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Spitzer's Test for the Cauchy Process on the Line

By

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DVORETZKY-ERDÖS [1] and F. SPITZER [4] obtained the exact estimates of the order of the inferior limits at time points ∞ and 0 for the oscillation of Brownian motion paths. One of the present authors, in [5], has extended the result of [1] to the symmetric stable processes in R^N of index α with $N > \alpha$. The purpose of this note is to give the criteria for the remaining case $N = \alpha = 1$ which correspond to the result of [4] for the case of $N = \alpha = 2$.

Let $\{X(t, w), P_x\}^{\star}$ be a symmetric Cauchy process in R^1, that is, a Markov process in R^1 homogeneous in time and space with the transition probability $P(t, x, E)$ defined by

$$P(t, x, E) = P_x\{X(t, w) \in E\} = \int_E p(t, x - y)\, dy\,,$$

where

$$p(t, x) = \frac{1}{\pi} \frac{t}{t^2 + x^2} \qquad t > 0, \quad x \in R^1.$$

We may assume as usual that its sample paths are right continuous and have left-hand limits everywhere. Let $g(t)$ be a function defined for all large t such that $g(t) > 0$ and $g(t) \downarrow 0$ as $t \uparrow +\infty$.

Definition. (i) We define $g(t) \in \mathfrak{U}_\infty$ or $g(t) \in \mathfrak{L}_\infty$ according as

$P_0\{w$; there exists some $M > 0$ such that $|X(t, w)| \geqq t g(t)$ for all $t \geqq M\}^{\star\star}$

$= 0$ or 1.

(ii) $g(t) \in \mathfrak{U}_0$ or $g(t) \in \mathfrak{L}_0$ according as

$P_0\{w$; there exists some $\delta > 0$ such that $|X(t, w)| \geqq t g(1/t)$ for all $0 < t < \delta\}^{\star\star\star}$

$= 0$ or 1.

Let E be a set on (t, x)-plane defined by

$$E = \left\{(t, x);\ |x| < t \cdot g\left(\frac{1}{t}\right),\ t \geqq 0\right\},$$

then $g(t) \in \mathfrak{U}_0$ if and only if the point $(0, 0)$ is a regular point of E for the space-time Cauchy process.

Theorem. (i) $g(t) \in \mathfrak{U}_\infty$ *or* $\mathfrak{L}_\infty$ *according as*

$\star$ P_x denotes the probability law governing paths starting at $x \in R^1$.

$\star\star$ This probability is always 0 or 1 by the 0—1 law of HEWITT and SAVAGE [2, pp. 493 to 494].

$\star\star\star$ This probability is always 0 or 1 by the 0—1 law of BLUMENTHAL.

$$\int^{\infty} \frac{dt}{t|\log g(t)|} = +\infty \quad or \quad < +\infty .$$

(ii) $g(t) \in \mathfrak{U}_0$ *or* $\mathfrak{L}_0$ *according as*

$$\int^{\infty} \frac{dt}{t|\log g(t)|} = +\infty \quad or \quad < +\infty .$$

Thus, for example, we have

Corollary.

$$P_0\left\{\varliminf_{t\to\infty} \frac{|X(t)|}{t^{-n}} = 0\right\} = 1 ,$$

$$P_0\left\{\varliminf_{t\downarrow 0} \frac{|X(t)|}{t^{n}} = 0\right\} = 1 \qquad \textit{for every} \quad n > 0$$

and

$$P_0\left\{\varliminf_{t\to\infty} \frac{|X(t)|}{t-(\log t)\alpha} = +\infty\right\} = 1 ,$$

$$P_0\left\{\varliminf_{t\downarrow 0} \frac{|X(t)|}{t(\log t^{-1})\alpha} = +\infty\right\} = 1 \quad \textit{for every} \quad \alpha > 0 .$$

The proof is based on the following lemma which corresponds to lemma 1 of [*4*]. Its proof runs essentially on the same line as SPITZER's.

Lemma 1. *Let*

$$H(t_1, t_2 ; \varepsilon) = P_0\left\{\inf_{t_1 \leqq s \leqq t_2} |X(s)| < \varepsilon\right\}.$$

Then

$$H(t_1, t_2; \varepsilon) \sim \frac{\log t_2 - \log t_1}{\log \varepsilon^{-1}}$$

when $\varepsilon \to 0$.*

Proof. Let $\sigma_\varepsilon = \inf\{t; |X(t)| < \varepsilon\}$, then by the well-known first passage time relation

$$P_x\{X(t) \in (-\varepsilon, \varepsilon)\} = \int_0^t \int_{-\varepsilon}^{\varepsilon} P_y\{X(t-s) \in (-\varepsilon, \varepsilon)\} \cdot P_x\{\sigma_\varepsilon \in ds, X(\sigma_\varepsilon) \in dy\} .$$

It follows from the shape of the Cauchy distribution that

$$P_0\{X(t-s) \in (0, 2\varepsilon)\} \leqq P_y\{X(t-s) \in (-\varepsilon, \varepsilon)\} \leqq P_0\{X(t-s) \in (-\varepsilon, \varepsilon)\}$$

provided $|y| < \varepsilon$. Hence

$$\int_0^t P_0\{X(t-s) \in (0, 2\varepsilon)\} P_x\{\sigma_\varepsilon \in ds\} \leqq P_x\{X(t) \in (-\varepsilon, \varepsilon)\}$$
$$\leqq \int_0^t P_0\{X(t-s) \in (-\varepsilon, \varepsilon)\} P_x\{\sigma_\varepsilon \in ds\} .$$

Taking Laplace transforms, we have

$$G_\lambda\{0, (0, 2\varepsilon)\} E_x(e^{-\lambda\sigma_\varepsilon}) \leqq G_\lambda\{x, (-\varepsilon, \varepsilon)\} \leqq G_\lambda\{0, (-\varepsilon, \varepsilon)\} E_x(e^{-\lambda\sigma_\varepsilon}) ,$$

* The symbol $f(\varepsilon) \sim g(\varepsilon)$ $(\varepsilon \downarrow 0)$ means $\lim_{\varepsilon\downarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 1$.

where

$$G_\lambda\{x, E\} = \int_0^\infty e^{-\lambda t} P(t, x, E)\, dt = \int_E g_\lambda(x-y)\, dy$$

with

$$g_\lambda(x) = \int_0^\infty e^{-\lambda t} p(t, x)\, dt = \frac{1}{\pi}\int_0^\infty \frac{\cos x\xi}{\lambda+\xi}\, d\xi .$$

Therefore

$$\frac{G_\lambda\{x, (-\varepsilon, \varepsilon)\}}{G_\lambda\{0, (-\varepsilon, \varepsilon)\}} \leqq E_x(e^{-\lambda\sigma_\varepsilon}) \leqq \frac{G_\lambda\{x, (-\varepsilon, \varepsilon)\}}{G_\lambda\{0, (0, 2\varepsilon)\}} . \tag{1}$$

Fix $t_1 > 0$, then $H(t_1, t_1 + t; \varepsilon)$ is an increasing function of $t \geqq 0$ whose Stieltjes measure is denoted by $H(t_1, t_1 + dt; \varepsilon)$. By the Markov property,

$$H(t_1, t_1+t;\, \varepsilon) = P_0\left\{\inf_{t_1 \leqq s \leqq t_1+t} |X(s)| < \varepsilon\right\} = \int_{-\infty}^\infty P_x\left\{\inf_{0\leqq s\leqq t} |X(s)| < \varepsilon\right\} p(t_1, x)\, dx$$
$$= \int_{-\infty}^\infty P_x\{\sigma_\varepsilon < t\}\, p(t_1, x)\, dx$$

and so

$$\int_0^\infty e^{-\lambda t} H(t_1, t_1 + dt;\, \varepsilon) = \int_{-\infty}^\infty E_x(e^{-\lambda\sigma_\varepsilon})\, p(t_1, x)\, dx .$$

By (1)

$$\frac{\int_{-\infty}^\infty p(t_1, x)\, G_\lambda\{x, (-\varepsilon, \varepsilon)\}\, dx}{G_\lambda\{0, (-\varepsilon, \varepsilon)\}} \leqq \int_0^\infty e^{-\lambda t} H(t_1, t_1 + dt;\, \varepsilon) \leqq \frac{\int_{-\infty}^\infty p(t_1, x)\, G_\lambda\{x, (-\varepsilon, \varepsilon)\}\, dx}{G_\lambda\{0, (0, 2\varepsilon)\}} .$$

Since

$$\int_{-\infty}^\infty p(t_1, x)\, G_\lambda\{x, (-\varepsilon, \varepsilon)\}\, dx = \int_0^\infty e^{-\lambda t} dt \int_{-\varepsilon}^{\varepsilon} dy \int_{-\infty}^\infty p(t_1, x)\, p(t, x-y)\, dx$$
$$= \int_0^\infty e^{-\lambda t} dt \int_{-\varepsilon}^{\varepsilon} p(t_1 + t, y)\, dy$$

we have

$$\frac{\int_0^\infty e^{-\lambda t} dt \int_{-\varepsilon}^{\varepsilon} p(t_1 + t, y)\, dy}{G_\lambda\{0, (-\varepsilon, \varepsilon)\}} \leqq \int_0^\infty e^{-\lambda t} H(t_1, t_1 + dt;\, \varepsilon) \leqq \frac{\int_0^\infty e^{-\lambda t} dt \int_{-\varepsilon}^{\varepsilon} p(t_1 + t, y)\, dy}{G_\lambda\{0, (0, 2\varepsilon)\}} .$$

Now, noting

$$p(t_1 + t, y) \leqq p(t_1 + t, 0) = \frac{1}{\pi}\frac{1}{t_1 + t} \leqq \frac{1}{\pi}\frac{1}{t_1},$$

we have

$$\int_0^\infty e^{-\lambda t} dt \int_{-\varepsilon}^{\varepsilon} p(t_1 + t, y)\, dy \sim \frac{2\varepsilon}{\pi}\int_0^\infty e^{-\lambda t} \frac{dt}{t_1 + t} \qquad (\varepsilon \downarrow 0)$$

and

$$G_\lambda\{0, (-\varepsilon, \varepsilon)\} = \int_{-\varepsilon}^{\varepsilon} g_\lambda(x)\, dx = \frac{1}{\pi}\int_0^\infty d\xi \int_{-\varepsilon}^{\varepsilon} \frac{\cos x\xi}{\lambda+\xi}\, dx = \frac{2}{\pi}\int_0^\infty \frac{\sin \varepsilon\xi}{(\lambda+\xi)\,\xi}\, d\xi$$
$$= \frac{2\varepsilon}{\pi}\int_0^\infty \frac{\sin\xi}{(\lambda\varepsilon + \xi)\,\xi}\, d\xi \sim \frac{2\varepsilon}{\pi}\int_0^1 \frac{d\xi}{\lambda\varepsilon+\xi} \sim \frac{2\varepsilon}{\pi}\log\frac{1}{\varepsilon} .$$

Similarly, we have

$$G_\lambda\{0, (0, 2\varepsilon)\} = \frac{1}{\pi}\int_0^\infty \frac{\sin 2\varepsilon\xi}{(\lambda+\xi)\xi}\,d\xi = \frac{2\varepsilon}{\pi}\int_0^\infty \frac{\sin\xi}{(2\lambda\varepsilon+\xi)\xi}\,d\xi \sim \frac{2\varepsilon}{\pi}\int_0^1 \frac{d\xi}{2\lambda\varepsilon+\xi} \sim \frac{2\varepsilon}{\pi}\log\frac{1}{\varepsilon}$$

and so

$$\lim_{\varepsilon\downarrow 0}\left(\log\frac{1}{\varepsilon}\right)\int_0^\infty e^{-\lambda t} H(t_1, t_1+dt;\varepsilon) = \int_0^\infty e^{-\lambda t}\frac{dt}{t_1+t}. \tag{2}$$

From (2) we have

$$\lim_{\varepsilon\downarrow 0}\left(\log\frac{1}{\varepsilon}\right) H(t_1, t_1+t;\varepsilon) = \int_0^t \frac{dt}{t_1+t} = \log\frac{t_1+t}{t_1}. \tag{3}$$

The implication "(2) → (3)" can be verified, for example, in the following way: let, for some $\lambda_0 > 0$, $\mu_\varepsilon(dt)$ be a measure on $[0, +\infty]$ defined by

$$\mu_\varepsilon(dt) = e^{-\lambda_0 t}\left(\log\frac{1}{\varepsilon}\right) H(t_1, t_1+dt;\varepsilon), \quad \mu_\varepsilon(\{+\infty\}) = 0,$$

then by (2) $\{\mu_\varepsilon\}$ is uniformly bounded and so for any sequence $\varepsilon_m\downarrow 0$ we can select some subsequence $\varepsilon_n\downarrow 0$ such that

$$\mu_{\varepsilon_n} \to \mu \quad \text{(weakly)}$$

where μ is a measure on $[0, +\infty]$. Then

$$\int_0^\infty e^{-\lambda t}\mu_{\varepsilon_n}(dt) \to \int_0^\infty e^{-\lambda t}\mu(dt)$$

and by (2)

$$\int_0^\infty e^{-\lambda t}\mu(dt) = \int_0^\infty e^{-(\lambda+\lambda_0)t}\frac{dt}{t_1+t},$$

and so by the uniqueness theorem for Laplace transforms,

$$\mu(dt) = \frac{e^{-\lambda_0 t}}{t_1+t}\,dt.$$

Thus

$$\int_0^t e^{\lambda_0 t}\mu_{\varepsilon_n}(dt) \to \int_0^t e^{\lambda_0 t}\mu(dt),$$

that is

$$\left(\log\frac{1}{\varepsilon_n}\right) H(t_1, t_1+t;\varepsilon_n) \to \int_0^t \frac{dt}{t_1+t}.$$

Proof of the theorem. We prove (i) first. Note that, by the space-time relation of the Cauchy process, we have

$$H(t_1, t_2;\varepsilon) = H(ct_1, ct_2; c\varepsilon), \qquad c > 0.$$

Upon setting

$$E_k = \{w;\ |X(t)| < tg(t) \quad \text{for some} \quad 2^k \leqq t \leqq 2^{k+1}\},$$

$$\bar{E}_k = \{w;\ |X(t)| < 2^{k+1}g(2^k) \quad \text{for some} \quad 2^k \leqq t \leqq 2^{k+1}\}$$

and

$$\underline{E}_k = \{w; |X(t)| < 2^k g(2^{k+1}) \quad \text{for some} \quad 2^k \leqq t \leqq 2^{k+1}\}$$

for every positive integer k, we see

$$\underline{E}_k \subset E_k \subset \bar{E}_k$$

and by lemma 1

$$P_0(\underline{E}_k) = H(2^k, 2^{k+1}; 2^k g(2^{k+1})) = H(1, 2; g(2^{k+1})) \sim \frac{\log 2}{|\log g(2^{k+1})|}, \quad k \to \infty.$$

$$P_0(\bar{E}_k) = H(2^k, 2^{k+1}; 2^{k+1} g(2^k)) \sim \frac{\log 2}{|\log g(2^k)|}.$$

Now we shall assume the convergence of the integral in our theorem. Then we have

$$\sum_{k=1}^{\infty} P_0(\bar{E}_k) = O(1) \sum_{k=1}^{\infty} \frac{1}{|\log g(2^k)|} = O(1) \int_1^{\infty} \frac{dt}{t|\log g(t)|} < +\infty,$$

a fortiori $\sum_{k=1}^{\infty} P_0(E_k) < +\infty$ and so $g \in \mathfrak{L}_\infty$ by the Borel-Cantelli lemma.

Next we consider the divergent case. We shall then prove $g \in \mathfrak{U}_\infty$ by using the lemma in J. LAMPERTI [3]. In the same way as above we get

$$\sum_{k=1}^{\infty} P_0(\underline{E}_k) = O(1) \int_1^{\infty} \frac{dt}{t|\log g(t)|} = \infty.$$

Now define

$$\sigma_j(w) = \begin{cases} \inf\{t; |X(t)| < 2^j g(2^{j+1}), 2^j \leqq t \leqq 2^{j+1}\} \\ +\infty \qquad\qquad \text{if there is no such } t. \end{cases}$$

For any $j < k$, it follows from the strong Markov property that

$$\begin{aligned} P_0(\underline{E}_j \cap \underline{E}_k) = & \int_{2^j}^{2^{j+1}} \int_{R^1} P_y\{|X(t)| < 2^k g(^{k+1}) \quad \text{for some} \quad 2^k - s \leqq t \leqq 2^{k+1} - s\} \\ & \qquad \times P_0\{\sigma_j \in ds, X(\sigma_j) \in dy\}. \\ \leqq & \int_{R^1} P_y\{|X(t)| < 2^k g(2^{k+1}) \quad \text{for some} \quad 2^k - 2^{j+1} \leqq t \leqq 2^{k+1} - 2^j\} \\ & \qquad \times P_0\{X(\sigma_j) \in dy; \sigma_j < \infty\}. \end{aligned}$$

Now we have

$$\begin{aligned} & P_y\{|X(t)| < 2^k g(2^{k+1}) \quad \text{for some} \quad 2^k - 2^{j+1} \leqq t \leqq 2^{k+1} - 2^j\} \\ \leqq & P_0\{|X(t)| < 2^k g(2^{k+1}) \quad \text{for some} \quad 2^k - 2^{j+1} \leqq t \leqq 2^{k+1} - 2^j\}. \end{aligned}$$

This fact seems intuitively clear, but we shall prove it in the following lemma 2. *Further by lemma 1 there exist positive constants c_1, c_2 and K such that*

$$\frac{c_1}{|\log g(2^{k+1})|} \leqq P_0(\underline{E}_k) = H(1, 2; g(2^{k+1})) \leqq H\left(\frac{1}{2}, 2; g(2^{k+1})\right) \leqq \frac{c_2}{|\log g(2^{k+1})|}$$

for all $k \geqq K$.

Then, if $k > j \geqq K$ and $k \geqq j + 2$,

$$\begin{aligned} P_0(\underline{E}_j \cap \underline{E}_k) & \leqq P_0(\underline{E}_j) \cdot H(2^k - 2^{j+1}, 2^{k+1} - 2^j; 2^k g(2^{k+1})) \\ & \leqq P_0(\underline{E}_j) \cdot H(1/2, 2; g(2^{k+1})) \\ & \leqq P_0(\underline{E}_j) \frac{c_2}{|\log g(2^{k+1})|} \leqq \frac{c_2}{c_1} P(\underline{E}_j) P(\underline{E}_k). \end{aligned}$$

Then if we set $A_m = \underline{E}_{2m}$ and $A'_m = \underline{E}_{2m+1}$, the lemma of [3] is applicable to either $\{A_m\}$ or $\{A'_m\}$ and so we have $P_0\,(\overline{\lim_k}\,\underline{E}_k) > 0$, *a fortiori* $P_0\,(\overline{\lim_k}\,E_k) > 0$. This proves $g \in \mathfrak{U}_\infty$.

As for the proof of (ii), set for instance

$$\bar{E}_k = \{w;\, |X(t)| < 2^{-k} g(2^k) \quad \text{for some} \quad 2^{-k-1} \leqq t \leqq 2^{-k}\}$$

and note

$$P_0(\bar{E}_k) = H(1/2, 1; g(2^k)) \sim \frac{\log 2}{|\log g(2^k)|}, \qquad k \to \infty.$$

The above proof may be carried over in the same way.

Lemma 2. *For fixed* $0 < t_1 < t_2$ *and* $r > 0$,

$$P_x\{|X(t)| < r \quad \textit{for some} \quad t_1 \leqq t \leqq t_2\}$$
$$\leqq P_0\{|X(t)| < r \quad \textit{for some} \quad t_1 \leqq t \leqq t_2\}.$$

Proof. Let

$$F(x) = P_x\{|X(t)| < r \quad \text{for some} \quad t_1 \leqq t \leqq t_2\} = \int \Phi(y)\, p(t_1, y - x)\, dy,$$

where

$$\Phi(y) = P_y\{|X(t)| < r \quad \text{for some} \quad 0 \leqq t \leqq t_2 - t_1\}.$$

Then $\Phi(y)$ is a monotone decreasing function of $|y|$. In fact, if $0 \leqq x \leqq y$,

$$\Phi(x) = P_x\{|X(t)| < r \qquad \text{for some} \quad 0 \leqq t \leqq t_2 - t_1\}$$
$$= P_y\left\{|X(t)| < \frac{y}{x} r \quad \text{for some} \quad 0 \leqq t \leqq \frac{y}{x}(t_2 - t_1)\right\} \geqq \Phi(y)$$

in view of the space-time relation of the Cauchy process. Therefore

$$F(0) - F(x) = \int_{-\infty}^{\infty} p(t_1, y)\,(\Phi(y) - \Phi(y - |x|))\, dy$$

$$= \int_{-\infty}^{0} \left[p\left(t_1, \frac{|x|}{2} + y\right) - p\left(t_1, \frac{|x|}{2} - y\right)\right]\left[\Phi\left(\frac{|x|}{2} + y\right) - \Phi\left(\frac{|x|}{2} - y\right)\right] dy \geqq 0.$$

Remark. Applying the above method to the case $\alpha < N$, we find that for fixed $0 < t_1 < t_2$ there exist positive constants K_1, K_2 such that

$$K_1\, \varepsilon^{N-\alpha} \leqq H(t_1, t_2; \varepsilon) \leqq K_2\, \varepsilon^{N-\alpha}$$

for sufficiently small ε. In this case we see

$$\lim_{\varepsilon \downarrow 0} \frac{G_\lambda\{0, I'_\varepsilon\}}{G_\lambda\{0, I_\varepsilon\}} < 1$$

where $I_\varepsilon = \{x;\, |x| < \varepsilon\}$, $I'_\varepsilon = \{x;\, |x - a| < \varepsilon\}$ and a is a point with $|a| = \varepsilon$ and so we could not find the exact estimate of the form

$$H(t_1, t_2; \varepsilon) \sim K_3\, \varepsilon^{N-\alpha}(t_1^{1-N/\alpha} - t_2^{1-N/\alpha}), \quad \varepsilon \downarrow 0.$$

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