Z. Wahrscheinlichkeitstheorie verw. Geb. 15, 177–179 (1970)

A Dissipative Transformation with a Trivial Tail-Algebra

F. H. SIMONS

Summary. In [1], an example was given of a measure-preserving dissipative transformation T in a σ-finite measure space $(X, \mathscr{R}, \mu)$, such that T is conservative in the measure space $(X, \mathscr{R}_\infty, \mu)$ where $\mathscr{R}_\infty = \bigcap_{n=0}^{\infty} T^{-n} \mathscr{R}$. Here we shall show that for this transformation we actually have $\mathscr{R}_\infty = \{\emptyset, X\}$ $[\mu]$.

For every integer n let I_n be the set $\{(x, n) \mid 0 \leqq x < 1\}$, $\mathscr{R}_n$ be the σ-algebra of Borelsets in I_n and μ_n be the Lebesgue measure on $(X_n, \mathscr{R}_n)$. Define

$$X = \bigcup_{n=-\infty}^{+\infty} I_n$$

$$\mathscr{R} = \{A \mid \forall_n (A \cap I_n \in \mathscr{R}_n)\}$$

$$\mu(A) = \sum_{n=-\infty}^{+\infty} \mu_n(A \cap I_n) \qquad \text{for all } A \in \mathscr{R},$$

then $(X, \mathscr{R}, \mu)$ is a σ-finite measure space.

For $(x, n) \in X$ define

$$T(x, n) = \begin{cases} (2x, n+1) & \text{if } 0 \leqq x < \frac{1}{2} \\ (2x-1, n+2) & \text{if } \frac{1}{2} \leqq x < 1, \end{cases}$$

then T is a measurable measure-preserving transformation in $(X, \mathscr{R}, \mu)$, and since each of the sets I_n is wandering, T is dissipative.

Let $\mathscr{L}$ be the class of all functions f on X such that f is integrable on each interval I_n. If f is $\mathscr{R}_\infty$-measurable, f is $T^{-k}\mathscr{R}$-measurable for every $k \geqq 0$; therefore for every $k \geqq 0$ we must have for every sequence $(b_{k+1}, b_{k+2}, \ldots)$ with $b_i = 0$ or $b_i = 1$

$$f\left(\sum_{i=1}^{k} \frac{a_i}{2^i} + \sum_{i=k+1}^{\infty} \frac{b_i}{2^i}, n\right) = f\left(\sum_{i=1}^{k} \frac{a_i'}{2^i} + \sum_{i=k+1}^{\infty} \frac{b_i}{2^i}, n'\right) \tag{1}$$

as soon as $n + \sum_{i=1}^{k} a_i = n' + \sum_{i=1}^{k} a_i'$, where a_i and a_i' are 0 or 1.

We shall prove the triviality of $\mathscr{R}_\infty$ by showing that every $\mathscr{R}_\infty$-measurable function $f \in \mathscr{L}$ necessarily must be constant almost everywhere on X. Because of (1) it is sufficient to show that such a function is almost everywhere constant on any of the intervals I_n.

Let $[a_1 \ldots a_k[$ $(a_i = 0$ or $a_i = 1)$ denote the dyadic interval of those points of I_n for which the dyadic expansion starts with $a_1 \ldots a_k$. For any $f \in \mathscr{L}$ which is $\mathscr{R}_\infty$-measurable we first show

$$\int_{[a_1 \ldots a_k 0[} f \, d\mu_n = \int_{[a_1 \ldots a_k 1[} f \, d\mu_n. \tag{2}$$

For any natural number m we have

$$\Big|\int_{[a_1\dots a_k 0[} f\,d\mu_n - \int_{[a_1\dots a_k 1[} f\,d\mu_n\Big|$$
$$= \Big|\sum \int_{[a_1\dots a_k 0 b_{k+2}\dots b_{k+2m}[} f\,d\mu_n - \sum \int_{[a_1\dots a_k 1 b_{k+2}\dots b_{k+2m}[} f\,d\mu_n\Big|$$

where both sums have to be taken over all sequences $(b_{k+2}\dots b_{k+2m})$ with $b_i=0$ or $b_i=1$. Let j be given such that $0\leqq j\leqq 2m$. In the first sum there are $\binom{2m-1}{j}$ intervals for which exactly j of the b_i are 1, and in the second sum there are $\binom{2m-1}{j-1}$ intervals for which exactly $j-1$ of the b_i are 1. Because of (1) the contribution of $2\min\left(\binom{2m-1}{j},\binom{2m-1}{j-1}\right)$ of these intervals to the difference vanishes. Hence

$$\Big|\int_{[a_1\dots a_k 0[} f\,d\mu_n - \int_{[a_1\dots a_k 1[} f\,d\mu_n\Big| \leqq \int_{B_m} |f|\,d\mu_n$$

where B_m is the union of $\sum_{j=0}^{2m}\left|\binom{2m-1}{j}-\binom{2m-1}{j-1}\right|$ intervals of measure 2^{-k-2m}.

$$\begin{aligned}
\mu_n(B_m) &= \sum_{j=0}^{2m}\left|\binom{2m-1}{j}-\binom{2m-1}{j-1}\right|\frac{1}{2^{k+2m}}\\
&= 2\sum_{j=0}^{m}\left(\binom{2m-1}{j}-\binom{2m-1}{j-1}\right)\frac{1}{2^{k+2m}}\\
&= \frac{1}{2^{k-1}}\sum_{j=0}^{m}\frac{1}{2^{2m}}\binom{2m}{j}\left(1-\frac{j}{m}\right)\\
&= \frac{1}{2^{k-1}}\left(\frac{1}{2^{2m}}\sum_{j=0}^{m}\binom{2m}{j}-\frac{1}{m\,2^{2m}}\sum_{j=0}^{m} j\binom{2m}{j}\right).
\end{aligned}$$

Since

$$\frac{1}{m\,2^{2m}}\sum_{i=0}^{2m} i\binom{2m}{i}=1$$

and

$$\sum_{i=1}^{m} i\binom{2m}{i}=\sum_{i=m+1}^{2m} i\binom{2m}{i} \qquad \left(\text{for } i\binom{2m}{i}=(2m-i+1)\binom{2m}{2m-i+1},\quad i=1,\dots,m\right),$$

the second sum between the brackets is $\frac{1}{2}$ for every m. For the first sum we have

$$\begin{aligned}
\frac{1}{2^{2m}}\sum_{j=0}^{m}\binom{2m}{j} &= \frac{1}{2^{2m+1}}\sum_{j=0}^{2m}\binom{2m}{j}+\frac{1}{2^{2m+1}}\binom{2m}{m}\\
&= \frac{1}{2}+\frac{\sqrt{m}\,(2m)!}{2^{2m}(m!)^2}\cdot\frac{1}{2\sqrt{m}}
\end{aligned}$$

which, because of Wallis's product, tends to $\frac{1}{2}$ if $m\to\infty$. This proves $\mu_n(B_m)\to 0$ if $m\to\infty$, and thereby (2).

Now let $f \in \mathscr{L}$ be $\mathscr{R}_\infty$-measurable. If for some n we have $\int_{I_n} f(x,n)\,d\mu \neq 0$, without loss of generality we may assume $\int_{I_n} f\,d\mu_n = 1$.

For every $A \in \mathscr{R}_n$ define $P(A) = \int_A f\,d\mu_n$. Let $F(x) = P([0, x[)$ be the distribution function of P on I_n, then it easily follows from (2) that we must have

$$F\left(\sum_{i=1}^{k} \frac{a_i}{2^i}\right) = \sum_{i=1}^{k} \frac{a_i}{2^i}$$

and therefore $F(x) = x$ on I_n.

It follows that $P = \mu_n$ on $(I_n, \mathscr{R}_n)$ and therefore $f(x,n) = 1$ μ_n-almost everywhere on I_n. This proves the triviality of $\mathscr{R}_\infty$.

Reference

1. Helmberg, G., Simons, F. H.: On the conservative parts of the Markov processes induced by a measurable transformation. Z. Wahrscheinlichkeitstheorie verw. Geb. **11**, 165–180 (1969).

F. H. Simons
Technological University Eindhoven
Department of Mathematics
P. O. Box 513
Eindhoven, Netherlands

(Received June 9, 1969)