denote L_F^p the set of F-valued random variables X such that $\|X\|^p$ is integrable. An F-valued random variable T is said to be Gaussian if for each $x^*\in F^*$, $x^*\circ T$ is a real-valued Gaussian random variable. It is known that if F is a Hilbert space, then each F-valued Gaussian random variable T satisfies the following condition.

"There exist a constant G such that for $s,\delta\geqq 0$ we have $P(s\leqq\|T\|\leqq s+\delta)\leqq G\delta$." (1.2)

Known examples (in l^∞) show that this is not true in general for an arbitrary Banach space. However, we don't know what is the situation when (1.1) is satisfied.

We denote by $E(Z)$ or EZ for the expectation of the real valued random variable Z.

Suppose $(X_i)_{i\leqq n}$ is a sequence in L_F^2. Since (1.1) implies that F is of type 2, there exists a Gaussian random variable T which has same covariance as X, (the covariance being the bilinear functional of F^* given by

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$(x^*, y^*) \to E(x^*(X)\, y^*(X))$ where $X = \sum_{i \leq n} X_i$. In [9], [11], bounds of $\Delta = \sup_t |P(\|X\| \leq t) - P(\|T\| \leq t)|$ are estimated under the hypothesis (1.1) and (1.2), when the (X_i) are independent random variables with mean zero and in L_F^3. In this work, under the assumption of (1.1) and (1.2) we shall find the bounds of Δ for m-dependent sequences $(X_i)_{i \leq n}$ of random variables with mean zero, i.e. sequences such that for $a, b \in [1, n]$, the sequences $(X_i)_{i \in [a,b]}$ and $(X_i)_{i \in A}$ are independent, where $A = [1, a-m-1] \cup [b+m+1, n]$ (with the convention $[p, q] = \emptyset$ if $q < p$). Using truncation ideas of Feller [3] we obtain these results assuming only $X_i \in L_F^2$. It is noted that, in contrast with the independent case, the covariance of X is not simply related to the covariance of the X_i. We find it is worthy to work out universal bounds, bounds which depend only on universal constants and the parameters. We have tried to get sharp bounds of Δ. However, we have not tried to find numerical values of the universal constant in the bound since the values obtained by our methods are too large to be interesting.

Part 2 recalls some elementary facts. In part 3, we establish bounds for independent random variables. The reward of having the courage to work out the explicit computations is that we improve a result of Kuelbs and Kurtz [9]. In part 4, we gather some technical tools. In part 5, we find bounds of Δ for m-dependent random variables case by using blocking techniques and combinatorial ideas.

2. Some Preliminaries

The results of this section are either well known or easy. Hence most of them are stated without proofs.

Lemma 1. *For $x \in F$, $x \neq 0$, $\lambda \neq 0$, we have $D_{\lambda x} = D_x$, $D_{\lambda x}^2 = \lambda^{-1} D_x^2$, $D_{\lambda x}^3 = \lambda^{-2} D_x^3$. Hence $\|D_x\| \leq R$, $\|D_x^2\| \leq R\, \|x\|^{-1}$, $\|D_x^3\| \leq R\, \|x\|^{-2}$.*

Lemma 2. *F is of type 2 with constant R, i.e. for all independent F-valued random variables $X_1, \ldots, X_n$ of mean zero in L_F^2, $E\, \|\Sigma X_i\|^2 \leq R \Sigma E\, \|X_i\|^2$.*

In fact, F is a "type G" in the terminology of [4], i.e. there exists a mapping g (given by $g(0) = 0$, $g(x) = \|x\|^2 D_x$ for $x \neq 0$) with the properties $\|g(x)\|_{F^*} = \|x\|_F$, $\langle g(x), x \rangle = \|x\|_F^2$, $\|g(x) - g(y)\|_{F^*} \leq R\, \|x - y\|_F$.

Lemma 3. *There exists a universal constant K_1 such that for $\delta > 0$, $s > 0$ there exists $f\colon \mathbb{R} \to [0, 1]$, $f(t) = 0$ if $t \leq s$, $f(t) = 1$ if $t \geq s + \delta$, f is three times continuously differentiable, $\|f^{(3)}\|_\infty \leq K_1 \delta^{-3}$.*

Lemma 4. *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is three times continously differentiable and $f(t) = 0$ if $t \leq 0$. Let $x, y \in F$, $h(\lambda) = f(\|x + \lambda y\|)$. Then h is three times continuously differentiable. If $x + \lambda y = 0$, $h(\lambda) = h'(\lambda) = h''(\lambda) = h^{(3)}(\lambda) = 0$. If $\|x + \lambda y\| \neq 0$,*

$$h'(\lambda) = D_{x+\lambda y}(y)\, f'(\|x + \lambda y\|)$$
$$h''(\lambda) = (D_{x+\lambda y}(y))^2 f''(\|x + \lambda y\|) + D_{x+\lambda y}^2(y, y)\, f'(\|x + \lambda y\|)$$

$$h^{(3)}(\lambda)=(D_{x+\lambda y}(y))^3 f^{(3)}(\|x+\lambda y\|)+3D_{x+\lambda y}(y)\, D^2_{x+\lambda y}(y,y) \cdot f''(\|x+\lambda y\|)+D^3_{x+\lambda y}(y,y,y)\, f'(\|x+\lambda y\|).$$

The following lemma will be used many times without quoting.

Lemma 5 (c_r-inequality [10]). *For $a_1, a_2, \ldots, a_n \geqq 0$ and $r \geqq 0$*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leqq A_{n,r} \sum_{i=1}^{n} a_i^r$$

where $A_{n,r}=n^{r-1}$ if $r\geqq 1$, $A_{n,r}=1$ if $r \leqq 1$. Hence if $X_1, \ldots, X_n$ are random variables in $L^r_{\mathbb{R}}$,

$$E\left|\sum_{i=1}^{n} X_i\right|^r \leqq A_{n,r} \sum_{i=1}^{n} E|X_i|^r.$$

Lemma 6. *Let A, $(B_i)_{i\leqq n}$, $(r_i)_{i\leqq n}$ be positive numbers. Then*

$$\operatorname*{Inf}_{\delta>0}(A\delta+\Sigma B_i \delta^{-r_i}) \leqq (n+1) \sum_{i\leqq n} A^{\frac{r_i}{1+r_i}} B^{\frac{1}{1+r_i}}.$$

Proof. It is of course true if $A=0$. If $A \neq 0$, let i_0 such that $\gamma_{i_0}=\operatorname{Sup}\{\gamma_i,\ i\leqq n\}$ where $\gamma_i=(B_i A^{-1})^{\frac{1}{1+r_i}}$. Then for all i,

$$B_i \gamma_{i_0}^{-r_i} \leqq B_i \gamma_i^{-r_i} = A\gamma_i \leqq A\gamma_{i_0} = A^{\frac{r_{i_0}}{1+r_{i_0}}} B^{\frac{1}{1+r_{i_0}}}$$

so

$$A\gamma_{i_0}+\sum_{i\leqq n} B_i \gamma_{i_0}^{-r_i} \leqq (n+1) A\gamma_{i_0} \leqq (n+1) \sum_{i\leqq n} A^{\frac{r_i}{1+r_i}} B^{\frac{1}{1+r_i}}. \quad \text{Q.E.D.}$$

The following is an easy consequence of the method of Fernique in [2].

Lemma 7. *There exists a universal constant K_2 such that for all Banach space valued Gaussian random variable X, one has:*

a) *for all $u \in \mathbb{R}$* $\quad P(\|X\| \geqq u) \leqq \exp\left(-\dfrac{u^2}{K_2 \|X\|_2^2}\right)$

b) *For all $1 \leqq p \leqq 4$* $\quad \|X\|_p \leqq K_2 \|X\|_2$.

3. Bounds for Independent Random Variables

Let $X=(X_i)_{i\leqq n}$ be a sequence of independent F-valued random variables in L^2_F with mean zero. Let $T_1, \ldots, T_n$ be independent F-valued Gaussian random variables such that for each i, T_i has the same covariance as X_i. The existence of T_i is shown in [6], Proposition 3.3 since F is of type 2, and moreover it is shown that $E\|T\|^2 \leqq RE\|X_i\|^2$. We want to find a bound for $\Delta=\Delta(X) =\operatorname*{Sup}_t |P(\|\sum_{i\leqq n} X_i\|<t)-P(\|\sum_{i\leqq n} T_i\|<t)|$.

The method will follow the Theorem 2.1 in [9]. However, since we don't assume that the T_i have same covariance, the computations have to be done with somewhat more care.

Suppose that for $i \leq n$ we have a decomposition $X_i = \bar{X}_i + X'_i$, where $\|\bar{X}_i\| \cdot \|X'_i\| = 0$, $\bar{X}_i \in L_F^{7/2}$, and each of the sequences $(\bar{X}_i)_{i \leq n}$ and $(X'_i)_{i \leq n}$ is independent. (such a decomposition is a generalization of truncations in the real-valued case). Set

$$b = \sum_{i \leq n} E\|X'_i\|^2; \quad c = \sum_{i \leq n} E\|\bar{X}_i\|^3; \quad d = \sum_{i \leq n} E\|\bar{X}_i\|^{7/2}; \quad e = \sum_{i \leq n} (E\|X'_i\|^2)^{7/4}$$

$$c_1 = c + \sum_{i \leq n} E\|T_i\|^3 \quad d_1 = d + \sum_{i \leq n} E\|T_i\|^{7/2}.$$

In order to get an interesting bound for Δ, it is reasonable to assume that $P(\|\sum_{i \leq n} T_i\| < t)$ does not vary too wildly as a function of t. We write $\sum_{i \leq n} T_i = W - V$, where W and V are Gaussian, such that there exists a constant G such that

$$\sup_{s \geq 0} P(s \leq \|W\| \leq s + \delta) \leq G\delta.$$

Let $M_V = \operatorname{Inf}_{\varepsilon > 0} \{G\varepsilon + P(\|V\| \geq \varepsilon)\}$.

The following lemma is the key of the method of successive improvements of the bound of Δ.

Lemma 8. *Let $(X_i)_{i \leq n}$ be a sequence of L_F^2. Suppose that for each sequence $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_n)$, where $\tilde{X}_i = X_i$ or $\tilde{X}_i = T_i$, we have $\Delta(\tilde{X}) \leq \Delta^n(X)$, where Δ^n is a function of b, c_1, d_1, e, G, M_V, R. Let $s \geq 0$, $\delta \geq 0$, and let f: $\mathbb{R} \to [0, 1]$ be a three times continuously differentiable function, with $f(\tau) = 0$ for $\tau \leq s$, $f(\tau) = 1$ for $\tau \geq s + \delta$, $\|f^{(3)}\|_\infty \leq K_1 \delta^{-3}$, and let*

$$\Delta f(X) = |Ef(\Sigma X_i\|) - Ef(\|\Sigma T_i\|)|.$$

Then for all sequences $\tilde{X}$, where $\tilde{X}_i = X_i$ or $\tilde{X}_i = T_i$, we have

$$\Delta f(\tilde{X}) \leq K_{11} R(\delta^{-2}(c_1 G + b) + \delta^{-3} c_1(\Delta^n(X) + M_V) + \delta^{-7/2}(d_1 + e)) \tag{3.1}$$

where K_{11} is a universal constant.

Proof. We are going first to prove (3.1) for $\tilde{X} = X$. It is of course possible to suppose that the T_i are independent of the $\bar{X}_i$ and of the X'_i. For $i \leq n$, let

$$U_i = \sum_{j < i} X_i + \sum_{j > i} T_i$$

so

$$f(\|\Sigma X_i\|) - f(\|\Sigma T_i\|) = \sum_{i \leq n} f(\|U_i + X_i\|) - f(\|U_i + T_i\|)$$

and hence $\Delta f(X) \leq \sum_{i \leq n} V_i$, where $V_i = |E(f(\|U_i + X_i\|) - f(\|U_i + T_i\|))|$. We fix i and evaluate V_i. For $\lambda \in \mathbb{R}$, set $g(\lambda) = f(\|U_i + \lambda X_i\|)$, $h(\lambda) = f(\|U_i + \lambda T_i\|)$. From Lemma 4, g and h are three times continuously differentiable. It is shown in [8] or [11] that $E(g'(0)) = E(h'(0))$, $E(g''(0)) = E(h''(0))$ so we get $V_i \leq V_i^1 + V_i^2$, where

$$V_i^1 = E|g(1) - g(0) - g'(0) - \tfrac{1}{2} g''(0)|; \qquad V_i^2 = E|h(1) - h(0) - h'(0) - \tfrac{1}{2} h''(0)|.$$

Now set

$$g_1(\lambda)=f(\|U_i+\lambda\bar{X}_i\|), \qquad g_2(\lambda)=f(\|U_i+\lambda X_i'\|).$$

Since $\|\bar{X}_i\|\,\|X_i'\|=0$, we have $g_1(\lambda)+g_2(\lambda)=g(\lambda)+g(0)$ and for $j=1,2$ $g_1^{(j)}(\lambda)+g_2^{(j)}(\lambda)=g^{(j)}(\lambda)$. So we get $V_i^1\leqq V_i^3+V_i^4+V_i^5$ where

$$V_i^3=E|g_1(1)-g_1(0)-g_1'(0)-\tfrac{1}{2}g''(0)|=E|\tfrac{1}{6}g_1^{(3)}(\tau_1)|$$
$$V_i^4=E|g_2(1)-g_2(0)-g_2'(0)|=E|\tfrac{1}{2}g_2''(\tau_2)|$$
$$V_i^5=E|\tfrac{1}{2}g_2''(0)|.$$

Note that for $1\leqq j\leqq 3$, $f^{(j)}(t)\leqq K_1\delta^{-3}t^{3-j}\chi_{[s,s+\delta]}(t)$ and $\|D_x^j\|\leqq R\|x\|^{-j+1}$. It follows then from Lemma 4, and since $\tau_1\leqq 1$, that

$$V_i^3\leqq K_1R\delta^{-3}E(\|\bar{X}_i\|^3\chi_{[s,s+\delta]}(\|U_i+\tau_1\bar{X}_i\|)).$$

We have

$$\chi_{[s,s+\delta]}(\|U_i+\tau_i\bar{X}_i\|)\leqq\chi_{[s-\delta,s+2\delta]}(\|U_i\|)+\chi_{[\delta,\infty[}(\|\bar{X}_i\|).$$

So, since U_i and $\bar{X}_i$ are independent, and since $\|\bar{X}_i\|^3\chi_{[\delta,\infty[}(\|\bar{X}_i\|)\leqq\delta^{-1/2}\|\bar{X}_i\|^{7/2}$, we have:

$$V_i^3\leqq K_1R\delta^{-3}(E(\|\bar{X}_i\|^3)E\chi_{[s-\delta,s+2\delta]}(\|U_i\|)+\delta^{-1/2}E(\|\bar{X}_i\|^{7/2})). \tag{3.2}$$

We have:

$$\chi_{[s-\delta,s+2\delta]}(\|U_i\|)\leqq\chi_{[s-2\delta,s+3\delta]}(\|U_i+X_i\|)+\chi_{[\delta,\infty[}\|X_i\|.$$

Now if $\tilde{X}$ denotes the sequence $(X_1,\dots,X_i,T_{i+1},\dots,T_n)$, we have by hypothesis $\Delta(\tilde{X})\leqq\Delta^n(X)$, so

$$E\chi_{[s-2\delta,s+3\delta]}(\|U_i+X_i\|)\leqq 2\Delta^n(X)+E\chi_{[s-2\delta,s+3\delta]}(\|\sum_{i\leqq n}T_i\|).$$

Moreover, for each $\varepsilon>0$

$$\chi_{[s-2\delta,s+3\delta]}(\|\sum_{i\leqq n}T_i\|)\leqq\chi_{[s-2\delta-\varepsilon,s+3\delta+\varepsilon]}(\|W\|)+\chi_{[\varepsilon,\infty[}(\|V\|).$$

Since $\chi_{[\delta,\infty[}\|X_i\|\leqq\delta^{-1/2}\|X_i\|^{1/2}$, we get, by substituting these relations into (3.2).

$$\begin{aligned}V_i^3\leqq K_1R\delta^{-3}(E\|\bar{X}_i\|^3(2\Delta^n(X)+5\delta G+2\varepsilon G+P(\|V\|\geqq\varepsilon)\\ +\delta^{-1/2}E\|X_i\|^{1/2})+\delta^{-1/2}E\|\bar{X}_i\|^{7/2}).\end{aligned} \tag{3.3}$$

Now, notice that $\|X_i\|^{1/2}=\|\bar{X}_i\|^{1/2}+\|X_i'\|^{1/2}$. By Hölder's inequality, $E\|\bar{X}_i\|^3E\|\bar{X}_i\|^{1/2}\leqq E\|\bar{X}_i\|^{7/2}$. Moreover, since for $p,q\geqq 1$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, we have $ab\leqq\dfrac{a^p}{p}+\dfrac{b^q}{q}\leqq a^p+b^q$, we get $E\|\bar{X}_i\|^3E\|X_i'\|^{1/2}\leqq(E\|\bar{X}_i\|^3)^{7/6}+(E\|X_i'\|^{1/2})^7\leqq E\|\bar{X}_i\|^{7/2}+(E\|X_i'\|^2)^{7/4}$. Since (3.3) is true for all $\varepsilon>0$, we get

$$V_i^3\leqq 5K_1R\delta^{-3}(E\|\bar{X}_i\|^3(\Delta^n(X)+\delta G+M_V)+\delta^{-1/2}(E\|\bar{X}_i\|^{7/2}+E\|X_i'\|^2)^{7/4}). \tag{3.4}$$

A very similar computation yields

$$V_i^2\leqq 5K_1R\delta^{-3}(E\|T_i\|^3(\Delta^n(X)+\delta G+M_V)+\delta^{-1/2}E\|T_i\|^3). \tag{3.5}$$

Moreover, easier computations give, using the fact that $f^{(2)}(t) \leqq K_1 \delta^{-2}$, $f'(t) \leqq K_1 \delta^{-2} t$:

$$V_i^4 \leqq K_1 RE \|X_i'\|^2; \quad V_i^5 \leqq K_1 RE \|X_i'\|^2 \tag{3.6}$$

and the result follows from (3.4), (3.5), (3.6), with $K_{11} = 5K_1$.

To see that the result still holds for $\tilde{X}$ instead of X, just note that if $\tilde{X}_i = T_i$, the corresponding V_i is zero. Q.E.D.

Lemma 9. *Suppose, under the same hypothesis as Lemma 8, that for each sequence $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_n)$ where $\tilde{X}_i = X_i$ or $\tilde{X}_i = T_i$ we have $\Delta(\tilde{X}) \leqq \Delta^n(X)$, where Δ^n is a function of $b, c_1, d_1, e, G, M_V, R$. Then we have $\Delta(\tilde{X}) \leqq \Delta^{n+1}(X)$, where*

$$\begin{aligned} \Delta^{n+1}(X) = K_{12}(R^{1/3}(c_1 G + b)^{1/3} G^{2/3} + R^{1/4} c_1^{1/4} (\Delta^n(X) + M_V)^{1/4} G^{3/4} \\ + R^{2/9}(d_1 + e)^{2/9} G^{7/9}) + 2M_V \end{aligned} \tag{3.7}$$

and K_{12} is a universal constant.

Proof. Let f: $\mathbb{R} \to [0, 1]$ be three times continuously differentiable and $f(t) = 0$ for $t \leqq s$, $f(t) = 1$ for $t \geqq s + \delta$ and $f^{(3)}(t) \leqq K_1 \delta^{-3}$. We have, if $h = 1 - f$:

$$\begin{aligned} p(\|\Sigma X_i\| \leqq s) &\leqq Eh(\|\Sigma X_i\|) \\ &\leqq \Delta f(X) + Eh(\|\Sigma T_i\|) \\ &\leqq \Delta f(X) + P(\|\Sigma T_i\| \leqq s + \delta) \\ &\leqq \Delta f(X) + P(\|W\| \leqq s + \delta + \varepsilon) + P(\|V\| \geqq \varepsilon) \\ &\leqq \Delta f(X) + P(\|W\| \leqq s - \varepsilon) + G(\delta + 2\varepsilon) + P(\|\bar{V}\| \geqq \varepsilon) \\ &\leqq \Delta f(X) + P(\|\Sigma T_i\| \leqq s) + G\delta + 2\varepsilon G + 2P(\|V\| \geqq \varepsilon). \end{aligned}$$

Since this is true for all $\varepsilon > 0$,

$$P(\|\Sigma X_i\| \leqq s) - P(\|\Sigma T_i\| \leqq s) \leqq \Delta f(X) + G\delta + 2M_V.$$

A similar computation yields

$$P(\|\Sigma T_i\| \leqq s) - P(\|\Sigma X_i\| \leqq s) \leqq \Delta f(X) + G\delta + 2M_V.$$

So $\Delta(X) \leqq \Delta f(X) + G\delta + 2M_V$. This is true for all $\delta > 0$. If we substitute the bound for $\Delta f(X)$ given by Lemma 8 and use Lemma 6, the result follows with $K_{12} = 4K_{11}^{1/3}$. Q.E.D.

Theorem 10. *Under the same hypothesis as Lemma 8, we have*

$$\Delta \leqq K_0(M_V + R^{5/6} c^{1/3} G + R^{5/6} b^{1/3} G^{2/3} + R^{11/18}(d + e)^{2/9} G^{7/9}) \tag{3.8}$$

where $K_0 \geqq 1$ is a universal constant.

Proof. Consider the sequence $\Delta^n(X)$ defined by (3.7) and $\Delta^0(X) = 1$. Let $\Delta^\infty(X) = \operatorname{Inf}_n \Delta^n(X)$. We have $\Delta(X) \leqq \Delta^\infty(X)$. Moreover, since $K_{12} \geqq 3$ and if we set

$$\begin{aligned} Y &= \Delta^\infty(X) + M_V \\ A &= K_{12}(M_V + R^{1/3}(c_1 G + b)^{1/3} G^{2/3} + R^{2/9}(d_1 + e)^{2/9} G^{7/9}) \\ B &= K_{12} R^{1/4} c_1^{1/4} G^{3/4}, \end{aligned}$$

then we have for all n:

$$Y \leqslant A + B(\Delta^n(X) + M_y)^{1/4}$$

and hence $Y \leqslant A + BY^{1/4}$. Since $BY^{1/4} \leqq \frac{3}{4}B^{4/3} + \frac{1}{4}Y$, we get $Y \leqq \frac{4}{3}A + B^{4/3}$, so, with $K_{13} = 2\sup(\frac{4}{3}K_{12}, K_{12}^{4/3})$, and since $(c_1 G + b)^{1/3} \leqq (c_1 G)^{1/3} + b^{1/3}$

$$\Delta \leqq K_{13}(M_V + R^{1/3}c_1^{1/3}G + R^{1/3}b^{1/3}G^{2/3} + R^{2/9}(d_1 + e)^{2/9}G^{7/9}).$$

It is possible to assume $bG^2 \leqq 1$. Otherwise, since $K_{13} \geqq 1$, $R \geqq 1$ (3.8) is automatically satisfied. Using Lemmas 5, 7 and Schwartz's inequality, we get

$$\begin{aligned} E\|T_i\|^3 &\leqq K_2^{3/2}(E\|T_i\|^2)^{3/2} \leqq K_2^{3/2}R^{3/2}(E\|X_i\|^2)^{3/2} \\ &\leqq K_2^{3/2}R^{3/2}((E\|\bar{X}_i\|^3)^{2/3} + E\|X_i'\|^2)^{3/2} \\ &\leqq K_2^{3/2}R^{3/2}\sqrt{2}(E\|\bar{X}_i\|^3 + E\|X_i'\|^2 b^{1/2}) \end{aligned}$$

so

$$G\sum_{i\leqq n} E\|T_i\|^3 \leqq K_2^{3/2}R^{3/2}\sqrt{2}(cG + b^{3/2}G) \leqq K_2^{3/2}R^{3/2}\sqrt{2}(cG + b).$$

Similar computation gives

$$\sum_{i\leqq n} E\|T_i\|^{7/4} \leqq K_2^{7/4}R^{7/4}2^{3/4}(d+e)$$

hence we get (3.8) with $K_0 = K_{13}(1 + K_2^{3/2}\sqrt{2})^{1/3}$.

Example: Let (Y_n) be a sequence of independent F-valued random variables with $E(\|Y_n\|^{7/2}) \leqq M$ for all n. Suppose that all the Y_n have the same covariance, and let T be a Gaussian random variable with this covariance. Suppose that $P(s \leqq \|T\| \leqq s + \delta) \leqq G\delta$, for $s, \delta \geqq 0$. Then Theorem 10 shows that

$$\begin{aligned} &\sup_t |P(\|n^{-1/2}\sum_{i\leqq n} Y_i\| \leqq t) - P(\|T\| \leqq t)| \\ &\quad \leqq K_{13}n^{-1/6}(R^{5/6}M^{6/7}G + R^{11/18}MG^{7/6}) = O(n^{-1/6}) \end{aligned}$$

(we take $\bar{X}_i = n^{-1/2}Y_i$, $X_i' = 0$, $V = 0$).

This shows that Theorem 10 is stronger than Theorem 2.1.0 in [9] for the case $r = 3$. Moreover, it is more precise since the bound includes all the parameters explicitly. Hence this bound can be used when the parameters vary (i.e. triangular arrays). The term M_V in Theorem 10 will be used later.

It should be noted that V. Paulauskas [11] shows that for independent identically distributed Hilbert space valued random variables with third moments, the rate of convergence of Δ is of order $n^{-1/6}$. His proof relies heavily on the fact that the variables are identically distributed. However, there is some hope that his method gives a bound similar to the bound of Theorem 10 for non-identically distributed random variables with only third moments involved. We have not been able to achieve this goal.

4. Some More Lemmas

Lemma 11. *For $p \geqq 2$ and a sequence (X_i) of independent F-valued random variables in L^p we have*

$$E\|\sum_{i\leq n} X_i\|^p \leqq N_p R^{p/2} E[(\sum_{i\leq n}\|X_i\|^2)^{p/2}]$$

where N_p is a universal constant.

Proof. Let $\varepsilon_1, \dots, \varepsilon_n$ be a Rademacher sequence independent of the X_i. To be more clear, we assume that the probability space is a product, and that the ε_i depends on the first coordinate ω_1 and the X_i on the second ω_2. A result of Kahane [7] asserts that for each elements $x_1, \dots, x_n$ of any Banach space,

$$\int \|\sum_{i\leq n} \varepsilon_i(\omega) x_i\|^p \leqq N'_p(\int \|\sum_{i\leq n} \varepsilon_i(\omega) x_i\|^2)^{p/2}$$

where N'_p is a universal constant. So we get

$$\begin{aligned}\int \|\sum_{i\leq n} \varepsilon_i(\omega) X_i(\omega)\|^p d\omega &\leqq \int(\int \|\Sigma \varepsilon_i(\omega_1) X_i(\omega_2)\|^p d\omega_1)\, d\omega_2 \\ &\leqq N'_p \int(\int \|\Sigma \varepsilon_i(\omega_1) X_i(\omega_2)\|^2)^{p/2} d\omega_2 \leqq N'_p R^{p/2} \int(\Sigma \|X_i(\omega_2)\|^2)^{p/2} d\omega_2 \\ &\leqq N'_p R^{p/2} E[(\Sigma \|X_i\|^2)^{p/2}]\end{aligned}$$

since F is of type 2 with constant R. But it follows from Corollary 4.2 in [5] that $E\|\sum_{i\leq n} X_i\|^p \leqq 2^p E\|\sum_{i\leq n} \varepsilon_i X_i\|^p$, whence the result with $N_p = 2^p N'_p$.

The following lemma is an extension to Banach spaces of a lemma of Egorov [1].

Lemma 12. *Let $X_1, \dots, X_n$ be m-dependent F-valued random variables, with $m \geqq 1$. Then*

$$E\|\sum_{i\leq n} X_i\|^p \leqq N_p^1 n^{p/2-1} m^{p/2} R^{p/2} \sum_{i\leq n} E\|X_i\|^p$$

where N_p^1 is a universal constant and R is defined in the introduction. We set $K_3 = \sup(N_2^1, N_3^1, N_{7/2}^1)$.

Proof. Write $\sum_{i\leq n} X_i = \sum_{j\leq m+1} Y_j$, where $Y_j = \sum_{1\leq q+(m+2)l\leq n} X_{q+(m+2)l}$. From Lemma 5 and 11 we get

$$\begin{aligned}E\|\sum_{i\leq n} X_i\|^p &\leqq (m+1)^{p-1} \sum_{j\leq m+1} E\|Y_j\|^p \\ &\leqq N_p (m+1)^{p-1} R^{p/2} \sum_{j\leq m+1} E[(\sum_{q+(m+2)l\leq n} \|X_{q+(m+2)l}\|^2)^{p/2}].\end{aligned}$$

Now for each j, l runs over at most $1+\dfrac{n}{m+2} \leqq \dfrac{2n}{m+1}$ integers since we can suppose $n \geqq m+1$ and hence by Lemma 5 again we have

$$E\|\sum_{i\leq n} X_i\|^p \leqq N_p (m+1)^{p/2} 2^{p/2} n^{p/2-1} R^{p/2} \sum_{i\leq n} E\|X_i\|^p$$

whence the result with $N_p^1 = N_p 2^p$. Q.E.D.

Lemma 13. *Let Y, Z, $X=Y+Z$ be F-valued random variables in L^2. Then there exist U, V, $T=U+V$ Gaussian F-valued random variables with the same covariance as Y, Z and X respectively.*

Proof. Let W be the $F\times F$-valued random variable $\omega\to(Y(\omega),\ Z(\omega))$. Since $F\times F$ is also a type 2, there exist a Gaussian random variable $\omega\to(U(\omega),\ V(\omega))$ with the same covariance. It is easy to check that for $\alpha,\beta\in\mathbb{R}$, $\alpha Y+\beta Z$ has the same covariance as $\alpha U+\beta V$. Q.E.D.

Lemma 14. *Suppose the probability space is diffuse. Let $p\geqslant 2$, $U\in L_F^p$, $V\in L_F^2$, $W=U+V$. Then we can write $W=\bar{W}+W'$ with $\|\bar{W}\|\cdot\|W'\|=0$ and*

$$E\|\bar{W}\|^p\leqslant 2^{p+2}E\|U\|^p,\qquad E\|W'\|^2\leqslant 2^{p+2}E\|V\|^2. \tag{4.1}$$

Proof. Suppose we can write $\|U\|+\|V\|=\bar{T}+T'$ where $E|\bar{T}|^p\leqslant 2^{p+2}E\|U\|^p$, $E'|T'|^2\leqslant 2^{p+2}E\|V\|^2$. If one set $\bar{W}=0$ for $\bar{T}=0$ and $\bar{W}=W$ otherwise, and $W'=W-\bar{W}$, then $\|\bar{W}\|\leqq\bar{T}$, $\|W'\|\leqq T'$ and hence $\bar{W}$ and W' satisfy (4.1). Hence one can assume $F=\mathbb{R}$, U, $V\geqslant 0$.

If $E(W^2)\leqq 2^{p+2}E(V^2)$, one can take $\bar{W}=0$, $W'=W$. It is hence possible to suppose $E(W^2)>2^{p+2}E(V^2)$. Define

$$\lambda=\operatorname{Inf}\{\tau\geqq 0;\ E(W^2\chi_{\{W\geqq\tau\}})\leqq 2^{p+2}E(V^2)\}.$$

We have $E(W^2\chi_{\{W>\lambda\}})\leqq 2^{p+2}E(V^2)$, and hence $\lambda>0$ (if $\lambda=0$, $W^2\chi_{\{W>\lambda\}}=W^2$). For $\tau<\lambda$, we have by definition of λ $E(W^2\chi_{\{W\geqq\tau\}})\geqq 2^{p+2}E(V^2)$ and hence

$$E(W^2\chi_{\{W\geqq\lambda\}})\geqq 2^{p+2}E(V^2). \tag{4.2}$$

It follows that there exists a measurable set A such that $\{W>\lambda\}\subset A\subset\{W\geqq\lambda\}$ and $E(W^2\chi_A)=2^{p+2}E(V^2)$. In fact,

$$E(W^2\chi_{\{W\geqq\lambda\}})=\lambda^2P(W=\lambda)+E(W^2\chi_{\{W>\lambda\}})$$

so from (4.2)

$$\lambda^2P(W=\lambda)\geqq 2^{p+2}E(V^2)-E(W^2\chi_{\{W>\lambda\}}) \tag{4.3}$$

and it is enough to take $A=\{W>\lambda\}\cup B'$, where $B'\subset\{W=\lambda\}$ and $\lambda^2P(B')=2^{p+2}E(V^2)-E(W^2\chi_{\{W>\lambda\}})$. (It is to ensure the existence of B' that we assume there are no atoms.)

We are going to show that $E(W^p\chi_{A^c})\leqq 2^{p+2}E(U^p)$ and hence that it is enough to take $\bar{W}=W\chi_{A^c}$, $W'=W\chi_A$.

Suppose

$$E(W^p\chi_{A^c})>2^{p+2}E(U^p).$$

From Lemma 5, we have $W^2\leqq 2(U^2+V^2)$, $W^p\leqq 2^{p-1}(U^p+V^p)$. Hence

$$\begin{gathered}2^{p+2}E(V^2)=E(W^2\chi_A)\leqq 2E(U^2\chi_A+V^2\chi_A)\leqq 2E(V^2)+2E(U^2\chi_A)\\ (2^{p+1}-1)\,E(V^2)\leqq E(U^2\chi_A).\end{gathered} \tag{4.4}$$

Similarly

$$2^{p+2}E(U^p)\leqq E(W^p\chi_{A^c})\leqslant 2^{p-1}E(U^p\chi_{A^c}+V^p\chi_{A^c})\leqslant 2^{p-1}E(U^p)+2^{p-1}E(V^p\chi_{A^c}).$$

So

$$7E(U^p) \leqq E(V^p \chi_{A^c}). \tag{4.5}$$

Now define

$$B = A \cap \{U \geqq V\} \quad \text{and} \quad C = A^c \cap \{V \geqq U\}.$$

So

$$E(U^2 \chi_{A \setminus B}) \leqq E(V^2)$$
$$E(V^p \chi_{A^c \setminus C}) \leqq E(U^p).$$

Hence, from (4.4) and (4.5)

$$(2^{p+1}-2)\,E(V^2\chi_C) \leqslant (2^{p+1}-2)\,E(V^2) \leqq E(U^2\chi_B) \tag{4.6}$$

$$6E(U^p\chi_B) \leqq 6E(U^p) \leqq E(V^p\chi_C). \tag{4.7}$$

On B, since $U \geqq V$, $U \geqq \frac{1}{2}(U+V) = \frac{1}{2}W$.
On C, since $V \geqq U$, $V \geqq \frac{1}{2}(U+V) = \frac{1}{2}W$.

Hence from (4.6) and (4.7)

$$(2^{p-1}-1)\,E(W^2\chi_C) \leqslant (2^{p-1}-2)\,E(V^2\chi_C) \leqq E(U^2\chi_B) \leqq E(W^2\chi_B) \tag{4.8}$$

$$2^{-p+2}E(W^p\chi_B) \leqq 4E(U^p\chi_B) \leqq E(V^p\chi_C) \leqq E(W^p\chi_C). \tag{4.9}$$

Since $C \subset A^c$, we have $W \leqq \lambda$ on C and since $B \subset A$, $W \geqq \lambda$ on B. From (4.9) we get

$$2^{-p+2}\lambda^{p-2}E(W^2\chi_B) \leqslant 2^{-p+2}E(W^p\chi_B) \leqq E(W^p\chi_C) \leqslant \lambda^{p-2}E(W^2\chi_C).$$

So $2^{-p+2}E(W^2\chi_B) \leqq E(W^2\chi_C)$. Since $(2^{p-1}-1)2^{-p+2} > 1$, together with (4.8), this implies that $E(W^2\chi_C) = 0$. Hence $V \leqq U$ on $A^c \cap \{W > 0\}$, so $E(W^p\chi_{A^c}) \leqq 2^p E(U\chi_{A^c}) \leqslant 2^p E(U^p)$ and this contradiction concludes the proof.

The following two lemmas prepare the basic Lemma 17.

Lemma 15. *Let p and k be two integers. Let $(I_i)_{i \leqq k}$ be a disjoint family of sets such that $\frac{3}{5}p \leqq \operatorname{card} I_i \leqq 3p$ for all i. Let $I = \bigcup_{i \leqq k} I_i$. Let $(a_j)_{j \in I}$ be a family of non-negative real numbers. Let $\theta \geqq 0$. Suppose $\operatorname{Sup}_{j \in I} a_j \leqq \theta\mu$ where $\mu = (\sum_{j \in I} a_j)/pk$. Then there exist for $i \leqq k$ an element $j_i \in I_i$ such that $k\mu/3 \leqq \sum_{i \leqq k} a_{j_i} \leqq (\theta + 2k)\,\mu$.*

Proof. Let us pick by induction on $i \leqq k$ an element $j_i \in I_i$, such that

$$\text{If } \sum_{i' < i} a_{j_{i'}} < k\mu/3, \quad \text{then } a_{j_i} = \operatorname{Sup}\{a_j : j \in I_i\} \tag{4.10}$$

$$\text{If } \sum_{i' < i} a_{j_{i'}} \geqq k\mu/3, \quad \text{then } a_{j_i} = \operatorname{Inf}\{a_j ; j \in I_i\}. \tag{4.11}$$

Let i_0 be the greatest integer such that $\sum_{i' < i_0} a_{j_{i'}} < k\mu/3$. If $i_0 < k$, then $\sum_{i \leqq k} a_{j_i} \geqq k\mu/3$. If $i_0 = k$, then

$$\sum_{i \leqq k} a_{j_i} = \sum_{i \leqq k} \sup\{a_j ; j \in I_i\} \geqq \frac{1}{3p} \sum_{j \in I} a_j \geqq \frac{\mu k}{3}.$$

On the other hand,

$$\sum_{i\leqq k} a_{j_i} = \sum_{i<i_0} a_{j_i} + a_{j_{i_0}} + \sum_{i>i_0} a_{j_i} \leqq \frac{k\mu}{3} + \theta\mu + \sum_{i\leqq k} \operatorname{Inf}\{a_j; j\in I_j\}$$
$$\leqq \frac{k\mu}{3} + \theta\mu + \frac{5}{3p}\sum_{j\in I} a_j \leqq (\theta+2k)\,\mu$$

which concludes the lemma.

For $t\in\mathbb{R}$ we denote $[t]$ the largest integer $\leqslant t$.

Lemma 16. *Let p and k be two integers. Let $(I_i)_{i\leqq k}$ be a disjoint family of sets such that $p\leqq \operatorname{card} I_i \leqq 3p$ for all i, and let $I=\bigcup_{i\leqq k} I_i$. Let $(a_j)_{j\in I}$ be a family of non-negative integers such that $\sup_{j\in I} a_j \leqq \theta\mu$, where $\mu=(\sum_{j\in I} a_j)/pk$. Then, if $r=\left[\frac{p}{2(\theta/k+2)}\right]$, there exist for each $i<k$ a family $(j_{i,l})_{l\leqq r}$ of distinct elements of I_j such that for all $l\leqq r$*

$$\frac{k\mu}{6} \leqq \sum_{i\leqq k} a_{j_{i,l}} \leqq (\theta+2k)\,\mu.$$

Proof. The construction goes by induction on $l\leqq r$. Note that $r\leqq p/4 \leqq \frac{2}{5}p$. If the points $j_{i,l'}$ have been constructed for all $i\leqq k$ and $l'<l$, set $I_i^l = I_i \setminus \{j_{i,l},\ldots,j_{i,l-1}\}$ and $I^l=\bigcup_{i\leqq k} I_i^l$. We have

$$\frac{3p}{5} \leqq p-l \leqq \operatorname{card} I_i^l \leqq 3p.$$

Moreover

$$\frac{\mu pk}{2} \leqq \mu pk - \mu(\theta+2k)\,r \leqq \sum_{j\in I} a_j - \sum_{\substack{l'<l\\ i\leqq k}} a_{j_{i,l'}} = \sum_{j\in I^l} a_j \leqq \mu pk$$

hence, if $\mu' = (\sum_{j\in I^l} a_j)/pk$, $\frac{\mu}{2} \leqq \mu' \leqq \mu$. Then the existence of the family $(j_{i,l})_{i\leqq k}$ follows from Lemma 5 which concludes the proof.

The following lemma will be essential to sharpen the blocking methods. It is one of the main ideas of this paper.

Lemma 17. *Let q and k be two integers, with $8k\leqq q$ and $k\geqq 5$. Let $(a_i)_{i\leqq q}$, $(f_i^\tau)_{i\leqq q}$, $\tau=1,2,3,4$ five families of non-negative real numbers. Let θ be a real, $\theta\geqq k$. Set $\mu=q^{-1}(\sum_{i\leqq q} a_i)$. Suppose*

$$\sum_{a_i\geqq\theta\mu} a_i \leqq \tfrac{1}{2}\sum_{i\leqq q} a_i. \tag{4.12}$$

Then there exist integers $j_1,\ldots,j_k\in[1,q]$ such that the following properties are satisfied if we set $j_0=1$, $j_{k+1}=q$.

$$\textit{For all } 0\leqq l\leqq k, \quad j_{l+1}-j_l \geqq 2. \tag{4.13}$$

$$\textit{For all } 0\leqq l\leqq k, \quad j_{l+1}-j_l \leqq \frac{3q}{k}. \tag{4.14}$$

$$\frac{k\mu}{18} \leqq \sum_{l=1}^{k} a_{j_l} \leqq 3(3\theta+2k)\mu. \tag{4.15}$$

$$For\ \tau=1,2,3,4, \qquad \sum_{l\leqq k} f_{j_l}^{\tau} \leqq \frac{30(3\theta+2k)}{q} \sum_{i\leqq q} f_i^{\tau}. \tag{4.16}$$

Proof. 1st Step: We are going to show that there exist $I_1, I_2, \ldots, I_k$ such that if $p=\left[\frac{q-q/k}{2k-1}\right]$ the following conditions are satisfied:

$$\text{For } 1\leqq i\leqq k-1, \qquad l\in I_i,\ l'\in I_{i+1} \Rightarrow l'-l\geqq 2. \tag{4.17}$$

$$\text{For } 1\leqq i\leqq k-1, \qquad l\in I_i,\ l'\in I_{i+1} \Rightarrow l'-l\leqq \frac{3q}{k}. \tag{4.18}$$

$$\text{For } 1\leqq i\leqq k-1, \qquad p\leqq \operatorname{card} I_i \leqq 3p. \tag{4.19}$$

$$\sum_{i\in I} a_i \geqq \frac{\mu q}{3}, \qquad \text{where } I=\bigcup_{i=1}^{k} I_i. \tag{4.20}$$

$$i\in I \Rightarrow a_i \leqq \theta\mu. \tag{4.21}$$

It is easily checked that $\frac{q}{3k}\leqq p\leqq\frac{q}{2k}$ (and hence $p\geqq 3$).

Let $J=\{i\in[1,q];\ a_i\leqq\theta\mu\}$. Since $\theta\geqq k$, it follows that $\operatorname{card} J\geqq q-\frac{q}{k}$. Moreover (4.12) implies that $\sum_{i\in J} a_i\geqq\frac{\mu q}{2}$. Since $p(2k-1)\leqq\operatorname{card} J$, we can enumerate in a increasing way the first $p(2k-1)$ elements of J by $n_1,\ldots,n_{p(2k-1)}$. For $1\leqq l\leqq k-1$, let $J_l=\{n_i,\ p(2l-1)\leqq i<2pl\}$. For each $1\leqq l\leqq k-1$, let $i_l\in J_l$ such that $a_{i_l}=\operatorname{Inf}\{a_i;\ i\in J_l\}$.

Then $pa_{i_l}\leqq\sum_{i\in I_l} a_i$ and hence $\sum_{l=1}^{k-1} a_{i_l}\leqq\frac{1}{p}\sum_{i\in J} a_i$. Now set $i_0=0$, $i_k=q$, set $I_l=]i_{l-1}, i_l[\cap J$ for $1\leqq l\leqq k$, $I=\bigcup_l I_l$ and let us check (4.17) to (4.20). First, (4.17) is obvious. If $1\leqq s\leqq k-1$; $l\in I_s$, $l'\in I_{s+1}$, we have, since $i_s\in J_s$, $i_{s-1}\in J_{s-1}$:

$$l'-l\leqq i_s-i_{s-1}\leqq n_{2ps}-n_{p(2s-3)}.$$

But since $\operatorname{card}\{[1,q]\setminus J\}\leqq q/k$ it is clear that $n_{2ps}-n_{p(2s-3)}\leqq 3p+\frac{q}{k}\leqq\frac{3q}{k}$ which shows (4.19). It is obvious that $p\leqq\operatorname{card} I_l\leqq 3p$ for each l. Finally, (4.20) comes from the fact that $\sum_{i\in I} a_i=\sum_{i\in J} a_i-\sum_{l=1}^{k} a_{i_l}\geqq\sum_{i\in J} a_i-\frac{1}{p}\sum_{i\in J} a_i\geqq\frac{2}{3}\sum_{i\in J} a_i\geqq\frac{\mu q}{3}$.

2nd Step: Let $\mu'=\frac{1}{kp}\sum_{i\in J} a_j$. Then $\frac{\mu}{3}\leqq\mu'\leqq 3\mu$ since $kp\geqq\frac{q}{3}$. For each $j\in J$, we have $a_j\leqq\theta\mu\leqq 3\theta\mu'$. Set $r=\left[\frac{q}{6(3\theta+2k)}\right]$. If $r\leqq 4$ then (4.16) is automatically satisfied and it is easy to conclude. If $r\geqq 1$, then it follows from Lemma 16 that for all

$i \leqq k$, there exists a family $(j_{i,l})_{l \leqq r}$ of distinct elements of I_i such that for all $l \leqq r$

$$\frac{k\mu}{18} \leqq \sum_{i \leqq k} a_{j_{i,l}} \leqq 3(3\theta + 2k)\mu.$$

Let $A_\tau = \left\{ l \leqq r, \ \sum_{i \leqq k} f_{j_{i,l}}^\tau > \frac{4}{r} \sum_{i \leqq n} f_i^\tau \right\}$. Then card $A_\tau < \frac{r}{4}$. Hence there exist $l_0 \leqq r$ such that $l_0 \notin A_1 \cup A_2 \cup A_3 \cup A_4$. If we set $j_l = j_{i,l_0}$ for $i \leqq k$, then it is clear that (4.15) and (4.16) are satisfied, since $r \geqq \left[\frac{q}{6(3\theta + 2k)}\right] \geqq 4$ and hence $\frac{1}{r} \leqq \frac{15(3\theta + 2k)}{2q}$.

We gave rather precise bounds in Lemma 17, because we feel that it is of independant interest, and that this can be done at a negligable extra cost. In the sequel we shall use it with $\theta = K_6 k$, where K_6 is a universal constant to be defined later. Hence there is a universal constant K_5 such that (4.15) and (4.16) become

$$K_5^{-1} \frac{k}{q} \sum_{i \leqq q} a_i \leqq \sum_{l \leqq k} a_{i_l} \leqq K_5 \frac{k}{q} \sum_{i \leqq q} a_i \tag{4.22}$$

$$\text{for } \tau = 1, 2, 3, 4, \qquad \sum_{l \leqq k} f_{i_l}^\tau \leqq K_5 \frac{k}{q} \sum_{i \leqq q} f_{i_l}^\tau. \tag{4.23}$$

5. Bounds for m-Dependent Random Variables

Let $X_1, \ldots, X_n$ be a sequence of m-dependent random variables with mean zero. Suppose that for each i we have a decomposition $X_i = \bar{X}_i + X'_i$ as in Sect. III. Let

$$b = \sum_{i \leqq n} E\|X'_i\|^2, \qquad c = \sum_{i \leqq n} E\|\bar{X}_i\|^3, \qquad d = \sum_{i \leqq n} E\|\bar{X}_i\|^{7/2},$$

$$e = \sum_{i \leqq n} (E\|X'_i\|^2)^{7/4}, \qquad \bar{B} = \sum_{i \leqq n} E\|X_i\|^2, \qquad B = E\|\sum_{i \leqq n} X_i\|^2.$$

Let T be a Gaussian random variable with the same covariance as $X = \sum_{i \leqq n} X_i$. Suppose that for $s, \delta \geqq 0$, $P(s \leqq \|T\| \leqq s + \delta) \leqq G\delta$. Set $\Delta = \sup_t |P(\|X\| \leqq t) - P(\|T\| \leqq t)|$.

Theorem 18.

$$\Delta \leqq K(R^{4/3} m^{1/3} G^{2/3} b^{1/3} + R^{10/9} m^{4/9} G^{8/9} \bar{B}^{1/9} c^{2/9} + R^{8/9} m^{10/27} G^{20/27} \bar{B}^{1/9} (d + f')^{4/27})$$

where K is a universal constant.

Proof. Let $q = [n/m] + 1$. For $1 \leqq i \leqq q-1$, let $A_i = \sum_{j=m(i-1)+1}^{mi} X_j$, $A_q = \sum_{j=m(q-1)+1}^{n} X_j$. Let $a_i = E\|A_i\|^2$. Let k be an integer, which will be chosen later, such that $5 \leqq k$ and $8k \leqq q$. By a much simpler form of Lemma 17, which is used by Egorov [1], and that we leave to the reader, there exist $i_1, \ldots, i_k \leqq q$ satisfying (4.17) and (4.18) and $\sum_{l \leqq k} a_{i_l} \leqq K_5 \frac{k}{q} \sum_{i \leqq q} a_i$. Let, for $j \leqq k$, $Z_j = A_{i_j}$ and for $j \leqq k+1$, $Y_j = \sum_{i_{j-1} < l < i_j} A_{l'}$. Since the X_i are m-dependent, the $(Z_j)_{j \leqq k}$ and the $(Y_j)_{j \leqq k+1}$ are independent. Let $Z = \sum_{j \leqq k} Z_j$, $Y = \sum_{j \leqq k+1} Y_j$. Since $X = Y + Z$, it follows from Lemma 13 that one can write $T = U + V$, where U and V are Gaussian and have the same covariance as Y and Z respectively.

For $t \in \mathbb{R}$ one has

$$P(\|X\| \leqq t) - P(\|T\| \leqq t) \leqq P(\|Y\| \leqq t + \varepsilon) + P(\|Z\| \leqq \varepsilon) - P(\|U\| \leqq t - \varepsilon) + P(\|V\| \leqq \varepsilon)$$

so

$$P(\|X\| \leqq t) - P(\|T\| \leqq t) \leqq 2\varepsilon G + P(\|Z\| \leqq \varepsilon) + P(\|V\| \leqq \varepsilon) + \Delta'$$

where $\Delta' = \sup_t |P(\|Y\| \leqq t) - P(\|U\| \leqq t)|$. Similar estimates in the other direction give

$$\Delta \leqq 2\varepsilon G + P(\|Z\| \leqq \varepsilon) + P(\|V\| \leqq \varepsilon) + \Delta'. \tag{5.2}$$

From Lemmas 12 and 14, it is clear that one can write for all j $Y_j = \bar{Y}_j + Y_j'$, where $\|\bar{Y}_j\| \, \|Y_j'\| = 0$, and

$$\sum_{j \leqq k+1} E\|Y_j'\|^2 \leqq K_3 K_4 m R b \tag{5.3}$$

$$\sum_{j \leqq k+1} E\|\bar{Y}_j\|^3 \leqq 3^{1/2} K_3 K_4 \left(\frac{q}{k}\right)^{1/2} m^{3/2} R^{3/2} c \tag{5.4}$$

$$\sum_{j \leqq k+1} E\|\bar{Y}_j\|^{7/2} \leqq 3^{3/4} K_3 K_4 \left(\frac{q}{k}\right)^{3/4} m^{7/4} R^{7/4} d. \tag{5.5}$$

With some easy computations using Lemma 5:

$$\sum_{j \leqq k+1} (E\|Y_j'\|^2)^{7/4} \leqq 3^{3/4} K_1^{7/4} K_2^{7/4} \left(\frac{q}{k}\right)^{3/4} m^{7/4} R^{7/4} e. \tag{5.6}$$

Hence Theorem 10 shows that there exists a universal constant K_{14} with

$$\Delta' \leqq K_{14} \left(M_V + \left(\frac{q}{k}\right)^{1/6} m^{1/2} R^{4/3} c^{1/3} G + m^{1/3} R^{4/3} b^{1/3} G^{2/3} \right.$$
$$\left. + \left(\frac{q}{k}\right)^{1/6} m^{7/18} R (d+e)^{2/9} G^{7/9} \right). \tag{5.7}$$

Let $\tilde{B} = \sum_{i \leqq q} a_i$. We have $E\|Z\|^2 \leqq K_5 \frac{k}{q} \tilde{B}$. Moreover, since V is Gaussian with the same covariance as Z, $E\|V\|^2 \leqq R E\|Z\|^2$. Hence

$$P(\|V\| \leqq \varepsilon) \leqq \frac{K_5}{\varepsilon^2} \frac{k}{q} R \tilde{B}$$

So we get

$$M_V=\operatorname{Inf}_{\varepsilon}(G\varepsilon+P(\|V\|\leqq\varepsilon))\leqq 2K_5^{1/3}\left(\frac{k}{q}\right)^{1/3} R^{1/3}\tilde{B}^{1/3}G^{2/3}$$

by taking $\varepsilon^3=K_5\frac{k}{q}mR\bar{B}G^{-1}$.

Since (5.1) is true for all ε; and $P(\|Z\|\leqq\varepsilon)\leqq\frac{K_5}{\varepsilon^2}\frac{k}{q}\tilde{B}$ we get, with $K_{15}=2K_5^{1/3}K_{14}$

$$\Delta\leqq K_{15}\left(\left(\frac{k}{q}\right)^{1/3} R^{1/3}\tilde{B}^{1/3}G^{2/3}+\left(\frac{q}{k}\right)^{1/6} m^{1/2}R^{4/3}c^{1/3}G\right.$$
$$\left.+\left(\frac{q}{k}\right)^{1/6} m^{7/18}R(d+e)^{2/9}G^{7/9}+m^{1/3}R^{4/3}b^{1/3}G^{2/3}\right). \tag{5.8}$$

Let

$$k=[q(m^{1/2}R\tilde{B}^{-1/3}c^{1/3}G^{1/3}+m^{7/18}R^{2/3}\tilde{B}^{-1/3}G^{1/9}(d+e)^{2/9})^2]. \tag{5.9}$$

We shall not prove in details that if K_{15} is large enough one can suppose $k\geqq 5$ and $8k\leqq q$. The argument is rather tedious. The method is to show that if $k<4$ or $8k>q$ the right-hand side of (5.1) is $\geqslant 1$, which needs a lot of calculations. It uses the fact that since $P(\|T\|^2\geqq 2E\|T\|^2)\leqq\frac{1}{2}$ we have

$$\tfrac{1}{2}\leqq P(\|T\|\leqq(2E\|T\|^2)^{1/2})\leqq(2E\|T\|^2)^{1/2}G$$

and hence $1\leqq 8E\|T\|^2G^2$.

We have $\tilde{B}\leqq K_3 mR\bar{B}$. If $t\geqq 4$, then $\frac{4t}{5}\leqq[t]\leqq t$. Moreover, for $a,b>0$, we have $(a+b)^{-1}\leqq a^{-1}+b^{-1}$. If we use these elementary inequalities it is easy to substitute (5.9) into (5.8) to get (5.11). Q.E.D.

Let us now specialize this result. Suppose that we have a sequence (X_i) of m-dependent random variables, with $\sup_i E\|X_i\|^{7/2}<\infty$. Let $B_n=E\|\sum_{i\leqq n}X_i\|^2$. Let T_n be a Gaussian random variable with same covariance as the covariance of $B_n^{-1/2}(\sum_{i\leqq n}X_i)$. Suppose T_n satisfies (1.2) with a constant G_n. Then

$$\Delta_n=\sup_t|P(B_n^{-1/2}\|\sum_{i\leqq n}X_i\|<t)-P(\|T_n\|<t)|=O(n^{1/3}B_n^{-4/9}(G_n^{8/9}+G_n^{20/27})).$$

In the optimal case where $B_n\geqq\alpha n$ and G_n is bounded, then $\Delta_n=O(n^{-1/9})$.

We are now going to show that under stronger hypothesis, we can establish an estimate for Δ which will give a sharper order of convergence. Let us assume the following

"There exists R' such that for each F-valued random variable X in L_F^2, the unique Gaussian random variable T with the same covariance as X satisfies $E\|X\|^2\leqq R'E\|T\|^2$" (5.10)

From the proof of Proposition (3.3) in [6], one sees that this assumption is equivalent to say that F is of co-type 2. Hence by known results, F is isomorphic to a Hilbert space. But since the definition of Δ heavily depends on the norm there is some extra generality by not assuming F to be isometric to a Hilbert space.

For a Gaussian random variable satisfying (1.2) let $G(T)$ be the smallest possible constant. We have, for $a>0$, $G(aT)=a^{-1}G(T)$. We have shown in the preceeding proof that $G^2(T)E\|T\|^2 \geqq \frac{1}{8}$. It is easy to show, even in Hilbert spaces that $G^2(T)E\|T\|^2$ can be large. It is also possible to show in Hilbert space that $G^2(T)E\|T\|^2$ remains bounded when T belongs to a finite dimensional vector space.

Let us keep the notations of Theorem 18 and its proof. For $i \leqq q-1$, let C_i be a Gaussian random variable with the same covariance as A_i, and such that the C_i are independent. Let us assume that there exists L such that

$$\text{for all } \alpha_1, \ldots, \alpha_{q-1} \in \mathbb{R}, \quad G^2\Big(\sum_{i \leqq q-1} \alpha_i C_i\Big) E\Big\| \sum_{i \leqq q-1} \alpha_i C_i \Big\|^2 \leqq L. \tag{5.11}$$

Theorem 19. *Under these assumptions*

$$\begin{aligned} \Delta \leqq K'(N^{1/2}\bar{B}^{1/8}(\operatorname{Log} Q)^{1/8}(R^{13/8}R'^{3/4}m^{5/8}c^{1/4} + R^{13/12}R'^{7/24}m^{5/12}(d+e)^{1/6} \\ + R^{3/2}R'^{1/3}m^{1/3}N^{1/3}b^{1/3}) \end{aligned} \tag{5.12}$$

where $N = G^2 + LB^{-1}$

$$Q = 3 + (\bar{B}^{1/8}N^{1/2}(m^{5/8}c^{1/4} + m^{5/12}(d+e)^{1/6}))^{-1}$$

and K' *is a universal constant.*

Proof. From Lemmas 12 and 14 we can write $A_i = \bar{A}_i + A'_i$, where $\|\bar{A}_i\| \, \|A'_j\| = 0$, the $(\bar{A}_i)$ are independent and the (A'_i) are independent, and such that

$$\sum_{i \leqq q} E\|A'_i\|^2 \leqq K_3 K_4 m R b \tag{5.13}$$

$$\sum_{i \leqq q} E\|\bar{A}_i\|^3 \leqq K_3 K_4 m^2 R^{3/2} c \tag{5.14}$$

$$\sum_{i \leqq q} E\|\bar{A}_i\|^{7/2} \leqq K_3 K_4 m^{5/2} R^{7/4} d \tag{5.15}$$

$$\sum_{i \leqq q} (E\|A'_i\|^2)^{7/4} \leqq K_3 K_4 m^{5/2} R^{7/4} e. \tag{5.16}$$

Let k be an integer such that $5 \leqq k$ and $5k \leqq q$, which will be specified later. Let $I = \left\{ i \leqq q;\ a_i \geqq \frac{k}{q} \sum_{i \leqq q} a_i \right\}$. The choice of k will also be such that $\sum_{i \in I} a_i \leqq \frac{1}{2} \sum_{i \leqq q} a_i$. Then let $i_1, \ldots, i_k$ the integers given by Lemma 17. Define Z_i, Y_i, Y_i, Y'_i as in the proof of Theorem 18. We have for all $\varepsilon > 0$.

$$\Delta \leqq 2\varepsilon G + P(\|Z\| \leqq \varepsilon) + P(\|V\| \leqq \varepsilon) + \Delta'$$

We have

$$P(\|Z\| \leqq \varepsilon) \leqq P(\|V\| \leqq \varepsilon) + \Delta''$$

where $\Delta'' = \sup |P(\|Z\| \leqq t) - P(\|V\| \leqq t)|$.

Let $\tilde{B}=\sum_{i\leq q}^{t} a_i$, $B=E\|\sum_{i\leq n} X_i\|^2$. Since the A_i are 1-dependent, we have $B\leq K_3 R\tilde{B}$, and $\tilde{B}\leq K_3 mR\tilde{B}$

$$E\|Z\|^2=\sum_{l\leq k} a_{i_l}\leq K_5\frac{k}{q}\tilde{B}$$

$$E\|Z\|^2\geq K_5^{-1}\frac{k}{q}\tilde{B}\geq (K_3K_5)^{-1}\frac{k}{q}R^{-1}B.$$

By (5.10) one had $E\|V\|^2\geq R'^{-1}E\|Z\|^2$, so by hypothesis (5.11)

$$G^2(Z)\leq K_3K_5RR'LB^{-1}\left(\frac{q}{k}\right).$$

Now, from (4.23), (5.13) to (5.16) and Theorem 10, one sees that there exists a universal constant K_{17} such that

$$\begin{aligned}\Delta'\leq K_{17}\Bigg(&R^{3/2}R'^{1/3}m^{1/3}(LB^{-1})^{1/3}b^{1/3}+R^{11/6}R'^{1/2}m^{2/3}(LB^{-1})^{1/2}c^{1/3}\left(\frac{q}{k}\right)^{1/6}\\&+R^{25/18}R'^{7/18}m^{7/18}(LB^{-1})^{7/18}(d+e)^{2/9}\left(\frac{q}{k}\right)^{1/6}\Bigg).\end{aligned}\tag{5.17}$$

We have $\Delta\leq 2M_V+\Delta'+\Delta''$. We have

$$P(\|V\|\geq\varepsilon)\leq\exp\left(-\frac{\varepsilon^2}{K_2\|V\|_2}\right)\leq\exp\left(-\frac{\varepsilon^2}{K_2K_3K_5\frac{k}{q}R\tilde{B}}\right).$$

Let $K_{18}=(K_2K_3K_5)^{1/2}$. We have $M_V\leq G\varepsilon_0+P(\|V\|\geq\varepsilon_0)$ for

$$\varepsilon_0=K_{18}\left(\frac{k}{q}\right)^{1/2}R\tilde{B}^{1/2}(\operatorname{Log}Q)^{1/2}$$

where $N=G^2+LB^{-1}$ and

$$Q=3+(\tilde{B}^{1/8}N^{1/2}(m^{1/2}c^{1/4}+m^{7/24}(d+e)^{1/6}))^{-1}.$$

Since $R\geq 1$, $R'\geq 1$, we get by substitution, and from (5.7) and (5.17) (using Lemma 4 again), that there exists a universal constant K_{19} such that

$$\begin{aligned}\Delta\leq K_{19}\Bigg(&\left(\frac{k}{q}\right)^{1/2}R^{1/2}\tilde{B}^{1/2}G(\log Q)^{1/2}+Q^{-1}+R^{3/2}R'^{1/3}m^{1/3}N^{1/3}b^{1/3}\\&+R^{11/6}R'^{1/2}m^{2/3}N^{1/2}c^{1/3}\left(\frac{q}{k}\right)^{1/6}\\&+R^{25/18}R'^{7/18}m^{7/18}N^{7/18}(d+e)^{2/9}\left(\frac{q}{k}\right)^{1/6}\Bigg).\end{aligned}\tag{5.18}$$

Now let

$$k=[q\tilde{B}^{-3/4}(\log Q)^{-3/4}(R^2R'^{3/4}mN^{3/4}G^{-3/2}c^{1/2}+R^{4/3}R'^{7/12}m^{7/12}N^{7/12}G^{-7/6}(d+e)^{1/6}].\tag{5.19}$$

Long and tedious computations show that if we suppose K' large enough, (5.12) is automaticly satisfied of $k \leqq 5$ or $8k \geqq q$. Still worse computations show that there exists a universal constant K_6 such that if the right-hand side of (5.1) is $\leqslant 1$ and if

$$I = \left\{ i \leqq q' : a_i \geqq K_6 \frac{k}{q} \tilde{B} \right\} \quad \text{we have } \sum_{i \in I} a_i \leqq \tfrac{1}{2} \tilde{B}.$$

Now we substitute (5.19) into (5.18) we use the facts that $\tilde{B} \leqq K_3 m R \bar{B}$ and the function $t \to t \log(3 + a t^{-1})$ is increasing in R^+. Then we obtain (5.12). Q.E.D.

To see what is the order of convergence obtained in the best cases let us for example suppose that X_n is a Hilbert-space valued sequence of m-dependent random variables, such that $\sup_n E\|X_n\|^{3/2} < \infty$. Let $B_n = E\|\sum_{i \leqq n} X_i\|^2$, and G_n be the constant associated in (1.2) with the Gaussian random variable T_n of the same covariance as $B_n^{-1/2} \sum_{i \leqq n} X_i$. Suppose that there exists a Gaussian random variable T' such that for all p, $\sum_{1=p}^{p+m} X_i$ has the same covariance as $\alpha_i T$ ($\alpha_i \in \mathbb{R}$). Then (5.11) holds, and Theorem 19 gives

$$\begin{aligned} \Delta_n &= \sup_t |P(B_n^{-1/2} \|\sum_{i \leqq n} X_i\| < t) - P(\|T_n\|) < t)| \\ &= O(G_n (n^{3/2} B_n^{-1/2} + n^{7/24} B_n^{-10/24}) (\log Q_n^{1/8}) \end{aligned}$$

where

$$Q_n = 3 + (G_n (n^{3/8} B_n^{-1/2} + n^{7/24} B_n^{-10/24})^{-1}).$$

Hence in the good case where G_n is bounded and $B_n \geqq \alpha n$, we get $\Delta_n = O(n^{-1/8} (\log n)^{1/8})$. Hence, due to the use of an optimal blocking method, through Lemma 17, this result is comparatively sharp.

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